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INSTITVTIONVM CALCVLI INTEGRALIS

VOLVMEN PRIMVM

IN QVO METHODVS INTEGRANDI A PRIMIS PRINCIPIIS VSQVE AD INTEGRATIONEM AEQVATIONVM DIFFERENTIALIVM PRIMI GRADVS PERTRACTATVR.

AVCTORE

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INDEX CAPITVM,

in Volumine primo contentorum.

Sectio prima, de integratione *formularum* differentialium.

Sectio

PRAENO-

PRAENOTANDA.

DE
CALCVLO INTEGRALI
IN GENERE.

Definitio 1.

1.

Calculus integralis est methodus ex data differentialium relatione inueniendi relationem ipsarum quantitatum: et operatio, qua hoc praestatur, integratio vocari solet.

Coroll. 1.

2. Cum igitur calculus differentialis ex data relatione quantitatum variabilium, relationem differentialium inuestigare doceat: calculus integralis methodum inuersam suppeditat.

Coroll. 2.

3. Quemadmodum ſcilicet in Analyſi perpetuo binae operationes ſibi opponuntur, veluti ſubtractio additioni, diuiſio multiplicationi, extractio radicum euectioni ad poteſtates, ita etiam ſimili ratione calculus integralis calculo differentiali opponitur.

Coroll. 3.

4. Propoſita relatione quacunque inter binas quantitates variabiles x et y, in calculo differentiali methodus traditur rationem differentialium $dy:dx$ inueſtigandi: ſin autem viciſſim ex hac differentialium ratione ipſa quantitatum x et y relatio ſit definienda, hoc opus calculo integrali tribuitur.

Scholion 1.

5. In calculo differentiali iam notaui, quaeſtionem de differentialibus non abſolute ſed relatiue eſſe intelligendam, ita vt, ſi y fuerit functio quaecunque ipſius x, non tam ipſum eius differentiale dy, quam eius ratio ad differentiale dx ſit definienda. Cum enim omnia differentialia per ſe ſint nihilo aequalia, quaecunque functio y fuerit ipſius x, ſemper eſt $dy = 0$, neque ſic quicquam amplius abſolute quaeri poſſet. Verum quaeſtio ita rite proponi debet, vt dum x incrementum capit infinite paruum adeoque euaneſcens dx, definiatur ratio incrementi functionis y, quod inde capiet, ad iſtud dx: etſi enim vtrumque eſt $= 0$, tamen ratio certa inter ea intercedit, quae in calculo differentiali proprie inueſti-

vestigatur. Ita si fuerit $y = xx$, in calculo differentiali ostenditur esse $\frac{dy}{dx} = 2x$ neque hanc incrementorum rationem esse veram, nisi incrementum dx, ex quo dy nascitur, nihilo aequale statuatur. Verum tamen hac vera differentialium notione obseruata locutiones communes, quibus differentialia quasi absolute enunciantur, tolerari possunt, dummodo semper in mente saltem ad veritatem referantur. Recte ergo dicimus, si $y = xx$, fore $dy = 2xdx$, tametsi falsum non esset si quis diceret $dy = 3xdx$, vel $dy = 4xdx$, quoniam ob $dx = 0$ et $dy = 0$, hae aequalitates aeque subsisterent; sed prima sola rationi verae $\frac{dy}{dx} = 2x$ est consentanea.

Scholion 2.

6. Quemadmodum calculus differentialis apud Anglos methodus fluxionum appellatur, ita calculus integralis ab iis methodus fluxionum inuersa vocari solet, quandoquidem a fluxionibus ad quantitates fluentes reuertitur. Quas enim nos quantitates variabiles vocamus, eas Angli nomine magis idoneo quantitates fluentes vocant, et earum incrementa infinite parua seu euanescentia fluxiones nominant, ita vt fluxiones ipsis idem sint, quod nobis differentialia. Haec diuersitas loquendi ita iam vsu inualuit vt conciliatio vix vnquam sit expectanda, equidem Anglos in formulis loquendi lubenter imitarer, sed signa quibus nos vtimur, illorum signis longe anteferenda videntur. Verum cum tot iam libri vtraque

ratione conſcripti prodierint, huiusmodi conciliatio nullum vſum eſſet habitura.

Definitio 2.

7. Cum functionis cuiuscunque ipſius x differentiale huiusmodi habeat formam $X dx$, propoſita tali forma differentiali $X dx$, in qua X ſit functio quaecunque ipſius x, illa functio, cuius differentiale eſt $= X dx$, huius vocatur integrale, et praefixo ſigno $\int$ indicari ſolet, ita vt $\int X dx$ eam denotet quantitatem variabilem, cuius differentiale eſt $= X dx$.

Coroll. 1.

8. Quemadmodum ergo propoſitae formulae differentialis $X dx$ integrale, ſeu ea functio ipſius x, cuius differentiale eſt $= X dx$, quae hac ſcriptura $\int X dx$ indicatur, inueſtigari debeat, in calculo integrali eſt explicandum.

Coroll. 2.

9. Vti ergo littera d ſignum eſt differentiationis, ita littera $\int$ pro ſigno integrationis vtimur, ſicque haec duo ſigna ſibi mutuo opponuntur, et quaſi ſe deſtruunt ſcilicet $\int dX$ erit $= X$, quia ea quantitas denotatur cuius differentiale eſt dX, quae vtique eſt X.

Coroll. 3.

10. Cum igitur harum ipſius x functionum x^2, x^n, $\sqrt{(aa-xx)}$ differentialia ſint $2xdx$, $nx^{n-1}dx$, $\frac{-xdx}{\sqrt{(aa-xx)}}$ ſigno integrationis $\int$ adhibendo patet fore

$\int 2x$

$\int 2x\,dx = xx$; $\int nx^{n-1}dx = x^n$; $\int \frac{-x\,dx}{\sqrt{(aa-xx)}} = \sqrt{(aa-xx)}$ vnde vſus huius ſigni clarius perſpicitur.

Scholion 1.

11. Hic vnica tantum quantitas variabilis in computum ingredi videtur, cum tamen ſtatuamus tam in calculo differentiali quam integrali ſemper rationem duorum pluriumue differentialium ſpectari. Verum etſi hic vna tantum quantitas variabilis x apparet, tamen reuera duae conſiderantur; altera enim eſt ipſa illa functio, cuius differentiale ſumimus eſſe $X\,dx$, quae ſi deſignetur littera y erit $dy = X\,dx$, ſeu $\frac{dy}{dx} = X$, ita vt hic omnino ratio differentialium $dy:dx$ proponatur, quae eſt $= X$, indeque erit $y = \int X\,dx$: hoc autem integrale non tam ex ipſo differentiali $X\,dx$, quod vtique eſt $= 0$, quam ex eius ratione ad dx inueniri eſt cenſendum. Caeterum hoc ſignum $\int$ vocabulo *ſummae* efferri ſolet, quod ex conceptu parum idoneo, quo integrale tanquam ſumma omnium differentialium ſpectatur, eſt natum; neque maiore iure admitti poteſt, quam vulgo lineae ex punctis conſtare concipi ſolent.

Scholion 2.

12. At calculus integralis multo latius quam ad huiusmodi formulas integrandas patet, quae vnicam variabilem complectuntur. Quemadmodum enim hic functio vnius variabilis x ex data differentialis forma inueſtigatur; ita calculus integralis quoque

 extendi

extendi debet ad functiones duarum pluriumue variabilium inuestigandas, cum relatio quaedam differentialium fuerit proposita. Deinde calculus integralis non solum ad differentialia primi ordinis adstringitur, sed etiam praecepta tradere debet, quorum ope functiones tam vnius quam duarum pluriumue variabilium inuestigari queant, cum relatio quaedam differentialium secundi altiorisue cuiusdam ordinis fuerit data. Atque hanc ob rem definitionem calculi integralis ita instruximus, vt omnes huiusmodi inuestigationes in se complecteretur; differentialia enim cuinsque ordinis intelligi debent, et voce relationis, quae inter ea proponatur, sum vsus, vt latius pateret voce rationis, quae tantum duorum differentialium comparationem indicare videatur. Ex his ergo diuisionem calculi integralis constituere poterimus.

Definitio 3.

13. Calculus integralis diuiditur in duas partes, quarum prior tradit methodum functionem vnius variabilis inueniendi ex data quadam relatione inter eius differentialia tam primi quam altiorum ordinum.

Pars autem altera methodum continet functionem duarum pluriumue variabilium inueniendi, cum relatio inter eius differentialia siue primi siue altioris cuiusdam gradus fuerit proposita.

Coroll. 1.

14. Prout ergo functio ex data differentialium rela-

relatione inuenienda vel vnicam variabilem complectitur, vel duas pluresue: inde calculus integralis commode in duas partes principales dispescitur, quibus exponendis duos libros destinamus.

Coroll. 2.

15. Semper igitur calculus integralis in inventione functionum vel vnius vel plurium variabilium versatur, cum scilicet relatio quaepiam inter eius differentialia siue altioris cuiuspiam ordinis fuerit proposita.

Scholion.

16. Cum hic primam partem calculi integralis in inuestigatione functionum vnicae variabilis ex data differentialium relatione constituamus, plures partes pro numero variabilium functionem ingredientium constitui debere videatur, ita vt pars secunda functiones duarum variabilium, tertia trium, quarta quatuor etc. complectatur. Verum pro his posterioribus partibus methodus fere eadem requiritur, ita vt si inuentio functionum duas variabiles inuoluentium fuerit in potestate, via ad eas, quae plures variabiles implicant, satis sit patefacta; vnde inuentionem eiusmodi functionum, quae duas pluresue variabiles continent, commode coniungimus, indeque vnicam partem calculi integralis constituimus posteriori libro tractandam.

Caeterum haec altera pars in elementis adhuc nusquam est tractata, etiamsi eius vsus in Mecha-

nica

nica ac nec pue in doctrina fluidorum maximi sit vsus. Quocirca cum in hoc genere praeter prima rudimenta vix quicquam sit exploratum, noster secundus liber de calculo integrali admodum erit sterilis, ac praeter commemorationem eorum, quae adhuc desiderantur, parum erit expectandum; verum hoc ipsum ad scientiae incrementum multum conferre videtur.

Definitio 4.

17. Vterque de calculo integrali liber commode subdiuiditur in partes pro gradu differentialium, ex quorum relatione functionem quaesitam inuestigari oportet. Ita prima pars versatur in relatione differentialium primi gradus, secunda in relatione differentialium secundi gradus, quorsum etiam differentialia altiorum graduum ob tenuitatem eorum, quae adhuc sunt inuestigata, referri possunt.

Coroll. 1.

18. Vterque ergo liber constabit duabus partibus, in quarum priore relatio inter differentialia primi gradus proposita considerabitur, in posteriore vero eiusmodi integrationes occurrent, vbi relatio inter differentialia secundi altiorumue graduum proponitur.

Coroll. 2.

19. In primi ergo libri parte prima eiusmodi functio variabilis x inuenienda proponitur, vt posita ea functione $= y$, et $\frac{dy}{dx} = p$, relatio quaecunque data

data inter has tres quantitates x, y et p adimpleatur: seu proposita quacunque aequatione inter has ternas quantitates, vt indoles functionis y seu aequatio inter x et y tantum, exclusa p, eruatur.

Coroll. 3.

20. Posterioris autem partis primi libri quaestiones ita erunt comparatae, vt posito $\frac{dy}{dx} = p$, $\frac{dp}{dx} = q$, $\frac{dq}{dx} = r$ etc. si proponatur aequatio quaecunque inter quantitates x, y, p, q, r etc. indoles functionis y per x, seu aequatio inter x et y eliciatur.

Scholion 1.

21. Quae adhuc in calculo integrali sunt elaborata maximam partem ad libri primi partem primam sunt referenda, in qua excolenda Geometrae imprimis operam suam collocarunt: pauca sunt quae in parte posteriore sunt praestita et alter liber, quem secundum fecimus, etiamnunc fere vacuus est relictus. Prima autem pars libri primi, in qua potissimum nostra tractatio consumetur, denuo in plures sectiones distinguitur, pro modo relationis, quae inter quantitates x, y et $p = \frac{dy}{dx}$ proponitur. Relatio enim prae caeteris simplicissima est, quando $p = \frac{dy}{dx}$ aequatur functioni cuipiam ipsius x, qua posita $= X$, vt sit $\frac{dy}{dx} = X$ seu $dy = X\,dx$; totum negotium in integratione formulae differentialis $X\,dx$ absoluitur: huius operationis iam supra mentionem fecimus,

mus, quae vulgo ſub titulo integrationis formularum differentialium ſimplicium, ſeu vnicam variabilem inuoluentium tractari ſolet. Eodem res rediret, ſi $p = \frac{dy}{dx}$ aequaretur functioni ipſius y tantum, quandoquidem quantitates x et y ita inter ſe reciprocantur, vt altera tanquam functio alterius ſpectari poſſit: haec ergo ad ſectionem primam referentur. Sin autem $p = \frac{dy}{dx}$ aequetur expreſſioni ambas quantitates x et y inuoluenti, aequatio habetur differentialis huius formae $Pdx + Qdy = 0$, vbi P et Q ſunt expreſſiones quaecunque ex x, y et conſtantibus conflatae. Quanquam autem Geometrae multum in huiusmodi aequationum integratione deſudarunt, tamen vix vltra quosdam caſus ſatis particulares ſunt progreſſi. Sin autem p magis complicate per x et y determinatur, vt eius valor explicite exhiberi nequeat, veluti ſi fuerit

$$p^5 = xxp^2 - xyp + x^5 - y^5$$

ne via quidem conſtat tentanda, quomodo inde relatio inter x et y inueſtigari queat, pauca ergo, quae hic tradere licebit, cum praecedentibus ſecundam ſectionem primae partis libri primi occupabunt. Ita ex vniuerſa noſtra tractatione magis patebit, quod adhuc in calculo integrali deſideretur, quam quid iam ſit expeditum, cum hoc prae illo vt minima quaedam particula ſit ſpectandum.

Scho-

Scholion 2.

22. In ſingulis partibus, quas enarrauimus, fieri etiam ſolet, vt non ſolum vna quaedam functio, ſed etiam ſimul plures inueſtigentur, ita vt neutra ſine reliquis definiri poſſit, quemadmodum in Algebra communi vſu venit, vt ad ſolutionem problematis plures incognitae in calculum ſint introducendae, quae deinceps per totidem aequationes determinentur. Veluti ſi eiusmodi binae functiones y et z ipſius x ſint inueniendae, vt ſit

$$xdy + azzdx = 0 \text{ et } xxdz + bxydy = cdy$$

hinc nouae ſubdiuiſiones noſtrae tractationis conſtitui poſſent. Verum quia hic vt in Algebra communi totum negotium ad eliminationem vnius litterae reuocatur, vt deinceps duae tantum variabiles in vna aequatione ſuperſint, hinc tractatio non multiplicanda videtur.

Scholion 3.

23. In ſecundo libro calculi integralis, quo functio duarum pluriumue variabilium ex data differentialium relatione inueſtigatur, multo maior quaeſtionum varietas locum habet. Sit enim z functio binarum variabilium x et t inueſtiganda, et cum $\left(\frac{dz}{dx}\right)$ denotet rationem eius differentialis ad dx ſi ſola x pro variabili habiatur, at $\left(\frac{dz}{dt}\right)$ rationem eius differentialis ad dt, ſi ſola t variabilis ſumatur, prima pars eiusmodi continebit quaeſtiones, in quibus certa quaedam relatio inter quantitates x, t, z

et $\left(\frac{dz}{dx}\right)$, $\left(\frac{dz}{dt}\right)$ proponitur, et quaeſtio huc redit, vt hinc aequatio inter ſolas quantitates x, t et z eruatur; inde enim qualis z ſit functio ipſarum x et t, patebit. In ſecunda parte praeter has formulas $\left(\frac{dz}{dx}\right)$ et $\left(\frac{dz}{dt}\right)$ etiam iſtae $\left(\frac{ddz}{dxdx}\right)$, $\left(\frac{ddz}{dxdt}\right)$ et $\left(\frac{ddz}{dtdt}\right)$, in computum ingredientur: quarum ſignificatio ita eſt intelligenda, vt poſitis prioribus $\left(\frac{dz}{dx}\right)=p$ et $\left(\frac{dz}{dt}\right)=q$, vbi p et q iterum certae erunt functiones ipſorum x et t, futurum ſit ſimili expreſſionis modo,

$$\left(\frac{ddz}{dx\,dx}\right)=\left(\frac{dp}{dx}\right);\ \left(\frac{ddz}{dx\,dt}\right)=\left(\frac{dp}{dt}\right)=\left(\frac{dq}{dx}\right);\ \left(\frac{ddz}{dt\,dt}\right)=\left(\frac{dq}{dt}\right)$$

Propoſita ergo relatione inter has formulas et praecedentes ſimulque ipſas quantitates x, t et z, aequatio inter ternas iſtas quantitates ſolas x, t et z erui debet. Huiusmodi quaeſtiones frequenter occurrunt in Mechanica et Hydraulica, quando motus corporum flexibilium et fluidorum indagatur, ex quo maxime eſt optandum, vt haec altera ſectio ſecundi libri calculi integralis omni cura excolatur. Neque vero opus erit vt hanc inueſtigationem ad differentialia altiora extendamus, cum nullae adhuc quaeſtiones ſint tractatae, quae tanta calculi incrementa deſiderent.

Definitio 5.

24. Si functiones, quae in calculo integrali ex relatione differentialium quaeruntur, algebraice exhiberi nequeant, tum eae vocantur *tranſcendentes*, quandoquidem earum ratio vires Analyſeos communis tranſcendit.

Coroll. 1.

Coroll. 1.

25. Quoties ergo integratio non fuccedit, toties functio quae per integrationem quaeritur, pro transcendente est habenda. Ita si formula differentialis $X dx$ integrationem non admittit, eius integrale, quod ita indicari solet $\int X dx$ est functio transcendens ipsius x.

Coroll. 2.

26. Hinc intelligitur, si y fuerit functio transcendens ipsius x, viciſſim fore x functionem transcendentem ipsius y, atque ex hac conuerſione nouae functiones transcendentes oriuntur.

Coroll. 3.

27. Pro variis partibus et sectionibus calculi integralis naſcuntur etiam plura genera functionum transcendentium, quorum adeo numerus in infinitum exſurgit, vnde patet quanta copia omnium quantitatum poſſibilium nobis adhuc ſit ignota.

Scholion 1.

28. Iam ante quam in Analyſin infinitorum penetrauimus, ſpecies quasdam functionum transcendentium cognoſcere licuit. Primam ſuppeditauit doctrina logarithmorum, ſi enim y denotet logarithmum ipſius x, vt ſit $y = lx$, erit y vtique functio transcendens ipſius x, ſicque logarithmi quaſi primam ſpeciem functionum transcendentium conſtituunt.

 Deinde

Deinde cum ex aequatione $y = lx$ viciſſim ſit $x = e^y$, erit x vtique etiam functio tranſcendens ipſius y, ac tales functiones vocantur exponentiales. Porro autem conſideratio angulorum aliud genus aperuit, veluti ſi angulus, cuius ſinus eſt $= s$, ponatur $= \Phi$ vt ſit $\Phi = \text{Arc.ſin.} s$, nullum eſt dubium, quin Φ ſit functio tranſcendens ipſius s et quidem infinitiformis: hincque cum conuertendo prodeat $s = \text{ſin.} \Phi$, erit etiam ſinus s functio tranſcendens anguli Φ. Quanquam autem hae functiones tranſcendentes ſine ſubſidio calculi integralis ſunt agnitae, tamen in ipſo quaſi limine calculi integralis ad eas deducimur: earumque indoles ita nobis iam eſt perſpecta, vt propemodum functionibus algebraicis accenſeri queant. Quare etiam perpetuo in calculo integrali, quoties functiones tranſcendentes ibi repertas ad logarithmos vel angulos reuocare licet, eas tanquam algebraicas ſpectare ſolemus.

Scholion 2.

29. Cum calculus integralis ex inuerſione calculi differentialis oriatur, perinde ac reliquae methodi inuerſae ad notitiam noui generis quantitatum nos perducit. Ita ſi a tyrone primorum elementorum nihil praeter notitiam numerorum integrorum poſitiuorum poſtulemus, apprehenſa additione, ſtatim atque ad operationem inuerſam, ſubtractionem ſcilicet, ducitur, notionem numerorum negatiuorum aſſequetur. Deinde multiplicatione tradita, cum ad diuiſionem progreditur, ibi notionem fractionum accipiet.

cipiet. Porro postquam euectionem ad potestates didicerit, si per operationem inuersam extractionem radicum suscipiat, quoties negotium non succedit, ideam numerorum irrationalium adipiscetur, haecque cognitio per totam Analysin communem sufficiens censetur. Simili ergo modo calculus integralis, quatenus integratio non succedit, nouum nobis genus quantitatum transcendentium aperit. Non enim, vti omnium differentialia exhiberi possunt, ita vicissim omnium differentialium integralia exhibere licet.

Scholion 3.

30. Neque vero statim ac primi conatus in integratione expedienda fuerint initi, functiones quaesitae pro transcendentibus sunt habendae; fieri enim saepe solet, vt integrale etiam algebraicum nonnisi per operationes artificiosas obtineri queat. Deinde quando functio quaesita fuerit transcendens, sollicite videndum est, num forte ad species illas simplicissimas logarithmorum vel angulorum reuocari possit, quo casu solutio algebraicae esset aequiparanda. Quod si minus successerit, formam tamen simplicissimam functionum transcendentium, ad quam quaesitam reducere liceat, indagari conueniet. Ad vsum autem longe commodissimum est, vt valores functionum transcendentium vero proxime exhibentur, quem in finem insignis pars calculi integralis in inuestigationem serierum infinitarum impenditur, quae valores earum functionum contineant.

Theo-

Theorema.

31. Omnes functiones per calculum integralem inuentae sunt indeterminatae, ac requirunt determinationem ex natura quaestionis, cuius solutionem suppeditant, petendam.

Demonstratio.

31. Cum semper infinitae dentur functiones, quarum idem est differentiale, siquidem functionis $P+C$, quicunque valor constanti C tribuatur, differentiale idem est $=dP$: vicissim etiam proposito differentiali dP, integrale est $P+C$, vbi pro C quantitatem constantem quamcunque ponere licet, vnde patet eam functionem, cuius differentiale datur $=dP$, esse indeterminatam, cum quantitatem constantem arbitrariam in se inuoluat. Idem etiam eueniat necesse est, si functio ex quacunque differentialium relatione sit determinanda, semperque complectetur quantitatem constantem arbitrariam, cuius nullum vestigium in relatione differentialium apparuit. Determinabitur ergo huiusmodi functio per calculum integralem inuenta, dum constanti illi arbitrariae certus valor tribuitur, quem semper natura quaestionis, cuius solutio ad illam functionem perduxerat, suppeditabit.

Coroll. 1.

32. Si ergo functio y ipsius x ex relatione quapiam differentialium definitur, per constantem arbitrariam ingressam ita determinari potest, vt posito

ſito $x=a$ fiat $y=b$: quo facto functio erit determinata, et pro quouis valore ipſi x tributo functio y determinatum obtinebit valorem.

Coroll. 2.

33. Si ex relatione differentialium ſecundi gradus functio y definiatur, binas inuoluet conſtantes arbitrarias, ideoque duplicem determinationem admittit, qua effici poteſt vt poſito $x=a$, non ſolum y obtineat datum valorem b ſed etiam ratio $\frac{dy}{dx}$ dato valori c fiat aequalis.

Coroll. 3.

34. Si y ſit functio binarum variabilium x et t ex relatione differentialium eruta, etiam conſtantem arbitrariam inuoluet, cuius determinatione effici poterit, vt poſito $t=a$, aequatio inter y et x prodeat data ſeu naturam datae cuiuſpiam curuae exprimat.

Scholion.

35. Iſta functionum integralium, ſeu quae per calculum integralem ſunt inuentae, determinatio quouis caſu ex natura quaeſtionis tractatae facile deducitur, neque vlla difficultate laborat, niſi forte praeter neceſſitatem ſolutio ad differentialia fuerit perducta, cum per Analyſin communem erui potuiſſet: quo caſu perinde atque in Algebra quaſi radices inutiles ingeruntur. Cum autem haec determinatio tantum in applicatione ad certos caſus inſtituatur, hic vbi integrandi methodum in genere tradimus,

 inte-

integralia in omni amplitudine eruere conabimur, ita vt constantes per integrationem ingressae maneant arbitrariae, neque nisi conditio quaedam vrgeat, eas determinabimus. Caeterum determinatio functionum ipsius x simplicissima est, qua eae casu $x = 0$, ipsae euanescentes redduntur.

Definitio 6.

36. Integrale *completum* exhiberi dicitur, quando functio quaesita omni extensione cum constante arbitraria repraesentatur. Quando autem ista constans iam certo modo est determinata, integrale vocari solet *particulare*.

Coroll. 1.

37. Quouis ergo casu datur vnicum integrale completum; integralia autem particularia infinita exhiberi possunt. Sic differentialis xdx integrale completum est $\frac{1}{2}xx + C$, integralia autem particularia $\frac{1}{2}xx$; $\frac{1}{2}xx + 1$, $\frac{1}{2}xx + 2$ etc. multitudine infinita.

Coroll. 2.

38. Integrale ergo completum omnia integralia particularia in se complectitur; ex eoque haec omnia facile formari possunt. Vicissim autem ex integralibus particularibus, integrale completum non innotescit. Saepenumero autem, vti deinceps patebit, habetur methodus ex integrali particulari completum inueniendi.

Scholion.

Scholion.

39. Interdum facile eſt integrale particulare coniectura vel diuinatione aſſequi. Veluti ſi eiusmodi functio ipſius x, quae ſit y quaeritur, vt ſit $dy + yydx = dx + xxdy$, huic aequationi manifeſto ſatisfit ſumendo $y = x$, quod ergo eſt integrale particulare, quoniam, in eo nulla ineſt conſtans arbitraria: at integrale completum reperitur $y = \frac{1 + Cx}{C + x}$, quod illud particulare in ſe continet, ſumendo $C = \infty$. Simili modo ſumendo $C = 0$, hinc aliud integrale obtinetur $y = \frac{1}{x}$, quod ſuperiori aequationi perinde ſatisfacit ac prius $y = x$. Omnia autem integralia particularia, quaecunque ſatisfaciunt, contineri neceſſe eſt in formula generali $y = \frac{1 + Cx}{C + x}$, prouti conſtanti arbitrariae C alii atque alii valores tribuantur, ita ſumto $C = 1$ fit etiam $y = 1$. Plerumque autem euenire ſolet, vt etiamſi integrale quoddam particulare ſit algebraicum, tamen integrale completum ſit tranſcendens. Veluti ſi propoſita ſit haec aequatio $dy + ydx = dx + xdx$, ſtatim patet ſatisfieri poſito $y = x$, quod ergo eſt integrale particulare; verum integrale completum conſtantem arbitrariam C inuoluens eſt $y = x + Ce^{-x}$, denotante e numerum cuius logarithmus $= 1$, niſi ergo hic ſumatur $C = 0$, functio y ſemper eſt tranſcendens. Haec in genere notaſſe ſufficiat, antequam ad tractationem ipſam calculi integralis aggrediamur, quandoquidem ad omnes integrationes pertinent, nunc igitur forma tractationis expoſita ad opus tractandum pergamus.

CONSPECTVS
VNIVERSI OPERIS
DE
CALCVLO INTEGRALI.

LIBER PRIOR: tradit methodum inueſtigandi functiones vnius variabilis ex data quadam relatione differentialium, continetque duas partes:

Pars prior: quando relatio illa data tantum differentialia primi gradus complectitur.

Pars poſterior: quando relatio illa data differentialia ſecundi altiorumue graduum complectitur.

LIBER POSTERIOR: tradit methodum inueſtigandi functiones duarum pluriumue variabilium ex data quadam relatione differentialium, continetque duas partes:

Pars prior: quando relatio illa data tantum differentialia primi gradus complectitur.

Pars poſterior: quando relatio illa data differentialia ſecundi altiorumue graduum complectitur.

CALCVLI INTEGRALIS.
LIBER PRIOR.

PARS PRIMA
SEV
METHODVS INVESTIGANDI FVNCTIONES
VNIVS VARIABILIS EX DATA RELATIONE QVACVN-
QVE DIFFERENTIALIVM PRIMI GRADVS.

SECTIO PRIMA
DE
INTEGRATIONE FORMVLARVM
DIFFERENTIALIVM.

CAPVT I.

DE
INTEGRATIONE FORMVLARVM DIFFERENTIALIVM RATIONALIVM.

Definitio.

40.

Formula differentialis *rationalis* eſt, quando variabilis x, cuius functio quaeritur, differentiale dx multiplicatur in functionem rationalem ipſius x, ſeu ſi X deſignet functionem rationalem ipſius x, haec formula differentialis Xdx, dicitur rationalis.

Coroll. 1.

41. In hoc ergo capite eiusmodi functio ipſius x quaeritur, quae ſi ponatur y, vt $\frac{dy}{dx}$ aequetur functioni rationali ipſius x, ſeu poſita tali functione $= X$ vt ſit $\frac{dy}{dx} = X$.

Coroll. 2.

Coroll. 2.

42. Hinc quaeritur eiusmodi functio ipsius x, cuius differentiale sit $= Xdx$; huius ergo integrale, quod ita indicari solet $\int Xdx$, praebebit functionem quaesitam.

Coroll. 3.

43. Quodsi P fuerit eiusmodi functio ipsius x, vt eius differentiale dP sit $= Xdx$, quoniam quantitatis $P + C$ idem est differentiale, formulae propositae Xdx integrale completum est $P + C$.

Scholion 1.

44. Ad libri primi partem priorem huiusmodi referuntur quaestiones, quibus functiones solius variabilis x, ex data differentialium primi gradus relatione quaeruntur. Scilicet si functio quaesita $= y$ et $\frac{dy}{dx} = p$, id praestari oportet, vt proposita aequatione quacunque inter ternas quantitates x, y et p, inde indoles functionis y, seu aequatio inter x et y elisa littera p inueniatur. Quaestio autem sic in genere proposita vires analyseos adeo superare videtur, vt eius solutio nunquam expectari queat. In casibus igitur simplicioribus vires nostrae sunt exercendae, inter quos primum occurrit casus, quo p functioni cuipiam ipsius x puta X aequatur, vt sit $\frac{dy}{dx} = X$, seu $dy = Xdx$, ideoque integrale $y = \int Xdx$ requiratur, in quo primam sectionem collocamus. Verum et hic casus pro varia indole functionis X latissime patet, ac plurimis difficultatibus implicatur, vnde

vnde in hoc capite eiusmodi tantum quaestiones euoluere instituimus, in quibus ista functio X est rationalis deinceps ad functiones irrationales atque adeo transcendentes progressuri. Hinc ista pars commode in duas sectiones subdiuiditur, in quarum altera integratio formularum simplicium, quibus $p = \frac{dy}{dx}$ functioni tantum ipsius x aequatur, est tradenda, in altera autem rationem integrandi doceri conueniet, cum proposita fuerit aequatio quaecunque ipsarum x, y et p. Et cum in his duabus sectionibus ac potissimum priore a Geometris plurimum sit elaboratum, eae fere maximam partem totius operis complebunt.

Scholion 2.

45. Prima autem integrationis principia ex ipso calculo differentiali sunt petenda, perinde ac principia diuisionis ex multiplicatione et principia extractionis radicum ex ratione euectionis ad potestates sumi solent. Cum igitur si quantitas differentianda ex pluribus partibus constet, vt $P + Q - R$, eius differentiale sit $dP + dQ - dR$, ita vicissim si formula differentialis ex pluribus partibus constet, vt $Pdx + Qdx - Rdx$, integrale erit $\int Pdx + \int Qdx - \int Rdx$, singulis scilicet partibus seorsim integrandis. Deinde cum quantitatis aP differentiale sit adP, formulae differentialis $aPdx$ integrale erit $a\int Pdx$: scilicet per quam quantitatem constantem formula differentialis multiplicatur, per eandem integrale multiplicari debet. Ita si formula differentialis sit $aPdx + bQdx$

 $+ cRdx$

$+cRdx$ quaecunque functiones ipsius x litteris P, Q, R designentur, integrale erit $a\int Pdx + b\int Qdx + c\int Rdx$, ita vt integratio tantum in singulis formulis Pdx, Qdx et Rdx, sit instituenda, hocque facto insuper adiici debet constans arbitraria C, vt integrale completum obtineatur.

Problema 1.

46. Inuenire functionem ipsius x, vt eius differentiale sit $= ax^n dx$, seu integrare formulam differentialem $ax^n dx$.

Solutio.

Cum potestatis x^m differentiale sit $mx^{m-1}dx$, erit vicissim $\int mx^{m-1}dx = m\int x^{m-1}dx = x^m$, ideoque $\int x^{m-1}dx = \frac{1}{m}x^m$; fiat $m-1=n$ seu $m=n+1$ erit $\int x^n dx = \frac{1}{n+1}x^{n+1}$ et $a\int x^n dx = \frac{a}{n+1}x^{n+1}$. Vnde formulae differentialis propositae $ax^n dx$ integrale completum erit $\frac{a}{n+1}x^{n+1} + C$, cuius ratio vel inde patet, quod eius differentiale reuera fit $= ax^n dx$. Atque haec integratio semper locum habet, quicunque numerus exponenti n tribuatur, siue positiuus siue negatiuus, siue integer siue fractus, siue etiam irrationalis.

Vnicus casus hinc excipitur, quo est exponens $n = -1$, seu haec formula $\frac{adx}{x}$ integranda proponitur. Verum in calculo differentiali iam ostendimus, si lx denotet logarithmum hyperbolicum ipsius x, fore eius differentiale $= \frac{dx}{x}$, vnde vicissim concludimus

mus esse $\int \frac{dx}{x} = lx$ et $\int \frac{a\,dx}{x} = alx$. Quare adiecta constante arbitraria, erit formulae $\frac{a\,dx}{x}$ integrale completum $= alx + C = lx^a + C$, quod etiam pro C ponendo lc ita exprimitur lcx^a.

Coroll. 1.

47. Formulae ergo differentialis $ax^n dx$ integrale semper est algebraicum, solo excepto casu quo $n = -1$, et integrale per logarithmos exprimitur, qui ad functiones transcendentes sunt referendi. Est scilicet $\int \frac{a\,dx}{x} = alx + C = lcx^a$.

Coroll. 2.

48. Si exponens n numeros positiuos denotet, sequentes integrationes vtpote maxime obuiae probe sunt tenendae:

$$\int a\,dx = ax + C; \int ax\,dx = \tfrac{a}{2}xx + C; \int ax^2 dx = \tfrac{a}{3}x^3 + C$$
$$\int ax^3 dx = \tfrac{a}{4}x^4 + C; \int ax^4 dx = \tfrac{a}{5}x^5 + C; \int ax^5 dx = \tfrac{a}{6}x^6 + C.$$

Coroll. 3.

49. Si n sit numerus negatiuus, posito $n = -m$ fit $\int \frac{a\,dx}{x^m} = \frac{a}{1-m}x^{1-m} + C = \frac{-a}{(m-1)x^{m-1}} + C$; vnde hi casus simplicores notentur:

$$\int \frac{a\,dx}{x^2} = \frac{-a}{x} + C;\ \int \frac{a\,dx}{x^3} = \frac{-a}{2xx} + C;\ \int \frac{a\,dx}{x^4} = \frac{-a}{3x^3} + C$$
$$\int \frac{a\,dx}{x^5} = \frac{-a}{4x^4} + C;\ \int \frac{a\,dx}{x^6} = \frac{-a}{5x^5} + C;\ \text{etc.}$$

Coroll. 4.

50. Quin etiam si n denotet numeros fractos, integralia hinc obtinentur. Sit primo $n = \frac{m}{2}$, erit $\int adx\sqrt{x^m} = \frac{2a}{m+2} x\sqrt{x^m} + C$, vnde casus notentur:

$\int adx\sqrt{x} = \frac{2a}{3} x\sqrt{x} + C$; $\int axdx\sqrt{x} = \frac{2a}{5} x^2\sqrt{x} + C$

$\int axxdx\sqrt{x} = \frac{2a}{7} x^3\sqrt{x} + C$; $\int ax^3dx\sqrt{x} = \frac{2a}{9} x^4\sqrt{x} + C$.

Coroll. 5.

51. Ponatur etiam $n = \frac{-m}{2}$, et habebitur

$$\int \frac{adx}{\sqrt{x^m}} = \frac{2a}{2-m} \cdot \frac{x}{\sqrt{x^m}} + C = \frac{-2a}{(m-2)\sqrt{x^{m-2}}} + C$$

vnde hi casus notentur:

$$\int \frac{adx}{\sqrt{x}} = 2a\sqrt{x} + C; \int \frac{adx}{x\sqrt{x}} = \frac{-2a}{\sqrt{x}} + C$$

$$\int \frac{adx}{xx\sqrt{x}} = \frac{-2a}{3x\sqrt{x}} + C; \int \frac{adx}{x^3\sqrt{x}} = \frac{-2a}{5x^2\sqrt{x}} + C$$

Coroll. 6.

52. Si in genere ponamus $n = \frac{\mu}{\nu}$, fiet $\int ax^{\frac{\mu}{\nu}} dx = \frac{\nu a}{\mu+\nu} x^{\frac{\mu+\nu}{\nu}} + C$, seu per radicalia

$$\int adx\sqrt[\nu]{x^\mu} = \frac{\nu a}{\mu+\nu} \sqrt[\nu]{x^{\mu+\nu}} + C$$

sin autem ponatur $n = \frac{-\mu}{\nu}$ habebitur:

$$\int \frac{adx}{x^{\frac{\mu}{\nu}}} = \frac{\nu a}{\nu-\mu} x^{\frac{\nu-\mu}{\nu}} + C$$ seu pro radicalia

$$\int \frac{adx}{\sqrt[\nu]{x^\mu}} = \frac{\nu a}{\nu-\mu} \sqrt[\nu]{x^{\nu-\mu}} + C$$

Scholion 1.

Scholion 1.

53. Quanquam in hoc capite functiones tantum rationales tractare instituteram, tamen istae irrationalitates tam sponte se obtulerunt, vt perinde ac rationales tractari possint. Caeterum hinc quoque formulae magis complicatae integrari possunt, si pro x functiones alius cuiuspiam variabilis z statuantur. Veluti si ponamus $x = f + gz$, erit $dx = gdz$: quare si pro a scribamus $\frac{a}{g}$, habebitur

$$\int adz(f+gz)^n = \frac{a}{(n+1)g}(f+gz)^{n+1} + C$$

casu autem singulari quo $n = -1$,

$$\int \frac{adz}{f+gz} = \frac{a}{g}\, l(f+gz) + C.$$

Tum si sit $n = -m$ fiet

$$\int \frac{adz}{(f+gz)^m} = \frac{-a}{(m-1)g(f+gz)^{m-1}} + C.$$

Ac posito $n = \frac{\mu}{\nu}$, prodit

$$\int adz(f+gz)^{\frac{\mu}{\nu}} = \frac{\nu a}{(\nu+\mu)g}(f+gz)^{\frac{\mu}{\nu}+1} + C$$

posito autem $n = -\frac{\mu}{\nu}$ obtinetur,

$$\int \frac{adz}{(f+gz)^{\frac{\mu}{\nu}}} = \frac{\nu a(f+gz)}{(\nu-\mu)g(f+gz)^{\frac{\mu}{\nu}}} + C.$$

Scholion 2.

54. Caeterum hic insignis proprietas annotari meretur. Cum hic quaeratur functio y, vt sit $dy = ax^n dz$, si ponamus $\frac{dy}{dx} = p$, haec habebitur re-

 latio

latio $p = ax^n$, ex qua functio y inuestigari debet. Quoniam igitur est $y = \frac{a}{n+1} x^{n+1} + C$, ob $ax^n = p$ erit quoque $y = \frac{px}{n+1} + C$, sicque casum habemus, vbi relatio differentialium per aequationem quandam inter x, y et p proponitur, cuique iam nouimus satisfieri per aequationem $y = \frac{a}{n+1} x^{n+1} + C$. Verum haec non amplius erit integrale completum pro relatione in aequatione $y = \frac{px}{n+1} + C$ contenta, sed tantum particulare, quoniam integrale illud non inuoluit nouam constantem, quae in relatione differentiali non insit. Integrale autem completum est $y = \frac{aD}{n+1} x^{n+1} + C$: nouam constantem D inuoluens: hinc enim fit $\frac{dy}{dx} = aDx^n = p$, ideoque $y = \frac{px}{n+1} + C$. Etsi hoc non ad praesens institutum pertinet, tamen notasse iuuabit.

Problema 2.

55. Inuenire functionem ipsius x, cuius differentiale sit $= X dx$, denotante X functionem quamcunque rationalem integram ipsius x, seu definire integrale $\int X dx$.

Solutio.

Cum X sit functio rationalis integra ipsius x in hac forma contineatur necesse est:

$$X = \alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \zeta x^5 \text{ etc.}$$

vnde per problema praecedens integrale quaesitum est

$$\int X dx = C + \alpha x + \tfrac{1}{2}\beta x^2 + \tfrac{1}{3}\gamma x^3 + \tfrac{1}{4}\delta x^4 + \tfrac{1}{5}\varepsilon x^5 + \tfrac{1}{6}\zeta x^6 \text{ etc.}$$

atque

atque in genere si sit $X = \alpha x^\lambda + \beta x^\mu + \gamma x^\nu$ etc.
erit $\int X dx = C + \frac{\alpha}{\lambda+1} x^{\lambda+1} + \frac{\beta}{\mu+1} x^{\mu+1} + \frac{\gamma}{\nu+1} x^{\nu+1}$ etc.
vbi exponentes λ, μ, ν etc. etiam numeros tam negatiuos quam fractos significare possunt, dummodo notetur, si fuerit $\lambda = -1$ fore $\int \frac{\alpha dx}{x} = \alpha lx$, qui est vnicus casus ad ordinem transcendentium referendus.

Problema 3.

56. Si X denotet functionem quamcunque rationalem fractam ipsius x, methodum describere, cuius ope formulae $X dx$ integrale inuestigari conveniat.

Solutio.

Sit igitur $X = \frac{M}{N}$, ita vt M et N futurae sint functiones integrae ipsius x, ac primo dispiciatur, num summa potestas ipsius x in numeratore M tanta sit, vel etiam maior quam in denominatore N? quo casu ex fractione $\frac{M}{N}$ partes integrae per diuisionem eliciantur, quarum integratio, cum nihil habeat difficultatis, totum negotium reducitur ad eiusmodi fractionem $\frac{M}{N}$, in cuius numeratore M summa potestas ipsius x minor sit quam denominatore N.

Tum quaerantur omnes factores ipsius denominatoris N; tam simplices si fuerint reales, quam duplices reales, vicem scilicet binorum simplicium imaginariorum gerentes; simulque videndum est, vtrum hi factores omnes sint inaequales nec ne? pro factorum enim aequalitate alio modo resolutio fractio-

fractionis $\frac{M}{N}$ in fractiones simplices est instituenda, quandoquidem ex singulis factoribus fractiones partiales nascuntur, quarum aggregatum fractioni propositae $\frac{M}{N}$ aequatur. Scilicet ex factore simplici $a+bx$ nascitur fractio $\frac{A}{a+bx}$; si bini sint aequales seu denominator N factorem habeat $(a+bx)^2$, hinc nascuntur fractiones $\frac{A}{(a+bx)^2}+\frac{B}{a+bx}$; ex huiusmodi autem factore $(a+bx)^3$ hae tres fractiones $\frac{A}{(a+bx)^3}+\frac{B}{(a+bx)^2}+\frac{C}{a+bx}$ et ita porro.

Factor autem duplex, cuius forma est $aa-2abx\cos.\zeta+bbxx$ nisi alius ipsi fuerit aequalis, dabit fractionem partialem $\frac{A+Bx}{aa-2abx\cos.\zeta+bbxx}$: si autem denominator N duos huiusmodi factores aequales inuoluat, inde nascuntur binae huiusmodi fractiones partiales:

$$\frac{A+Bx}{(aa-2abx\cos.\zeta+bbxx)^2}+\frac{C+Dx}{aa-2abx\cos.\zeta+bbxx}$$

at si cubus adeo $(aa-2abx\cos.\zeta+bbxx)^3$ fuerit factor denominatoris N, ex eo oriuntur huiusmodi tres fractiones partiales:

$$\frac{A+Bx}{(aa-2abx\cos.\zeta+bbxx)^3}+\frac{C+Dx}{(aa-2abx\cos.\zeta+bbxx)^2}+\frac{E+Fx}{aa-2abx\cos.\zeta+bbxx}$$

et ita porro.

Cum igitur hoc modo fractio proposita $\frac{M}{N}$ in omnes suas fractiones simplices fuerit resoluta, omnes continebuntur in alterutra harum formarum

$$\text{vel } \frac{A}{(a+bx)^n} \text{ vel } \frac{A+Bx}{(aa-2abx\cos.\zeta+bbxx)^n}$$

ac

ac ſingulos iam per dx multiplicatos integrari oportet, erit omnium horum integralium aggregatum valor functionis quaeſitae $\int X\,dx = \int \frac{M}{N}\,dx$.

Coroll. 1.

57. Pro integratione ergo omnium huiusmodi formularum $\frac{M}{N}\,dx$, totum negotium reducitur ad integrationem huiusmodi binarum formularum

$$\int \frac{A\,dx}{(a+bx)^n} \text{ et } \int \frac{(A+Bx)\,dx}{(aa-2abx\cos.\zeta+bbxx)^n}$$

dum pro n ſucceſſiue ſcribuntur numeri 1, 2, 3, 4 etc.

Coroll. 2.

58. Ac prioris quidem formae integrale iam ſupra (52) eſt expeditum, vnde patet fore

$$\int \frac{A\,dx}{a+bx} = \frac{A}{b}\,l(a+bx) + \text{Conſt.}$$

$$\int \frac{A\,dx}{(a+bx)^2} = \frac{-A}{b(a+bx)} + \text{Conſt.}$$

$$\int \frac{A\,dx}{(a+bx)^3} = \frac{-A}{2b(a+bx)^2} + \text{Conſt.}$$

et generatim:

$$\int \frac{A\,dx}{(a+bx)^n} = \frac{-A}{(n-1)b(a+bx)^{n-1}} + \text{Conſt.}$$

Coroll. 3.

59. Ad propoſitum ergo abſoluendum nihil aliud ſupereſt, niſi vt integratio huius formulae

$$\int \frac{(A+Bx)\,dx}{(aa-2abx\cos.\zeta+bbxx)^n}$$

 docea-

doceatur, primo quidem casu $n=1$, tum vero casibus $n=2$, $n=3$, $n=4$ etc.

Scholion 1.

60. Nisi vellemus imaginaria euitare, totum negotium ex iam traditis confici posset: denominatore enim N in omnes suos factores simplices resoluto, siue sint reales siue imaginarii, fractio proposita semper resolui poterit in fractiones partiales huius formae $\frac{A}{a+bx}$, vel huius $\frac{A}{(a+bx)^n}$, quarum integralia cum sint in promtu, totius formae $\frac{M}{N}dx$, integrale habetur. Tum autem non parum molestum foret binas partes imaginarias ita coniungere, vt expressio realis resultaret, quod tamen rei natura absolute exigit.

Scholion 2.

61. Hic vtique postulamus resolutionem cuiusque functionis integrae in factores nobis concedi, etiamsi algebra neutiquam adhuc eo sit perducta, vt haec resolutio actu institui possit. Hoc autem in Analysi vbique postulari solet, vt quo longius progrediamur, ea quae retro sunt relicta, etiamsi non satis fuerint explorata, tanquam cognita assumamus: sufficere scilicet hic potest, omnes factores per methodum approximationum quantumuis prope assignari posse. Simili modo cum in calculo integrali longius processerimus, integralia omnium huiusmodi formula-

mularum $X dx$, quaecunque functio ipsius x littera X significetur, tanquam cognita spectabimus, plurimumque nobis praestitisse videbimur, si integralia magis abscondita ad eas formas reducere valuerimus: atque hoc etiam in vsu practico nihil turbat, cum valoros talium formularum $\int X dx$, quantumuis prope assignare liceat, vti in sequentibus ostendemus. Caeterum ad has integrationes resolutio denominatoris N, in suos factores absolute est necessaria, propterea quod singuli hi factores in expressionem integralis ingrediuntur: paucissimi sunt casus, iique maxime obuii, quibus ista resolutione carere possumus, veluti si proponatur haec formula $\frac{x^{n-1}dx}{1+x^n}$, statim patet posito $x^n = v$ eam abire in $\frac{dv}{n(1+v)}$, cuius integrale est $\frac{1}{n}l(1+v) = \frac{1}{n}l(1+x^n)$ vbi resolutione in factores non fuerat opus. Verum huiusmodi casus per se tam sunt perspicui, vt eorum tractatio nulla peculiari explicatione indigeat.

Problema 4.

62. Inuenire integrale huius formulae:

$$y = \int \frac{(A+Bx)dx}{aa - 2abx\cos.\zeta + bbxx}.$$

Solutio.

Cum numerator duabus constet partibus $Adx + Bxdx$, haec posterior $Bxdx$ sequenti modo tolli poterit. Cum sit

$$l(aa - 2abx\cos.\zeta + bbxx) = \int \frac{-2abdx\cos.\zeta + 2bbxdx}{aa - 2abx\cos.\zeta + bbxx},$$

multiplicetur haec aequatio per $\frac{B}{2bb}$ et a propofita auferatur: fic enim prodibit

$$y - \frac{B}{2bb} l(aa - 2abx\,\text{cof.}\zeta + bbxx) = \int \frac{(A + \frac{Ba\,\text{cof.}\zeta}{b})\,dx}{aa - 2abx\,\text{cof.}\zeta + bbxx}$$

ita vt haec tantum formula integranda fuperfit. Ponatur breuitatis gratia $A + \frac{Ba\,\text{cof.}\zeta}{b} = C$, vt habeatur haec formula:

$$\int \frac{C\,dx}{aa - 2abx\,\text{cof.}\zeta + bbxx},$$

quae ita exhiberi poteft

$$\int \frac{C\,dx}{aa\,\text{fin.}\zeta^2 + (bx - a\,\text{cof.}\zeta)^2}.$$

Statuatur $bx - a\,\text{cof.}\zeta = av\,\text{fin.}\zeta$, hincque $dx = \frac{a\,dv\,\text{fin.}\zeta}{b}$ vnde formula noftra erit:

$$\int \frac{C\,a\,dv\,\text{fin.}\zeta : b}{aa\,\text{fin.}\zeta^2(1+vv)} = \frac{C}{ab\,\text{fin.}\zeta} \int \frac{dv}{1+vv}.$$

Ex calculo autem differentiali nouimus effe $\int \frac{dv}{1+vv} = \text{Arc. tang.}\, v = \text{Arc. tang.}\, \frac{bx - a\,\text{cof.}\zeta}{a\,\text{fin.}\zeta}$; vnde ob $C = \frac{Ab + Ba\,\text{cof.}\zeta}{b}$, erit noftrum integrale $\frac{Ab + Ba\,\text{cof.}\zeta}{abb\,\text{fin.}\zeta}$ Arc. tang. $\frac{bx - a\,\text{cof.}\zeta}{a\,\text{fin.}\zeta}$. Quocirca formulae propofitae $\frac{(A + Bx)\,dx}{aa - 2abx\,\text{cof.}\zeta + bbxx}$ integrale eft
$\frac{B}{2bb} l(aa - 2abx\,\text{cof.}\zeta + bbxx) + \frac{Ab + Ba\,\text{cof.}\zeta}{abb\,\text{fin.}\zeta}$ Arc. tang. $\frac{bx - a\,\text{cof.}\zeta}{a\,\text{fin.}\zeta}$, quod vt fiat completum conftans arbitraria C info- per addatur.

Coroll. 1.

63. Si ad Arc. tang. $\frac{bx - a\,\text{cof.}\zeta}{a\,\text{fin.}\zeta}$ addamus Arc. tang $\frac{\text{cof.}\zeta}{\text{fin.}\zeta}$ quippe qui in conftante addenda contentus conci-

concipiatur, prodibit Arc. tang. $\frac{bx\,\text{fin.}\,\zeta}{a-bx\,\text{cof.}\,\zeta}$, ficque habebimus:

$$\int\frac{(A+Bx)dx}{aa-2abx\,\text{cof.}\,\zeta+bbxx}=\frac{B}{2bb}l(aa-2abx\,\text{cof.}\,\zeta+bbxx)$$
$$+\frac{Ab+Ba\,\text{cof.}\,\zeta}{abb\,\text{fin.}\,\zeta}\text{ Arc. tang. }\frac{bx\,\text{fin.}\,\zeta}{a-bx\,\text{cof.}\,\zeta}$$

adiecta conftante C.

Coroll. 2.

64. Si velimus vt integrale hoc euanefcat pofito $x=0$ conftans C fumi debet $=\frac{-B}{2bb}laa$, ficque fiet:

$$\int\frac{(A+Bx)dx}{aa-2abx\,\text{cof.}\,\zeta+bbxx}=\frac{B}{bb}l\frac{\sqrt{(aa-2abx\,\text{cof.}\,\zeta+bbxx)}}{a}$$
$$+\frac{Ab+Ba\,\text{cof.}\,\zeta}{abb\,\text{fin.}\,\zeta}\text{ Arc. tang. }\frac{bx\,\text{fin.}\,\zeta}{a-bx\,\text{cof.}\,\zeta}.$$

Pendet ergo hoc integrale partim a logarithmis, partim ab arcubus circularibus feu angulis.

Coroll. 3.

65. Si littera B euanefcat, pars a logarithmis pendens euanefcit, fitque

$$\int\frac{A\,dx}{aa-2abx\,\text{cof.}\,\zeta+bbxx}=\frac{A}{ab\,\text{fin.}\,\zeta}\text{ Arc. tang. }\frac{bx\,\text{fin.}\,\zeta}{a-bx\,\text{cof.}\,\zeta}+C$$

ficque per folum angulum definitur.

Coroll. 4.

66. Si angulus ζ fit rectus, ideoque $\text{cof.}\,\zeta=0$, et $\text{fin.}\,\zeta=1$, habebitur:

$$\int\frac{(A+Bx)dx}{aa+bbxx}=\frac{B}{bb}l\frac{\sqrt{(aa+bbxx)}}{a}+\frac{A}{ab}\text{ Arc. tang. }\frac{bx}{a}+C$$

si angulus ζ sit 60°, ideoque $\cos.\zeta = \frac{1}{2}$ et $\sin.\zeta = \frac{\sqrt{3}}{2}$, erit

$$\int \frac{(A + Bx)dx}{aa - abx + bbxx} = \frac{B}{bb} l \frac{\sqrt{(aa - abx + bbxx)}}{a} + \frac{2Ab + Ba}{abb\sqrt{3}} \text{Arc. tang.} \frac{bx\sqrt{3}}{2a - bx}.$$

At si $\zeta = 120^\circ$ ideoque $\cos.\zeta = -\frac{1}{2}$ et $\sin.\zeta = \frac{\sqrt{3}}{2}$, erit

$$\int \frac{(A + Bx)dx}{aa + abx + bbxx} = \frac{B}{bb} l \frac{\sqrt{(aa + abx + bbxx)}}{a} + \frac{2Ab - Ba}{abb\sqrt{3}} \text{Arc. tang.} \frac{bx\sqrt{3}}{2a + bx}.$$

Scholion 1.

67. Omnino hic notatu dignum euenit, quod casu $\zeta = 0$, quo denominator $aa - 2abx + bbxx$, fit quadratum, ratio anguli ex integrali discedat. Posito enim angulo ζ infinite paruo, erit $\cos.\zeta = 1$ et $\sin.\zeta = \zeta$; vnde pars logarithmica fit $\frac{B}{bb} l \frac{a - bx}{a}$, et altera pars:

$$\frac{Ab + Ba}{abb\zeta} \text{Arc. tang} \frac{bx\zeta}{a - bx} = \frac{(Ab + Ba)x}{ab(a - bx)}$$

quia arcus infinite parui $\frac{bx\zeta}{a - bx}$ tangens ipsi est aequalis, sicque haec pars fit algebraica. Quocirca erit

$$\int \frac{(A + Bx)dx}{(a - bx)^2} = \frac{B}{bb} l \frac{a - bx}{a} + \frac{(Ab + Ba)x}{ab(a - bx)} + \text{Const.}$$

cuius veritas ex praecedentibus est manifesta: est enim

$$\frac{A + Bx}{(a - bx)^2} = \frac{-B}{b(a - bx)} + \frac{Ab + Ba}{b(a - bx)^2}.$$

Iam vero est

$$\int \frac{-Bdx}{b(a - bx)} = \frac{B}{bb} l(a - bx) - \frac{B}{bb} la = \frac{B}{bb} l \frac{a - bx}{a}$$

$$\int \frac{(Ab + Ba)dx}{b(a - bx)^2} = \frac{Ab + Ba}{bb(a - bx)} - \frac{(Ab + Ba)}{abb} = \frac{(Ab + Ba)x}{ab(a - bx)},$$

siquidem vtraque integratio ita determinetur vt casu $x = 0$, integralia euanescant.

Scholion 2.

Scholion 2.

68. Simili modo, quo hic vſi ſumus, ſi in formula differentiali fracta $\frac{M\,dx}{N}$, ſumma poteſtas ipſius x, in numeratore M vno gradu minor ſit quam in denominatore N, etiam is terminus tolli poterit. Sit enim

$$M = Ax^{n-1} + Bx^{n-2} + Cx^{n-3} \text{ etc.}$$

$$\text{et } N = \alpha x^{n} + \beta x^{n-1} + \gamma x^{n-2} \text{ etc.}$$

ac ponatur $\frac{M\,dx}{N} = dy$: Cum iam ſit

$$dN = n\alpha x^{n-1}dx + (n-1)\beta x^{n-2}dx + (n-2)\gamma x^{n-3}dx \text{ etc.}$$

erit $\frac{A\,dN}{n\alpha N} = \frac{dx}{N}\left(Ax^{n-1} + \frac{(n-1)A\beta}{n\alpha}x^{n-2} + \frac{(n-2)A\gamma}{n\alpha}x^{n-3} \text{ etc.}\right)$

quo valore inde ſubtracto remanebit

$$dy - \frac{A\,dN}{n\alpha N} = \frac{dx}{N}\left(\left(B - \frac{(n-1)A\beta}{n\alpha}\right)x^{n-2} + \left(C - \frac{(n-2)A\gamma}{n\alpha}\right)x^{n-3} \text{ etc.}\right)$$

Quare ſi breuitatis gratia ponatur

$$B - \frac{(n-1)A\beta}{n\alpha} = \mathfrak{B};\; C - \frac{(n-2)A\gamma}{n\alpha} = \mathfrak{C};\; D - \frac{(n-3)A\delta}{n\alpha} = \mathfrak{D} \text{ etc.}$$

obtinebitur,

$$y = \frac{A}{n\alpha}lN + \int \frac{dx(\mathfrak{B}x^{n-2} + \mathfrak{C}x^{n-3} + \mathfrak{D}x^{n-4} + \text{etc.})}{\alpha x^{n} + \beta x^{n-1} + \gamma x^{n-2} + \delta x^{n-3} \text{ etc.}} = \int \frac{M\,dx}{N}.$$

Hoc igitur modo omnes formulae differentiales fractae eo reduci poſſunt, vt ſumma poteſtas ipſius x, in numeratore duobus pluribusue gradibus minor ſit quam in denominatore.

Problema 5.

69. Formulam integralem $\int \frac{(A + Bx)dx}{(aa - 2abx\,\text{coſ.}\,\zeta + bbxx)^{n+1}}$

ad

ad aliam similem reducere, vbi potestas denominatoris fit vno gradu inferior.

Solutio.

Sit breuitatis gratia $aa - 2abx\cos.\zeta + bbxx = X$, ac ponatur $\int \frac{(A + Bx)dx}{X^{n+1}} = y$. Cum ob $dX = -2abdx\cos.\zeta + 2bbxdx$ fit

$$d.\frac{C + Dx}{X^n} = \frac{-n(C + Dx)dX}{X^{n+1}} + \frac{Ddx}{X^n}$$ ideoque

$$\frac{C + Dx}{X^n} = \int \frac{2nb(C + Dx)(a\cos.\zeta - bx)dx}{X^{n+1}} + \int \frac{Ddx}{X^n},$$

habebimns

$$y + \frac{C+Dx}{X^n} = \int \frac{dx(A + 2nCab\cos.\zeta + x(B + 2nDab\cos.\zeta - 2nCbb) - 2nDbbxx)}{X^{n+1}} + \int \frac{Ddx}{X^n}.$$

Iam in formula priori litterae C et D ita definiantur, vt numerator per X fiat diuisibilis: oportet ergo fit $= -2nDXdx$ vnde nanciscimur:

$A + 2nCab\cos.\zeta = -2nDaa$ et

$B + 2nDab\cos.\zeta - 2nCbb = 4nDab\cos.\zeta$

seu $B - 2nCbb = 2nDab\cos.\zeta$ hincque

$2nDa = \frac{B - 2nCbb}{b\cos.\zeta}$ at ex priori conditione est

$2nDa = \frac{-A - 2nCab\cos.\zeta}{a}$, quibus aequatis fit

$Ba + Ab\cos.\zeta - 2nCabb\sin.\zeta^2 = 0$ seu $C = \frac{Ba + Ab\cos.\zeta}{2nabb\sin.\zeta^2}$ vnde $B - 2nCbb = \frac{Ba\sin.\zeta^2 - Ba - Ab\cos.\zeta}{a\sin.\zeta^2} = \frac{-Ab\cos.\zeta - Ba\cos.\zeta^2}{a\sin.\zeta^2}$ ita vt reperiatur $D = \frac{-Ab - Ba\cos.\zeta}{2naab\sin.\zeta^2}$. Sumtis ergo litteris $C = \frac{Ba + Ab\cos.\zeta}{2nabb\sin.\zeta^2}$ et $D = \frac{-Ab - Ba\cos.\zeta}{2naab\sin.\zeta^2}$ crit

$y +$

$$y+\frac{C+Dx}{X^n}=\int\frac{-2nDdx}{X^n}+\int\frac{Ddx}{X^n}=-(2n-1)D\int\frac{dx}{X^n}$$

ideoque

$$\int\frac{(A+Bx)dx}{X^{n+1}}=\frac{-C-Dx}{X^n}-(2n-1)D\int\frac{dx}{X^n}\text{ siue}$$

$$\int\frac{A+Bx)dx}{X^{n+1}}=\frac{-Baa-Aab\,\text{cos}.\zeta+(Abb+Bab\,\text{cos}.\zeta)x}{2naabb\,\text{sin}.\zeta^2X^n}$$

$$+\frac{(2n-1)(Ab+Ba\,\text{cos}.\zeta)}{2naab\,\text{sin}.\zeta^2}\int\frac{dx}{X^n}.$$

Quare si formula $\int\frac{dx}{X^n}$ constet, etiam integrale hoc $\int\frac{(A+Bx)dx}{X^{n+1}}$ assignari poterit.

Coroll. 1.

70. Cum igitur manente $X=aa-2abx\,\text{cos}.\zeta+bbxx$, fit $\int\frac{dx}{X}=\frac{1}{ab\,\text{sin}.\zeta}\text{Arc. tang.}\frac{bx\,\text{sin}.\zeta}{a-bx\,\text{cos}.\zeta}+\text{Const.}$ erit

$$\int\frac{(A+Bx)dx}{X^2}=\frac{-Baa-Aab\,\text{cos}.\zeta+(Abb+Bab\,\text{cos}.\zeta)x}{2aabb\,\text{sin}.\zeta^2X}$$

$$+\frac{Ab+Ba\,\text{cos}.\zeta}{2a^3bb\,\text{sin}.\zeta^3}\text{Arc. tang.}\frac{bx\,\text{sin}.\zeta}{a-bx\,\text{cos}.\zeta}+\text{Const.}$$

ideoque posito $B=0$ et $A=1$ fiet

$$\int\frac{dx}{X^2}=\frac{-a\,\text{cos}.\zeta+hx}{2aab\,\text{sin}.\zeta^2X}+\frac{1}{2a^3b\,\text{sin}.\zeta^3}\text{Arc. tang.}\frac{hx\,\text{sin}.\zeta}{a-bx\,\text{cos}.\zeta}+\text{Const.}$$

Integrale ergo $\int\frac{(A+Bx)dx}{X^2}$ logarithmos non inuoluit.

Coroll. 2.

71. Hinc ergo cum sit

$$\int\frac{dx}{X^3}=\frac{-a\,\text{cos}.\zeta+bx}{4aab\,\text{sin}.\zeta^2.X^2}+\frac{3}{4aa\,\text{sin}.\zeta^2}\int\frac{dx}{X^2}+\text{Const.}$$

 erit

erit illum valorem ſubſtituendo

$$\int\frac{dx}{X^3}=\frac{-a\cos.\zeta+bx}{4aab\sin.\zeta^2.X^2}+\frac{3(-a\cos.\zeta+bx)}{2.4a^4b\sin.\zeta^4.X}+\frac{1.3}{2.4a^5b\sin.\zeta^5}\text{Arc. tang.}\frac{bx\sin.}{a-bx\cos.\zeta}$$

hincque porro concluditur:

$$\int\frac{dx}{X^4}=\frac{-a\cos.\zeta+bx}{6aab\sin.\zeta^2.X^3}+\frac{5(-a\cos.\zeta+bx)}{4.6a^4b\sin.\zeta^4.X^2}+\frac{3.5(-a\cos.\zeta+bx)}{2.4.6a^6b\sin.\zeta^6.X}$$
$$+\frac{1.3.5}{2.4.6a^7b\sin.\zeta^7}\text{Arc. tang.}\frac{bx\sin.\zeta}{a-bx\cos.\zeta}.$$

Coroll. 3.

72. Sic vlterius progrediendo omnium huiusmodi formularum integralia obtinebuntur:

$$\int\frac{dx}{X},\ \int\frac{dx}{X^2},\ \int\frac{dx}{X^3},\ \int\frac{dx}{X^4}\ \text{etc.}$$

quorum primum arcu circulari ſolo exprimitur, reliqua vero praeterea partes algebraicas continent.

Scholion.

73. Sufficit autem integralia $\int\frac{dx}{X^{n+1}}$ noſſe, quia formula $\int\frac{(A+Bx)dx}{X^{n+1}}$ facile eo reducitur, ita enim repraeſentari poteſt:

$$\frac{1}{2bb}\int\frac{2Abbdx+2Bbbxdx-2Babdx\cos.\zeta+2Babdx\cos.\zeta}{X^{n+1}}$$

quae ob $2bbxdx-2abdx\cos.\zeta=dX$ abit in hanc

$$\frac{1}{2bb}\int\frac{BdX}{X^{n+1}}+\frac{1}{b}\int\frac{(Ab+Ba\cos.\zeta)dx}{X^{n+1}}.$$

At

At $\int \frac{dX}{X^{n+1}} = -\frac{1}{nX^n}$, vnde habebitur

$$\int \frac{(A+Bx)dx}{X^{n+1}} = \frac{-B}{2nbbX^n} + \frac{Ab+Ba\operatorname{cof.}\zeta}{b} \int \frac{dx}{X^{n+1}}$$

vnde tantum opus eſt noſſe integralia $\int \frac{dX}{X^{n+1}}$, quae modo exhibuimus. Atque haec ſunt omnia ſubſidia quibus indigemus ad omnes formulas fractas $\frac{M}{N}dx$ integrandas, dummodo M et N ſunt functiones integrae ipſius x. Quocirca in genere integratio omnium huiusmodi formularum $\int V dx$, vbi V eſt functio, rationalis ipſius x quaecunque, eſt in poteſtate: de quibus notandum eſt, niſi integralia fuerint algebraica, ſemper vel per logarithmos vel angulos exhiberi poſſe. Nihil aliud igitur ſupereſt, niſi vt hanc methodum aliquot exemplis illuſtremus.

Exemplum 1.

74. *Propoſita formula differentiali* $\frac{(A+Bx)dx}{\alpha+\beta x+\gamma xx}$, *definire eius integrale.*

Cum in numeratore variabilis x pauciores habeat dimenſiones, quam in denominatore, haec fractio nullas partes integras complectitur. Hinc denominatoris indoles perpendatur, vtrum habeat duos factores ſimplices reales nec ne? ac priori caſu num factores ſint aequales: ex quo tres habebimus caſus euoluendos.

I. Habeat denominator ambos factores aequales, sitque $=(a+bx)^2$, et fractio $\frac{A+Bx}{(a+bx)^2}$ resoluitur in has duas, $\frac{Ab-Ba}{b(a+bx)^2}+\frac{B}{b(a+bx)}$, vnde fit

$$\int\frac{(A+Bx)dx}{(a+bx)^2}=\frac{Ba-Ab}{bb(a+bx)}+\frac{B}{bb}l(a+bx)+\text{Const.}$$

si integrale ita determinetur, vt euanescat posito $x=0$, reperitur

$$\int\frac{(A+Bx)dx}{(a+bx)^2}=\frac{(Ab-Ba)x}{ab(a+bx)}+\frac{B}{bb}l\frac{a+bx}{a}.$$

II. Habeat denominator duos factores inaequales, sitque proposita haec formula $\frac{A+Bx}{(a+bx)(f+gx)}dx$, et haec fractio resoluitur in has partiales:

$$\frac{Ab-Ba}{bf-ag}\cdot\frac{dx}{a+bx}+\frac{Ag-Bf}{ag-bf}\cdot\frac{dx}{f+gx},$$

vnde obtinetur integrale quaesitum:

$$\int\frac{(A+Bx)dx}{(a+bx)(f+gx)}=\frac{Ab-BA}{b(bf-ag)}l\frac{a+bx}{a}+\frac{Ag-Bf}{g(ag-bf)}l\frac{f+gx}{f}+\text{Const.}$$

Ponatur $\frac{Ab-Ba}{b(bf-ag)}=m+n$ et $\frac{Bf-Ag}{g(bf-ag)}=m-n$ vt integrale fiat $ml\frac{(a+bx)(f+gx)}{af}+nl\frac{f(a+bx)}{a(f+gx)}$, erit $2m=\frac{B(bf-ag)}{bg(bf-ag)}=\frac{B}{bg}$ et

$$2n=\frac{2Abg-Bag-Bbf}{bg(bf-ag)},\text{ erit ergo}$$

$$\int\frac{(A+Bx)dx}{(a+bx)(f+gx)}=\frac{B}{2bg}l\frac{(a+bx)(f+gx)}{af}+\frac{2Abg-B(ag+bf)}{2bg(bf-ag)}l\frac{f(a+bx)}{a(f+gx)}.$$

III. Sint denominatoris factores simplices ambo imaginarii quo casu formam habebit $aa-2abx\cos.\zeta+bbxx$; qui casus cum supra iam sit tractatus, erit

$$\int\frac{(A+Bx)dx}{aa-2abx\cos.\zeta+bbxx}=\frac{B}{bb}l\frac{\sqrt{(aa-2abx\cos.\zeta+bbxx)}}{a}$$
$$+\frac{Ab+Ba\cos.\zeta}{abb\sin.\zeta}\text{Arc. tang.}\frac{bx\sin\zeta}{a-bx\cos.\zeta}.$$

Coroll. 1.

Coroll. 1.

75. Casu secundo quo $f = a$, et $g = -b$, erit

$$\int \frac{(A + Bx)\,dx}{aa - bbxx} = \frac{-B}{2bb} l \frac{aa - bbxx}{aa} + \frac{A}{2ab} l \frac{a + bx}{a - bx},$$

hinc seorsim sequitur:

$$\int \frac{A\,dx}{aa - bbxx} = \frac{A}{2ab} l \frac{a + bx}{a - bx} + C \text{ et}$$

$$\int \frac{Bx\,dx}{aa - bbxx} = \frac{-B}{2bb} l \frac{aa - bbxx}{aa} = \frac{B}{bb} l \frac{a}{\sqrt{(aa - bbxx)}} + C.$$

Coroll. 2.

76. Casu tertio si ponamus cos. $\zeta = 0$, habemus

$$\int \frac{(A + Bx)\,dx}{aa + bbxx} = \frac{B}{bb} l \frac{\sqrt{(aa + bbxx)}}{a} + \frac{A}{ab} \text{Arc. tang.} \frac{bx}{a} + C$$

hincque sigillatim:

$$\int \frac{A\,dx}{aa + bbxx} = \frac{A}{ab} \text{Arc. tang.} \frac{bx}{a} + C, \text{ et}$$

$$\int \frac{Bx\,dx}{aa + bbxx} = \frac{B}{bb} l \frac{\sqrt{(aa + bbxx)}}{a} + C.$$

Exemplum 2.

77. *Proposita formula differentiali* $\frac{x^{m-1}dx}{1 + x^n}$, *siquidem exponens* $m - 1$ *minor sit quam* n *integrale definire.*

In capite vltimo Institut. Calculi Differential. inuenimus fractiones simplices, in quas haec fractio $\frac{x^m}{1 + x^n}$ resoluitur, sumto π pro mensura duorum

 angulo-

angulorum rectorum, in hac forma generali contineneri:

$$\frac{2\,\text{fin.}\frac{(2k-1)\pi}{n}\,\text{fin.}\frac{m(2k-1)\pi}{n} - 2\,\text{cof.}\frac{m(2k-1)\pi}{n}\left(x - \text{cof.}\frac{(2k-1)\pi}{n}\right)}{n\left(1 - 2x\,\text{cof.}\frac{(2k-1)\pi}{n} + xx\right)}$$

vbi pro k fucceffiue omnes numeros 1, 2, 3, etc. fubftitui conuenit, quoad $2k-1$ numerum n fuperare incipiat. Hac ergo forma in dx ducta et cum generali noftra: $\frac{(A + Bx)dx}{aa - 2abx\,\text{cof.}\,\zeta + bbxx}$ comparata, fit

$$a = 1,\ b = 1,\ \zeta = \frac{(2k-1)\pi}{n};\ \text{et}$$

$$A = \tfrac{2}{n}\,\text{fin.}\frac{(2k-1)\pi}{n}\,\text{fin.}\frac{m(2k-1)\pi}{n} + \tfrac{2}{n}\,\text{cof.}\frac{(2k-1)\pi}{n}\,\text{cof.}\frac{m(2k-1)\pi}{n}$$

feu $A = \frac{2}{n}\,\text{cof.}\frac{(m-1)(2k-1)\pi}{n}$

et $B = -\frac{2}{n}\,\text{cof.}\frac{m(2k-1)\pi}{n}$, vnde fit

$$Ab + Ba\,\text{cof.}\,\zeta = \tfrac{2}{n}\,\text{fin.}\frac{(2k-1)\pi}{n}\,\text{fin.}\frac{m(2k-1)\pi}{n},$$

ac propterea huius partis integrale erit

$$-\tfrac{2}{n}\,\text{cof.}\frac{m(2k-1)\pi}{n}\,l\sqrt{\left(1 - 2x\,\text{cof.}\frac{(2k-1)\pi}{n} + xx\right)}$$

$$+ \tfrac{2}{n}\,\text{fin.}\frac{m(2k-1)\pi}{n}\,\text{Arc. tang.}\frac{x\,\text{fin.}\frac{(2k-1)\pi}{n}}{1 - x\,\text{cof.}\frac{(2k-1)\pi}{n}}.$$

Ac fi n fit numerus impar praeterea accedit fractio $\frac{\pm dx}{n(1+x)}$ cuius integrale eft $\pm \frac{1}{n} l(1+x)$ vbi fignum fuperius valet, fi m impar, inferius vero fi m par. Quocirca integrale quaefitum $\int \frac{x^{m-1}dx}{1+x^n}$, fequenti modo exprimetur:

$-\frac{2}{n}$ cof.

$$-\frac{2}{n}\cos.\frac{m\pi}{n}\,l\sqrt{(1-2x\cos.\frac{\pi}{n}+xx)}+\frac{2}{n}\sin.\frac{m\pi}{n}\,\text{Arc.tang.}\frac{x\sin.\frac{\pi}{n}}{1-x\cos.\frac{\pi}{n}}$$

$$-\frac{2}{n}\cos.\frac{3m\pi}{n}\,l\sqrt{(1-2x\cos.\frac{3\pi}{n}+xx)}+\frac{2}{n}\sin.\frac{3m\pi}{n}\,\text{Arc.tang.}\frac{x\sin.\frac{3\pi}{n}}{1-x\cos.\frac{3\pi}{n}}$$

$$-\frac{2}{n}\cos.\frac{5m\pi}{n}\,l\sqrt{(1-2x\cos.\frac{5\pi}{n}+xx)}+\frac{2}{n}\sin.\frac{5m\pi}{n}\,\text{Arc.tang.}\frac{x\sin.\frac{5\pi}{n}}{1-x\cos.\frac{5\pi}{n}}$$

$$-\frac{2}{n}\cos.\frac{7m\pi}{n}\,l\sqrt{(1-2x\cos.\frac{7\pi}{n}+xx)}+\frac{2}{n}\sin.\frac{7m\pi}{n}\,\text{Arc.tang.}\frac{x\sin.\frac{7\pi}{n}}{1-x\cos.\frac{7\pi}{n}}$$

etc.

ſecundum numeros impares ipſo n minores, ſicque totum obtinetur integrale ſi n fuerit numerus par; ſin autem n ſit numerus impar, inſuper accedit haec pars $\pm\frac{1}{n}l(1+x)$, prout m ſit numerus vel impar vel par: vnde ſi $m=1$, accedit inſuper $+\frac{1}{n}l(1+x)$.

Coroll. 1.

78. Sumamus $m=1$, vt habeatur forma $\int\frac{dx}{1+x^n}$, et pro variis caſibus ipſius n adipiſcimur:

I. $\int\frac{dx}{1+x}=l(1+x)$

II. $\int\frac{dx}{1+x^2}=\text{Arc tang.}\,x$

III. $\int\frac{dx}{1+x^3}=-\frac{2}{3}\cos.\frac{\pi}{3}\,l\sqrt{(1-2x\cos.\frac{\pi}{3}+xx)}+\frac{2}{3}\sin.\frac{\pi}{3}\,\text{Arc.tang.}\frac{x\sin.\frac{\pi}{3}}{1-x\cos.\frac{\pi}{3}}$
$+\frac{1}{3}l(1+x)$

IV.

$$\text{IV.}\int\frac{dx}{1+x^4}=\begin{cases}-\tfrac{2}{4}\text{cof.}\tfrac{\pi}{4}\,l\sqrt{(1-2x\,\text{cof.}\tfrac{\pi}{4}+xx)}+\tfrac{2}{4}\text{fin.}\tfrac{\pi}{4}\,\text{Arc.tang.}\dfrac{x\,\text{fin.}\frac{\pi}{4}}{1-x\,\text{cof.}\frac{\pi}{4}}\\ -\tfrac{2}{4}\text{cof.}\tfrac{3\pi}{4}\,l\sqrt{(1-2x\,\text{cof.}\tfrac{3\pi}{4}+xx)}+\tfrac{2}{4}\text{fi.}\tfrac{3\pi}{4}\,\text{Arc.tang.}\dfrac{x\,\text{fin.}\frac{3\pi}{4}}{1-x\,\text{cof.}\frac{3\pi}{4}}\end{cases}$$

$$\text{V.}\int\frac{dx}{1+x^5}=\begin{cases}-\tfrac{2}{5}\text{cof.}\tfrac{\pi}{5}\,l\sqrt{(1-2x\,\text{cof.}\tfrac{\pi}{5}+xx)}+\tfrac{2}{5}\text{fin.}\tfrac{\pi}{5}\,\text{Arc tang.}\dfrac{x\,\text{fin.}\frac{\pi}{5}}{1-x\,\text{cof.}\frac{\pi}{5}}\\ -\tfrac{2}{5}\text{cof.}\tfrac{3\pi}{5}\,l\sqrt{(1-2x\,\text{cof.}\tfrac{3\pi}{5}+xx)}+\tfrac{2}{5}\text{fi.}\tfrac{3\pi}{5}\,\text{Arc.tang.}\dfrac{x\,\text{fin.}\frac{3\pi}{5}}{1-x\,\text{cof.}\frac{3\pi}{5}}\\ +\tfrac{1}{5}l(1+x)\end{cases}$$

$$\text{VI.}\int\frac{dx}{1+x^6}=\begin{cases}-\tfrac{2}{6}\text{cof.}\tfrac{\pi}{6}\,l\sqrt{(1-2x\,\text{cof.}\tfrac{\pi}{6}+xx)}+\tfrac{2}{6}\text{fin.}\tfrac{\pi}{6}\,\text{Arc.tang.}\dfrac{x\,\text{fin}\,\frac{\pi}{6}}{1-x\,\text{cof.}\frac{\pi}{6}}\\ -\tfrac{2}{6}\text{cof.}\tfrac{3\pi}{6}\,l\sqrt{(1-2x\,\text{cof.}\tfrac{3\pi}{6}+xx)}+\tfrac{2}{6}\text{fi.}\tfrac{3\pi}{6}\,\text{Arc.tang.}\dfrac{x\,\text{fin.}\frac{3\pi}{6}}{1-x\,\text{cof.}\frac{3\pi}{6}}\\ -\tfrac{2}{6}\text{cof.}\tfrac{5\pi}{6}\,l\sqrt{(1-2x\,\text{cof.}\tfrac{5\pi}{6}+xx)}+\tfrac{2}{6}\text{fi.}\tfrac{5\pi}{6}\,\text{Arc.tang.}\dfrac{x\,\text{fin.}\frac{5\pi}{6}}{1-x\,\text{cof.}\frac{5\pi}{6}}.\end{cases}$$

Coroll. 2.

79. Loco finuum et cofinuum valores, vbi commode fieri poteft, fubftituendo obtinemus:

$$\int\frac{dx}{1+x^3}=-\tfrac{1}{3}l\sqrt{(1-x+xx)}+\tfrac{1}{\sqrt{3}}\,\text{Arc. tang.}\frac{x\sqrt{3}}{2-x}+\tfrac{1}{3}l(1+x)$$

feu $\int\frac{dx}{1+x^3}=\frac{1}{3}l\frac{1+x}{\sqrt{(1-x+xx)}}+\frac{1}{\sqrt{3}}\,\text{Arc. tang}\,\frac{x\sqrt{3}}{2-x}$

Deinde ob $\text{fin.}\frac{\pi}{4}=\text{cof.}\frac{\pi}{4}=\frac{1}{\sqrt{2}}=\text{fin.}\frac{3\pi}{4}=-\text{cof.}\frac{3\pi}{4}$ fit

$$\int\frac{dx}{1+x^4}=+\frac{1}{2\sqrt{2}}\,l\frac{\sqrt{(1+x\sqrt{2}+xx)}}{\sqrt{(1-x\sqrt{2}+xx)}}+\frac{1}{2\sqrt{2}}\,\text{Arc. tang.}\frac{x\sqrt{2}}{1-xx},$$

tum vero

$$\int\frac{dx}{1+x^6}=\frac{1}{2\sqrt{3}}\,l\frac{\sqrt{(1+x\sqrt{3}+xx)}}{\sqrt{(1-x\sqrt{3}+xx)}}+\tfrac{1}{6}\,\text{Arc. tang.}\frac{3x(1-xx)}{1-4xx+x^4}.$$

Exem-

Exemplum 3.

80. *Propoſita formula differentiali* $\frac{x^{m-1}dx}{1-x^n}$ *ſiquidem exponens* m − 1 *ſit minor quam* n, *eius integrale definire.*

Functionis fractae $\frac{x^{m-1}}{1-x^n}$ pars ex factore quocunque oriunda hac forma continetur

$$\frac{2\,\mathrm{ſin.}\frac{2k\pi}{n}\,\mathrm{ſin.}\frac{2mk\pi}{n} - \mathrm{coſ.}\frac{2mk\pi}{n}\left(x - \mathrm{coſ.}\frac{2k\pi}{n}\right)}{n\left(1 - 2x\,\mathrm{coſ.}\frac{2k\pi}{n} + xx\right)}$$

quae cum forma noſtra $\frac{A+Bx}{aa-2abx\,\mathrm{coſ}.\zeta+bbxx}$ comparata dat $a=1$, $b=1$, $\zeta=\frac{2k\pi}{n}$;

$$A = \frac{2}{n}\,\mathrm{ſin.}\frac{2k\pi}{n}\,\mathrm{ſin.}\frac{2\pi k\pi}{n} + \frac{2}{n}\,\mathrm{coſ.}\frac{2k\pi}{n}\,\mathrm{coſ.}\frac{2mk\pi}{n}$$

$B = -\frac{2}{n}\,\mathrm{coſ.}\frac{2mk\pi}{n}$, hincque $Ab + Ba\,\mathrm{coſ.}\zeta = \frac{2}{n}\,\mathrm{ſin.}\frac{2k\pi}{n}\,\mathrm{ſin.}\frac{2mk\pi}{n}$.

Ex quo integrale hinc oriundum erit

$$-\frac{2}{n}\,\mathrm{coſ.}\frac{2km\pi}{n}\,l\sqrt{\left(1-2x\,\mathrm{coſ.}\frac{2k\pi}{n}+xx\right)} + \frac{2}{n}\,\mathrm{ſin.}\frac{2km\pi}{n}\,\mathrm{Arc.tang.}\frac{x\,\mathrm{ſin.}\frac{2k\pi}{n}}{1-x\,\mathrm{coſ.}\frac{2k\pi}{n}}$$

vbi pro k ſucceſſiue omnes numeri 0, 1, 2, 3 etc. ſubſtitui debent, quamdiu $2k$ non ſuperat n. At caſu $k=0$ fit integralis pars $-\frac{1}{n}l(1-x)$: et quando n eſt numerus par, vltima pars oritur ex $2k=n$, quae ergo erit

$$-\frac{2}{n}\,\mathrm{coſ.}\,m\pi\,l\sqrt{(1+2x+xx)} = -\frac{\mathrm{coſ.}\,m\pi}{n}\,l(1+x)$$

ergo ſi m eſt par erit $\mathrm{coſ.}\,m\pi = +1$, at ſi m impar, fit

$\cos. m\pi = -1$. Quocirca integrale $\int \frac{x^{m-1}dx}{1-x^n}$, hoc modo exprimitur:

$$-\tfrac{1}{n}l(1-x)$$

$$-\tfrac{2}{n}\cos.\tfrac{2m\pi}{n}\,l\surd(1-2x\cos.\tfrac{2\pi}{n}+xx)+\tfrac{2}{n}\sin.\tfrac{2m\pi}{n}\,\text{Arc. tang.}\,\frac{x\sin.\frac{2\pi}{n}}{1-x\cos.\frac{2\pi}{n}}$$

$$-\tfrac{2}{n}\cos.\tfrac{4m\pi}{n}\,l\surd(1-2x\cos.\tfrac{4\pi}{n}+xx)+\tfrac{2}{n}\sin.\tfrac{4m\pi}{n}\,\text{Arc. tang.}\,\frac{x\sin.\frac{4\pi}{n}}{1-x\cos.\frac{4\pi}{n}}$$

$$-\tfrac{2}{n}\cos.\tfrac{6m\pi}{n}\,l\surd(1-2x\cos.\tfrac{6\pi}{n}+xx)+\tfrac{2}{n}\sin.\tfrac{6m\pi}{n}\,\text{Arc. tang.}\,\frac{x\sin.\frac{6\pi}{n}}{1-x\cos.\frac{6\pi}{n}}$$

etc.

Corollarium.

81. Sit $m=1$ et pro n successiue numeri 1, 2, 3 etc. substituantur, vt nanciscamur sequentes integrationes:

$$\text{I.}\int\frac{dx}{1-x}=-l(1-x)$$

$$\text{II.}\int\frac{dx}{1-xx}=-\tfrac{1}{2}l(1-x)+\tfrac{1}{2}l(1+x)=\tfrac{1}{2}l\frac{1+x}{1-x}$$

$$\text{III.}\int\frac{dx}{1-x^3}=-\tfrac{1}{3}l(1-x)-\tfrac{2}{3}\cos.\tfrac{2}{3}\pi\,l\surd(1-2x\cos.\tfrac{2}{3}\pi+xx)+\tfrac{2}{3}\sin.\tfrac{2}{3}\pi\,\text{Arc.tang.}\frac{x\sin.\frac{2}{3}\pi}{1-x\cos.\frac{2}{3}\pi}$$

$$\text{IV.}\int\frac{dx}{1-x^4}=-\tfrac{1}{4}l(1-x)-\tfrac{2}{4}\cos.\tfrac{2}{4}\pi\,l\surd(1-2x\cos.\tfrac{2}{4}\pi+xx)+\tfrac{2}{4}\sin.\tfrac{2}{4}\pi\,\text{Arc.tang.}\frac{x\sin\frac{2}{4}\pi}{1-x\cos.\frac{2}{4}\pi}$$
$$+\tfrac{1}{4}l(1+x)$$

$$\text{V.}\int\frac{dx}{1-x^5}=-\tfrac{1}{5}l(1-x)-\tfrac{2}{5}\cos.\tfrac{2}{5}\pi\,l\surd(1-2x\cos.\tfrac{2}{5}\pi+xx)+\tfrac{2}{5}\sin.\tfrac{2}{5}\pi\,\text{Arc.tang.}\frac{x\sin.\frac{2}{5}\pi}{1-x\cos.\frac{2}{5}\pi}$$
$$-\tfrac{2}{5}\cos.\tfrac{4}{5}\pi\,l\surd(1-2x\cos.\tfrac{4}{5}\pi+xx)+\tfrac{2}{5}\sin.\tfrac{4}{5}\pi\,\text{Arc.tang.}\frac{x\sin.\frac{4}{5}\pi}{1-x\cos.\frac{4}{5}\pi}$$

VI.

$$\text{VI.} \int \frac{dx}{1-x^6} = \begin{cases} -\frac{1}{6}l(1-x) - \frac{2}{6}\cos.\frac{2}{6}\pi\, l\sqrt{(1-2x\cos.\frac{2}{6}\pi+xx)} + \frac{2}{6}\sin.\frac{2}{6}\pi \,\text{Arc.tang.}\frac{x\sin.\frac{2}{6}\pi}{1-x\cos.\frac{2}{6}\pi} \\ +\frac{1}{6}l(1+x) - \frac{2}{6}\cos.\frac{4}{6}\pi\, l\sqrt{(1-2x\cos\frac{4}{6}\pi+xx)} + \frac{2}{6}\sin.\frac{4}{6}\pi \,\text{Arc.tang.}\frac{x\sin.\frac{4}{6}\pi}{1-x\cos\frac{4}{6}\pi}. \end{cases}$$

Exemplum 4.

82. *Proposita formula differentiali* $\frac{(x^{m-1}+x^{n-m-1})dx}{1+x^n}$ *existente* $n > m-1$, *eius integrale definire.*

Ex exemplo 2 patet integralis partem quamcunque in genere esse, sumto i pro numero quocunque impare non maiore quam n,

$$-\frac{2}{n}\cos.\frac{im\pi}{n}\, l\sqrt{(1-2x\cos.\frac{i\pi}{n}+xx)} + \frac{2}{n}\sin.\frac{im\pi}{n}\,\text{Arc.tang.}\frac{x\sin\frac{i\pi}{n}}{1-x\cos\frac{i\pi}{n}}$$

$$-\frac{2}{n}\cos.\frac{i(n-m)\pi}{n}\, l\sqrt{(1-2x\cos.\frac{i\pi}{n}+xx)} + \frac{2}{n}\sin.\frac{i(n-m)\pi}{n}\,\text{Arc.tang.}\frac{x\sin.\frac{i\pi}{n}}{1-x\cos.\frac{i\pi}{n}}.$$

Verum est $\cos\frac{i(n-m)\pi}{n} = \cos.(i\pi - \frac{in\pi}{n}) = -\cos.\frac{in\pi}{n}$ et

$\sin.\frac{i(n-m)\pi}{n} = \sin.(i\pi - \frac{im\pi}{n}) = +\sin.\frac{im\pi}{n}$

vnde partes logarithmicae se destruent, eritque pars integralis in genere,

$$+\frac{4}{n}\sin.\frac{im\pi}{n}\,\text{Arc. tang.}\frac{x\sin.\frac{i\pi}{n}}{1-x\cos\frac{i\pi}{n}}.$$

Ponatur commoditatis ergo angulus $\frac{\pi}{n} = \omega$, eritque

$$\int \frac{(x^{m-1} + x^{n-m-1})\,dx}{1 + x^n} = + \frac{2}{n} \text{ fin } m\,\omega \text{ Arc. tang. } \frac{x\,\text{fin.}\,\omega}{1 - x\,\text{cof.}\,\omega}$$
$$+ \frac{2}{n} \text{ fin. } 3\,m\,\omega \text{ Arc. tang } \frac{x\,\text{fin.}\,3\omega}{1 - x\,\text{cof.}\,3\omega}$$
$$+ \frac{2}{n} \text{ fin. } 5\,m\,\omega \text{ Arc. tang } \frac{x\,\text{fin.}\,5\omega}{1 - x\,\text{cof.}\,5\omega}$$
$$\vdots$$
$$+ \frac{2}{n} \text{ fin. } i\,m\,\omega \text{ Arc. tang. } \frac{x\,\text{fin.}\,i\omega}{1 - x\,\text{cof.}\,i\omega}$$

fumto pro i maximo numero impare, exponentem n non excedente. Si ipfe numerus n fit impar pars ex pofitione $i = n$ oriunda, ob fin. $m\pi = 0$, euanefcet. Notetur ergo hic totum integrale per meros angulos exprimi.

Corollarium.

83. Simili modo fequens integrale elicitur, vbi foli logarithmi relinquentur, manente $\frac{\pi}{n} = \omega$.

$$\int \frac{(x^{m-1} - x^{n-m-1})\,dx}{1 + x^n} = - \frac{2}{n} \text{ cof. } m\,\omega\, l\sqrt{(1 - 2x \text{ cof } \omega + xx)}$$
$$- \frac{2}{n} \text{ cof. } 3m\omega\, l\sqrt{(1 - 2x \text{ cof. } 3\omega + xx)}$$
$$- \frac{2}{n} \text{ cof. } 5m\omega\, l\sqrt{(1 - 2x \text{ cof. } 5\omega + xx)}$$
$$\vdots$$
$$- \frac{2}{n} \text{ cof. } i\,m\,\omega\, l\sqrt{(1 - 2x \text{ cof. } i\omega + xx)}$$

donec

donec scilicet numerus impar i non superet exponentem n.

Exemplum 5.

84. *Proposita formula differentiali* $\frac{(x^{m-1} - x^{n-m-1})dx}{1 - x^n}$ *existente* $n > m - 1$, *eius integrale definire.*

Ex exemplo 3. integralis pars quaecunque concluditur, siquidem breuitatis gratia $\frac{\pi}{n} = \omega$ statuamus:

$$-\tfrac{2}{n}\,\text{cos.}\,2km\omega\, l\sqrt{(1 - 2x\,\text{cos.}\,2k\omega + xx)} + \tfrac{2}{n}\,\text{sin.}\,2km\omega\;\text{Arc. tang.}\,\frac{x\,\text{sin.}\,2k\omega}{1 - x\,\text{cos.}\,2k\omega}$$

$$+\tfrac{2}{n}\,\text{cos.}\,2k(n-m)\omega\, l\sqrt{(1 - 2k\,\text{cos.}\,2k\omega + xx)} - \tfrac{2}{n}\,\text{sin.}\,2k(n-m)\omega\;\text{Arc. tang.}\,\frac{x\,\text{sin.}\,2k\omega}{1 - x\,\text{cos.}\,2k\omega}.$$

At est $\text{cos.}\,2k(n-m)\omega = \text{cos.}(2k\pi - 2km\omega) = \text{cos.}\,2km\omega$ et

$$\text{sin.}\,2k(n-m)\omega = \text{sin.}(2k\pi - 2km\omega) = -\text{sin.}\,2km\omega,$$

vnde ista pars generalis abit in $\frac{4}{n}\,\text{sin.}\,2km\omega$ $\text{Arc. tang.}\,\frac{x\,\text{sin.}\,2k\omega}{1 - x\,\text{cos.}\,2k\omega}$, quare hinc ista integratio colligitur:

$$\int \frac{(x^{m-1} - x^{n-m-1})dx}{1 - x^n} = +\tfrac{4}{n}\,\text{sin.}\,2m\omega\;\text{Arc. tang.}\,\frac{x\,\text{sin.}\,2\omega}{1 - x\,\text{cos.}\,2\omega}$$

$$+\tfrac{4}{n}\,\text{sin.}\,4m\omega\;\text{Arc. tang.}\,\frac{x\,\text{sin.}\,4\omega}{1 - x\,\text{cos.}\,4\omega}$$

$$+\tfrac{4}{n}\,\text{sin.}\,6m\omega\;\text{Arc. tang.}\,\frac{x\,\text{sin.}\,6\omega}{1 - x\,\text{cos.}\,6\omega}$$

numeris paribus tamdiu ascendendo, quoad exponentem n non superent.

Corollarium.

85. Indidem etiam haec integratio abſoluitur, manente $\frac{\pi}{n} = \omega$:

$$\int \frac{(x^{m-1}+x^{n-m-1})dx}{1-x^n} = -\frac{1}{n}l(1-x)$$
$$-\frac{4}{n}\text{cof.}\,2m\omega\, l\sqrt{(1-2x\,\text{cof.}\,2\omega+xx)}$$
$$-\frac{4}{n}\text{cof.}\,4m\omega\, l\sqrt{(1-2x\,\text{cof.}\,4\omega+xx)}$$
$$-\frac{4}{n}\text{cof.}\,6m\omega\, l\sqrt{(1-2x\,\text{cof.}\,6\omega+xx)}$$

vbi etiam numeri pares non vltra terminum n ſunt continuandi.

Exemplum 6.

86. *Propoſita formula differentiali* $dy = \frac{dx}{x^3(1+x)(1-x^4)}$, *eius integrale inuenire.*

Functio fracta per dx affecta ſecundum denominatoris factores eſt $\frac{1}{x^3(1+x)^2(1-x)(1-xx)}$, quae in has fractiones ſimplices reſoluitur :

$$\frac{1}{x^3}-\frac{2}{x^2}+\frac{1}{x}-\frac{1}{4(1+x)^2}-\frac{9}{8(1+x)}+\frac{1}{8(1-x)}+\frac{1+x}{4(1+xx)}=\frac{dy}{dx}$$

vnde per integrationem elicitur :

$$y = -\frac{1}{2xx}+\frac{2}{x}+lx+\frac{1}{4(1+x)}-\frac{9}{8}l(1+x)-\frac{1}{8}l(1-x)$$
$$+\frac{1}{8}l(1+xx)+\frac{1}{4}\text{Arc. tang.}\,x$$

quae expreſſio in hanc formam transmutatur

$$y = C\,\frac{-2+6x+9xx}{4xx(1+x)}-l\frac{1+x}{x}+\frac{1}{8}l\frac{1+xx}{1-xx}+\frac{1}{4}\text{Arc. tang.}\,x.$$

Scholion.

Scholion.

87. Hoc igitur caput ita pertractare licuit, vt nihil amplius in hoc genere desiderari possit. Quoties ergo eiusmodi functio y ipsius x quaeritur, vt $\frac{dy}{dx}$ aequetur functioni rationali ipsius x, toties integratio nihil habet difficultatis, nisi forte ad denominatoris singulos factores eliciendos Algebrae praecepta non sufficiant: verum tum defectus ipsi Algebrae, non vero methodo integrandi, quam hic tractamus, est tribuendus. Deinde etiam potissimum notari conuenit, semper, cum $\frac{dy}{dx}$ functioni rationali ipsius x aequale ponitur, functionem y, nisi sit algebraica, alias quantitates transcendentes non inuoluere praeter logarithmos et angulos: vbi quidem obseruandum est, hic perpetuo logarithmos hyperbolicos intelligi oportere, cum ipsius lx differentiale non sit $= \frac{dx}{x}$, nisi logarithmus hyperbolicus sumatur: at horum reductio ad vulgares est facillima, ita vt hinc applicatio calculi ad praxin nulli impedimento sit obnoxia. Quare progrediamur ad eos casus, quibus formula $\frac{dy}{dx}$ functioni irrationali ipsius x aequatur, vbi quidem primo notandum est, quoties ista functio per idoneam substitutionem ad rationalitatem perduci poterit, casum ad hoc caput reuolui. Veluti si fuerit $dy = \frac{(1+\sqrt{x}-\sqrt[3]{xx})}{1+\sqrt[3]{x}}dx$, euidens est ponendo $x = z^6$, vnde fit $dx = 6z^5 dz$, fore $dy =$

$dy = \frac{(1+z^3-z^4)}{1+zz} . 6z^5dz$, ideoque $\frac{dy}{dz} = -6z^7+6z^6$ $+6z^5-6z^4+6zz-6+\frac{6}{1+zz}$, vnde integrale

$y = -\frac{3}{4}z^8+\frac{6}{7}z^7+z^6-\frac{6}{5}z^5+2z^3-6z+6\,\text{Arc. tang.}\,z$

et reſtituto valore

$$y = -\tfrac{3}{4}x\sqrt[3]{x}+\tfrac{6}{7}x\sqrt[6]{x}+x-\tfrac{6}{5}\sqrt[6]{x^5}+2\sqrt{x}-6\sqrt[6]{x}$$
$$+6\,\text{Arc. tang.}\sqrt[6]{x}+C.$$

CAPVT II.

CAPVT II.

DE INTEGRATIONE FORMVLARVM IRRATIONALIVM.

Problema 6.

88.

Propofita formula differentiali $dy = \frac{dx}{\sqrt{(\alpha + \beta x + \gamma xx)}}$, eius integrale inuenire.

Solutio.

Quantitas $\alpha + \beta x + \gamma xx$, vel habet duos factores reales vel fecus.

I. Priori cafu formula propofita erit huiusmodi $dy = \frac{dx}{\sqrt{(a + bx)(f + gx)}}$: ftatuatur ad irrationalitatem tollendam $(a + bx)(f + gx) = (a + bx)^2 zz$, erit $x = \frac{f - azz}{bzz - g}$, ideoque

$dx = \frac{2(ag - bf)zdz}{(bzz - g)^2}$ et $\sqrt{(a + bx)(f + gx)} = -\frac{(ag - bf)z}{bzz - g}$

vnde fit

$$dy = \frac{-2dz}{bzz - g} = \frac{2dz}{g - bzz}, \text{ atque } z = \sqrt{\frac{f + gx}{a + bx}}.$$

Quare fi litterae b et g paribus fignis funt affectae, integrale per logarithmos, fin autem fignis difparibus, per angulos exprimetur.

II. Posteriori casu habebimus $dy = \frac{dx}{\sqrt{(aa - 2abx\cos.\zeta + bbxx)}}$, statuatur $bbxx - 2abx\cos.\zeta + aa = (bx - az)^2$, erit $-2bx\cos.\zeta + a = -2bxz + azz$ et $x = \frac{a(1 - zz)}{2b(\cos.\zeta - z)}$; hinc

$$dx = \frac{adz(1 - 2z\cos.\zeta + zz)}{2b(\cos.\zeta - z)^2}, \text{ et}$$

$\sqrt{(aa - 2abx\cos.\zeta + bbxx)} = \frac{a(1 - 2z\cos.\zeta + zz)}{2(\cos.\zeta - z)}$ ergo

$$dy = \frac{dz}{b(\cos.\zeta - z)}, \text{ et } y = -\tfrac{1}{b}l\,(\cos.\zeta - z).$$

At est $z = \frac{bx - \sqrt{(aa - 2abx\cos.\zeta + bbxx)}}{a}$, ideoque

$$y = -\tfrac{1}{b}l\,\frac{a\cos.\zeta - bx + \sqrt{(aa - 2abx\cos.\zeta + bbxx)}}{a} \text{ vel}$$

$$y = \tfrac{1}{b}l(-a\cos.\zeta + bx + \sqrt{(aa - 2abx\cos.\zeta + bbxx)}) + C.$$

Coroll. 1.

89. Casus vltimus latius patet, et ad formulam $dy = \frac{dx}{\sqrt{(\alpha + \beta x + \gamma xx)}}$, accomodari potest, dummodo fuerit γ quantitas positiua: namque ob $b = \sqrt{\gamma}$ et $a\cos.\zeta = \frac{-\beta}{2\sqrt{\gamma}}$ oritur,

$$y = \tfrac{1}{\sqrt{\gamma}}l(\tfrac{\beta}{2\sqrt{\gamma}} + x\sqrt{\gamma} + \sqrt{(\alpha + \beta x + \gamma xx)}) + C$$

seu

$$y = \tfrac{1}{\sqrt{\gamma}}l(\tfrac{1}{2}\beta + \gamma x + \sqrt{\gamma(\alpha + \beta x + \gamma xx)}) + C.$$

Coroll. 2.

90. Pro casu priori cum sit $\int\frac{2dz}{g - bzz} = \frac{1}{\sqrt{bg}}l\frac{\sqrt{g} + z\sqrt{b}}{\sqrt{g} - z\sqrt{b}}$ et $\int\frac{2dz}{g + bzz} = \frac{2}{\sqrt{gb}}$ Arc. tang. $\frac{z\sqrt{b}}{\sqrt{g}}$, habebimus hos casus:

$$\int \frac{dx}{\sqrt{(a+bx)(f+gx)}} = \frac{1}{\sqrt{bg}} l \frac{\sqrt{g(a+bx)} + \sqrt{b(f+gx)}}{\sqrt{g(a+bx)} - \sqrt{b(f+gx)}} + C$$

$$\int \frac{dx}{\sqrt{(bx-a)(f+gx)}} = \frac{1}{\sqrt{bg}} l \frac{\sqrt{g(bx-a)} + \sqrt{b(f+gx)}}{\sqrt{g(bx-a)} - \sqrt{b(f+gx)}} + C$$

$$\int \frac{dx}{\sqrt{(bx-a)(gx-f)}} = \frac{1}{\sqrt{bg}} l \frac{\sqrt{g(bx-a)} + \sqrt{b(gx-f)}}{\sqrt{g(bx-a)} - \sqrt{b(gx-f)}} + C$$

$$\int \frac{dx}{\sqrt{(a-bx)(f-gx)}} = \frac{-1}{\sqrt{bg}} l \frac{\sqrt{g(a-bx)} + \sqrt{b(f-gx)}}{\sqrt{g(a-bx)} - \sqrt{b(f-gx)}} + C$$

$$\int \frac{dx}{\sqrt{(a-bx)(f+gx)}} = \frac{2}{\sqrt{bg}} \text{ Arc. tang. } \frac{\sqrt{b(f+gx)}}{\sqrt{g(a-bx)}} + C$$

$$\int \frac{dx}{\sqrt{(a-bx)(gx-f)}} = \frac{2}{\sqrt{bg}} \text{ Arc. tang. } \frac{\sqrt{b(gx-f)}}{\sqrt{g(a-bx)}} + C.$$

Coroll. 3.

91. Harum ſex integrationum quatuor priores omnes in caſu Coroll. 1. continentur, binae autem poſtremae in hac formula $dy = \frac{dx}{\sqrt{(\alpha+\beta x-\gamma xx)}}$, continentur: ſit enim pro penultima

$$af = \alpha, \; ag - bf = \beta, \; bg = \gamma,$$

vnde colligitur

$$y = \frac{1}{\sqrt{\gamma}} \text{ Arc. tang. } \frac{2\sqrt{\gamma(\alpha+\beta x-\gamma xx)}}{\beta - 2\gamma x};$$

ſi ſcilicet ille arcus duplicetur. Per coſinum autem erit

$$y = \frac{1}{\sqrt{\gamma}} \text{ Arc. coſ. } \frac{\beta - 2\gamma x}{\sqrt{(\beta\beta + 4\alpha\gamma)}} + C;$$

cuius veritas ex differentiatione patet.

Scholion 1.

92. Ex ſolutione huius problematis patet etiam hanc formulam latius patentem $\frac{X\,dx}{\sqrt{(\alpha+\beta x+\gamma xx)}}$, ſi

si X fuerit functio rationalis quaecunque ipsius x, per praecepta capitis praecedentis integrari posse. Introducta enim loco x variabili z, qua formula radicalis rationalis redditur, etiam X abibit in functionem rationalem ipsius z. Idem adhuc generalius locum habet, si posito $\sqrt{(\alpha+\beta x+\gamma xx)}=u$, fuerit X functio quaecunque rationalis binarum quantitatum x et u, tum enim per substitutionem adhibitam, quia tam pro x quam pro u formulae rationales ipsius z scribuntur, prodibit formula differentialis rationalis. Hoc idem etiam ita enunciari potest, vt dicamus formulae $X dx$, si functio X, nullam aliam irrationalem praeter $\sqrt{(\alpha+\beta x+\gamma xx)}$ inuoluat, integrale assignari posse, propterea quod ea ope substitutionis in formulam differentialem rationalem transformari potest.

Scholion 2.

93. Proposita autem formula differentiali quacunque irrationali, ante omnia videndum est, num ea ope cuiuspiam substitutionis in rationalem transformari possit? quod si succedat, integratio per praecepta capitis praecedentis absolui poterit, vnde simul intelligitur, integrale nisi sit algebraicum, alias quantitates transcendentes non inuoluere praeter logarithmos et angulos. Quodsi autem nulla substitutio ad hoc idonea inueniri possit, ab integrationis labore est desistendum, quandoquidem integrale neque algebraice neque per logarithmos vel angulos expri-

exprimere valemus. Veluti si $X dx$ fuerit eiusmodi formula differentialis, quae nullo pacto ad rationalitatem reduci queat, eius integrale $\int X dx$, ad nouum genus functionum transcendentium erit referendum, in quo nihil aliud nobis relinquitur, nisi vt eius valorem vero proxime assignare conemur. Admisso autem nouo genere quantitatum transcendentium, innumerabiles aliae formulae eo reduci atque integrari poterunt. Imprimis igitur in hoc erit elaborandum vt pro quolibet genere formula simplicissima notetur, qua concessa reliquarum formularum integralia definire liceat. Hinc deducimur ad quaestionem maximi momenti, quomodo integrationem formularum magis complicatarum ad simpliciores reduci oporteat. Quod antequam aggrediamur, alias eiusmodi formulas perpendamus, quae ope idoneae substitutionis ab irrationalitate liberari queant, quemadmodum iam ostendimus, quoties X fuerit functio rationalis quantitatum x et $u = \sqrt{(\alpha + \beta x + \gamma x x)}$, ita vt alia irrationalitas non ingrediatur praeter radicem quadratam huiusmodi formulae $\alpha + \beta x + \gamma x x$, toties formulam differentialem $X dx$ in rationalem transformari posse.

Problema 7.

94. Proposita formula differentiali $X dx (a + bx)^{\frac{\mu}{\nu}}$, in qua X denotet functionem quamcunque rationalem ipsius x; eam ab irrationalitate liberare.

Solutio.

Statuatur $a+bx=z^{\nu}$, vt fiat $(a+bx)^{\frac{\mu}{\nu}}=z^{\mu}$, tum quia $x=\frac{z^{\nu}-a}{b}$, facta hac substitutione functio X abibit in functionem rationalem ipsius z, quae sit Z, et ob $dx=\frac{\nu}{b}z^{\nu-1}dz$, formula nostra differentialis induet hanc formam $\frac{\nu}{b}Zz^{\mu+\nu-1}dz$, quae cum sit rationalis per caput superius integrari potest, et integrale, nisi sit algebraicum, per logarithmos et angulos exprimetur.

Coroll. 1.

95. Hac substitutione generalius negotium confici poterit, si posito $(a+bx)^{\frac{1}{\nu}}=u$, littera V denotet functionem quamcunque rationalem binarum quantitatum x et u; cum enim posito $x=\frac{u^{\nu}-a}{b}$, fiat V functio rationalis ipsius u, formula $Vdx=\frac{\nu}{b}Vu^{\nu-1}du$, erit rationalis.

Coroll. 2.

96. Quin etiam si binae irrationalitates eiusdem quantitatis $a+bx$, scilicet $(a+bx)^{\frac{1}{\nu}}=u$ et $(a+bx)^{\frac{1}{\mu}}=v$, ingrediantur in formulam Xdx, posito $a+bx=z^{\mu\nu}$ fit $x=\frac{z^{\mu\nu}-a}{b}$, $u=z^{\mu}$, et $v=z^{\nu}$; vnde

vnde cum X fiat functio rationalis ipsius z, et $dx = \frac{n\nu}{b} z^{n\nu - 1} dz$, hac substitutione formula $X dx$ euadet rationalis.

Coroll. 3.

97. Eodem modo intelligitur, si posito $(a+bx)^{\frac{1}{\lambda}} = u$, $(a+bx)^{\frac{1}{\mu}} = v$, $(a+bx)^{\frac{1}{\nu}} = t$ etc. littera X denotet functionem quamcunque rationalem quantitatum x, u, v, t etc. formulam differentialem $X dx$, rationalem reddi facto $a + bx = z^{\lambda\mu\nu}$; erit enim $x = \frac{z^{\lambda\mu\nu} - a}{b}$; $u = z^{\mu\nu}$; $v = z^{\lambda\nu}$; $t = z^{\lambda\mu}$ etc. et $dx = \frac{\lambda\mu\nu}{b} z^{\lambda\mu\nu - 1} dz$.

Exemplum.

98. Proposita hac formula $dy = \frac{x\,dx}{\sqrt[3]{(1+x)} - \sqrt{(1+x)}}$, facto $1 + x = z^6$, reperitur $dy = -\frac{6z^8 dz(1-z^6)}{1-z}$, seu

$$dy = -6dz(z^3 + z^4 + z^5 + z^6 + z^7 + z^8),$$

hincque integrando

$$y = C - \tfrac{3}{2}z^4 - \tfrac{6}{5}z^5 - z^6 - \tfrac{6}{7}z^7 - \tfrac{3}{4}z^8 - \tfrac{2}{3}z^9,$$

et restituendo

$$y = C - \tfrac{3}{2}\sqrt[3]{(1+x)^2} - \tfrac{6}{5}\sqrt[6]{(1+x)^5} - 1 - x - \tfrac{6}{7}(1+x)\sqrt[6]{(1+x)}$$
$$- \tfrac{3}{4}(1+x)\sqrt[3]{(1+x)} - \tfrac{2}{3}(1+x)\sqrt{(1+x)}$$

ita vt integrale adeo algebraice exhibeatur.

Proble-

Problema 8.

99. Propoſita formula differentiali $X\,dx\left(\frac{a+bx}{f+gx}\right)^{\frac{\mu}{\nu}}$ denotante X functionem rationalem quamcunque ipſius x, eam ab irrationalitate liberare.

Solutio.

Poſito $\frac{a+bx}{f+gx} = z^\nu$, fit $\left(\frac{a+bx}{f+gx}\right)^{\frac{\mu}{\nu}} = z^\mu$, et $x = \frac{a-fz^\nu}{gz^\nu - b}$ atque $dx = \frac{\nu(bf-ag)z^{\nu-1}dz}{(gz^\nu-b)^2}$, ſicque loco X prodibit functio rationalis ipſius z, qua poſita $= Z$ erit formula noſtra differentialis $= \frac{\nu(bf-ag)Zz^{\mu+\nu-1}dz}{(gz^\nu-b)^2}$, quae cum ſit rationalis per praecepta Cap. I. integrari poterit.

Coroll. 1.

100. Poſito $\left(\frac{a+bx}{f+gx}\right)^{\frac{1}{\nu}} = u$, ſi X fuerit functio quaecunque rationalis binarum quantitatum x et u, formula differentialis $X\,dx$ per ſubſtitutionem vſurpatam in rationalem transformabitur, cuius propterea integratio conſtat.

Coroll. 2.

101. Si X fuerit functio rationalis tam ipſius x, quam quantitatum quotcunque huiusmodi

$$\left(\frac{a+bx}{f+gx}\right)^{\frac{1}{\lambda}} = u\,,\ \left(\frac{a+bx}{f+gx}\right)^{\frac{1}{\mu}} = v\,,\ \left(\frac{a+bx}{f+gx}\right)^{\frac{1}{\nu}} = t$$

tum

tum formula differentialis $X\,dx$ rationalis reddetur adhibita substitutione $\frac{a+bx}{f+gx}=z^{\lambda\mu\nu}$, vnde fit $x=\frac{a-fz^{\lambda\mu\nu}}{gz^{\lambda\mu\nu}-b}$, et $u=z^{\mu\nu}$; $v=z^{\lambda\nu}$; $t=z^{\lambda\mu}$.

Scholion 1.

102. His casibus reductio ad rationalitatem ideo succedit, etiamsi plures formulae irrationales insint, quod eae omnes simul per eandem substitutionem rationales efficiantur, indeque etiam ipsa quantitas x per nouam variabilem z, rationaliter exprimetur. Sin autem differentiale propositum duas eiusmodi formulas irrationales contineat, quae non ambae simul ope eiusdem substitutionis rationales reddi queant, etiamsi hoc in vtraque seorsim fieri possit, reductio locum non habet, nisi forte ipsum differentiale in duas partes dispesci liceat, quarum vtraque vnam tantum formulam irrationalem complectatur. Veluti si proposita sit haec formula differentialis $dy=\frac{dx}{\sqrt{(1+xx)}-\sqrt{(1-xx)}}$ eius numeratorem ac denominatorem per $\sqrt{(1+xx)}+\sqrt{(1-xx)}$ multiplicando fit $dy=\frac{dx\sqrt{(1+xx)}}{2xx}+\frac{dx\sqrt{(1-xx)}}{2xx}$, cuius vtraque pars seorsim rationalis reddi et integrari potest. Reperitur autem:

$$y=C-\frac{\sqrt{(1-xx)}-\sqrt{(1+xx)}}{2x}+\tfrac{1}{2}l(x+\sqrt{(1+xx)})$$
$$-\tfrac{1}{2}\text{Arc. tang.}\frac{x}{\sqrt{(1-xx)}}.$$

Commodissime autem ibi irrationalitas tollitur, si in parte priori ponatur $\sqrt{(1+xx)}=px$, in posteriori

riori $\sqrt{(1-xx)}=qx$. Etsi enim hinc sit $x=\frac{1}{\sqrt{(pp-1)}}$ et $x=\frac{1}{\sqrt{(1+qq)}}$, tamen oritur rationaliter

$$dy=\frac{-ppdp}{2(pp-1)}-\frac{qqdq}{2(1+qq)}.$$

Scholion 2.

103. Circa formulas generales, quae ab irrationalitate liberari queant, vix quicquam amplius praecipere licet: dummodo hunc casum addamus, quo functio X binas huiusmodi formulas radicales $\sqrt{(a+bx)}$ et $\sqrt{(f+gx)}$ complectitur. Posito enim $(a+bx)=(f+gx)tt$, fit $x=\frac{a-ftt}{gtt-b}$ atque

$$\sqrt{(a+bx)}=\frac{t\sqrt{(ag-bf)}}{\sqrt{(gtt-b)}};\ \sqrt{(f+gx)}=\frac{\sqrt{(ag-bf)}}{\sqrt{(gtt-b)}}$$

in formula differentiali vnica tantum formula irrationalis $\sqrt{(gtt-b)}$, quae noua substitutione facile tolletur, per ea quae Problemate 6. tradidimus. Vt igitur ad alia pergamus, imprimis considerari meretur haec formula differentialis $x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$, cuius ob simplicitatem vsus per vniuersam analysin est amplissimus; vbi quidem sumimus litteras m, n, μ, ν numeros integros denotare, nisi enim tales essent, facile ad hanc formam reducerentur. Veluti si haberemus $x^{-\frac{1}{3}}dx(a+b\sqrt{x})^{\frac{\mu}{\nu}}$, statui oportet $x=u^6$ hinc $dx=6u^5du$, vnde prodit $6u^3du(a+bu^3)^{\frac{\mu}{\nu}}$. Tum vero pro n valorem positiuum assumere licet, si enim esset negatiuus puta $x^{m-1}dx(a+bx^{-n})^{\frac{\mu}{\nu}}$, pona-

ponatur $x = \frac{1}{u}$ fietque formula $-u^{-m-1}du(a+bu^n)^{\frac{\mu}{\nu}}$ ſimilis principali; quae ergo quibus caſibus ab irrationalitate liberari queat, inueſtigemus.

Problema 9.

104. Definire caſus, quibus formulam differentialem $x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$, ad rationalitatem perducere liceat.

Solutio.

Primo patet ſi fuerit $\nu = 1$ ſeu $\frac{\mu}{\nu}$ numerus integer, formulam per ſe fore rationalem, neque ſubſtitutione opus eſſe. At ſi $\frac{\mu}{\nu}$ ſit fractio, ſubſtitutione eſt vtendum, eaque duplici.

I. Ponatur $a+bx^n = u^\nu$, vt fiat $(a+bx^n)^{\frac{\mu}{\nu}} = u^\mu$, erit $x^n = \frac{u^\nu - a}{b}$, hinc $x^m = \left(\frac{u^\nu - a}{b}\right)^{\frac{m}{n}}$, ideoque $x^{m-1}dx = \frac{\nu}{n}u^{\nu-1}du\left(\frac{u^\nu - a}{b}\right)^{\frac{m-n}{n}}$, vnde formula noſtra fiet $\frac{\nu}{n}u^{\mu+\nu-1}du\left(\frac{u^\nu - a}{b}\right)^{\frac{m-n}{n}}$. Hinc ergo patet, quoties exponens $\frac{m-n}{n}$ ſeu $\frac{m}{n}$ fuerit numerus integer ſiue poſitiuus, ſiue negatiuus, hanc formulam eſſe rationalem.

II. Ponatur $a+bx^n = x^n z^\nu$, vt fiat $x^n = \frac{a}{z^\nu - b}$, et $(a+bx^n)^{\frac{\mu}{\nu}} = \frac{a^{\frac{\mu}{\nu}} z^\mu}{(z^\nu - b)^{\frac{\mu}{\nu}}}$; tum $x^m = \frac{a^{\frac{m}{n}}}{(z^\nu - b)^{\frac{m}{n}}}$,

 hinc

hinc $x^{m-1}dx = \frac{-\nu a^{\frac{m}{n}} z^{\nu-1} dz}{n(z^\nu - b)^{\frac{m}{n}+1}}$, ideoque formula noſtra erit $\frac{-\nu a^{\frac{m}{n}+\frac{\mu}{\nu}} z^{\mu+\nu-1} dz}{n(z^\nu - b)^{\frac{m}{n}+\frac{\mu}{\nu}+1}}$. Ex quo patet hanc formam fore rationalem, quoties $\frac{m}{n}+\frac{\mu}{\nu}$ fuerit numerus integer. Facile autem intelligitur alias ſubſtitutiones huic ſcopo idoneas excogitari non poſſe.

Quare concludimus formulam irrationalem hanc $x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$ ab irrationalitate liberari poſſe, ſi fuerit vel $\frac{m}{n}$ vel $\frac{m}{n}+\frac{\mu}{\nu}$ numerus integer.

Coroll. 1.

105. Si ſit $\frac{m}{n}$ numerus integer, caſus per ſe eſt facilis; ponatur enim $m = in$, et ſit $x^n = v$, erit $x^m = v^i$; ideoque formula noſtra $\frac{1}{n} v^{i-1} dv(a+bv)^{\frac{\mu}{\nu}}$, quae per Problema 7. expeditur.

Coroll. 2.

106. At ſi $\frac{m}{n}$ non eſt numerus integer, vt reductio ad rationalitatem locum habeat, neceſſe eſt vt $\frac{m}{n}+\frac{\mu}{\nu}$ ſit numerus integer: quod fieri nequit, niſi ſit $\nu = n$, ideoque $m+\mu$ multiplum debet eſſe ipſius $n = \nu$.

Coroll. 3.

Coroll. 3.

107. Quod si ergo haec formula $x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$, ad rationalitatem reduci queat, etiam haec formula $x^{m\pm\alpha n-1}dx(a+bx^n)^{\frac{\mu}{\nu}\pm\beta}$, eandem reductionem admittet; quicunque numeri integri pro α et β assumantur. Vnde ad casus reducibiles cognoscendos sufficit ponere $m<n$ et $\mu<\nu$.

Coroll. 4.

108. Si $m=0$ haec formula $\frac{dx}{x}(a+bx^n)^{\frac{\mu}{\nu}}$ semper per casum primum ad rationalitatem reducitur, ponendo $x^n=\frac{u^\nu-a}{b}$; transformatur enim in hanc $\frac{\nu b u^{\mu+\nu-1}du}{n(u^\nu-a)}$.

Scholion 1.

109. Quoniam formula $x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$, quoties est $m=in$, denotante i numerum integrum siue positiuum siue negatiuum quemcunque, semper ad rationalitatem reduci potest, hique casus per se sunt perspicui, reliquos casus hanc reductionem admittentes accuratius contemplari operae pretium videtur. Quem in finem statuamus $\nu=n$ et $m<n$, item $\mu<n$, ac necesse est vt sit $m+\mu=n$: vnde sequentes formae in genere suo simplicissimae, quae quidem ad rationalitatem reduci queant, obtinentur.

I. $dx(a+bx^2)^{\frac{1}{2}}$

II. $dx(a+bx^3)^{\frac{2}{3}}$; $xdx(a+bx^3)^{\frac{1}{3}}$

III. $dx(a+bx^4)^{\frac{3}{4}}$; $xxdx(a+bx^4)^{\frac{1}{4}}$

IV. $dx(a+bx^5)^{\frac{4}{5}}$; $xdx(a+bx^5)^{\frac{3}{5}}$; $x^2dx(a+bx^5)^{\frac{2}{5}}$; $x^3dx(a+bx^5)^{\frac{1}{5}}$

V. $dx(a+bx^6)^{\frac{5}{6}}$; $x^4dx(a+bx^6)^{\frac{1}{6}}$

vnde etiam hae reductionem admittent:

$x^{\pm 2\alpha}dx(a+bx^2)^{\frac{1}{2}\pm\beta}$

$x^{\pm 3\alpha}dx(a+bx^3)^{\frac{2}{3}\pm\beta}$; $x^{1\pm 3\alpha}dx(a+bx^3)^{\frac{1}{3}\pm\beta}$

$x^{\pm 4\alpha}dx(a+bx^4)^{\frac{3}{4}\pm\beta}$; $x^{2\pm 4\alpha}dx(a+bx^4)^{\frac{1}{4}\pm\beta}$

$x^{\pm 5\alpha}dx(a+bx^5)^{\frac{4}{5}\pm\beta}$; $x^{1\pm 5\alpha}dx(a+bx^5)^{\frac{3}{5}\pm\beta}$

$x^{2\pm 5\alpha}dx(a+bx^5)^{\frac{2}{5}\pm\beta}$; $x^{3\pm 5\alpha}dx(a+bx^5)^{\frac{1}{5}\pm\beta}$

$x^{\pm 6\alpha}dx(a+bx^6)^{\frac{5}{6}\pm\beta}$; $x^{4\pm 6\alpha}dx(a+bx^6)^{\frac{1}{6}\pm\beta}$.

Scholion 2.

110. Verum etiamſi formula $x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$, ab irrationalitate liberari nequeat, tamen ſemper omnium harum formularum $x^{m\pm n\alpha-1}dx(a+bx^n)^{\frac{\mu}{\nu}\pm\beta}$, integrationem ad eam reducere licet, ita vt illius integrali tanquam cognito ſpectato, etiam harum inte-

integralia assignari queant. Quae reductio cum in Analysi summam afferat vtilitatem, eam hic exponere necesse erit. Caeterum hic affirmare haud dubitamus, praeter eos casus, quos reductionem ad rationalitatem admittere hic ostendimus, nullos alios existere, qui vlla substitutione adhibita ab irrationalitate liberari queant. Proposita enim hac formula $\frac{dx}{\sqrt{(a+bx^2)}}$, nulla functio rationalis ipsius z loco x poni potest, vt $a+bx^2$ extractionem radicis quadratae admittat: obiici quidem potest, scopo satisfieri posse etiamsi loco x functio irrationalis ipsius z substituatur; dummodo similis irrationalitas in denominatore $\sqrt{(a+bx^2)}$ contineatur, qua illa numeratorem dx afficiens destruatur: quemadmodum fit in hac formula $\frac{dx}{\sqrt[3]{(a+bx^3)}}$ adhibendo substitutionem $x=\frac{\sqrt[3]{a}}{\sqrt[3]{(z^3-b)}}$, verum quod hic commode vsu venit, nullo modo perspicitur, quomodo idem illo casu euenire possit. Hoc tamen minime pro demonstratione haberi volo.

Problema 10.

111. Integrationem formulae $\int x^{m+n-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$, perducere ad integrationem huius formulae $\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$.

Solutio.

Solutio.

Confideretur functio $x^m(a+bx^n)^{\frac{\mu}{\nu}+1}$ cuius differentiale cum fit $(max^{m-1}dx+mbx^{m+n-1}dx + \frac{n(\mu+\nu)}{\nu}bx^{m+n-1}dx)(a+bx^n)^{\frac{\mu}{\nu}}$ erit $x^m(a+bx^n)^{\frac{\mu}{\nu}+1} = ma\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}} + \frac{(m\nu+n\mu+n\nu)b}{\nu}\int x^{m+n-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$ vnde elicitur:

$$\int x^{m+n-1}dx(a+bx^n)^{\frac{\mu}{\nu}} = \frac{\nu x^m(a+bx^n)^{\frac{\mu}{\nu}+1}}{(m\nu+n\mu+n\nu)b} - \frac{m\nu a}{(m\nu+n\mu+n\nu)b}\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}.$$

Coroll. 1.

112. Cum inde quoque fit

$$\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}} = \frac{x^m(a+bx^n)^{\frac{\mu}{\nu}+1}}{ma} - \frac{(m\nu+n\mu+n\nu)b}{m\nu a}\int x^{m+n-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$$

loco m fcribamus $m-n$, et habebimus hanc reductionem:

$$\int x^{m-n-1}dx(a+bx^n)^{\frac{\mu}{\nu}} = \frac{x^{m-n}(a+bx^n)^{\frac{\mu}{\nu}+1}}{(m-n)a} - \frac{(m\nu+n\mu)b}{(m-n)\nu a}\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}.$$

Coroll. 2.

Coroll. 2.

113. Concesso ergo integrali $\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$, etiam harum formularum $\int x^{m\pm n-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$, similique modo vlterius progrediendo omnium harum formularum $\int x^{m\pm an-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$ integralia exhiberi possunt.

Problema 11.

114. Integrationem formulae $\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}+1}$ ad integrationem huius $\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$ perducere.

Solutio.

Functionis $x^m(a+bx^n)^{\frac{\mu}{\nu}+1}$ differentiale hoc modo exhiberi potest

$$\left(ma-\frac{(m\nu+n\mu+n\nu)a}{\nu}\right)x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$$
$$+\frac{m\nu+n\mu+n\nu}{\nu}x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}+1}$$

vnde concluditur

$$x^m(a+bx^n)^{\frac{\mu}{\nu}+1}=-\frac{(n\mu+n\nu)a}{\nu}\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$$
$$+\frac{m\nu+n\mu+n\nu}{\nu}\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}+1}$$

quocirca habebimus:

$$\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}+1}=\frac{\nu x^m(a+bx^n)^{\frac{\mu}{\nu}+1}}{m\nu+n(\mu+\nu)}$$
$$+\frac{n(\mu+\nu)a}{m\nu+n(\mu+\nu)}\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}.$$

Coroll. 1.

115. Deinde ex eadem aequatione elicimus:

$$\int x^{m-1} dx (a + b x^n)^{\frac{\mu}{\nu}} = \frac{-\nu x^m (a + b x^n)^{\frac{\mu}{\nu}+1}}{n(\mu+\nu)a} + \frac{m\nu + n(\mu+\nu)}{n(\mu+\nu)a} \int x^{m-1} dx (a + b x^n)^{\frac{\mu}{\nu}+1}$$

scribamus iam $\mu - \nu$ loco μ vt nanciscamur hanc reductionem.

$$\int x^{m-1} dx (a + b x^n)^{\frac{\mu}{\nu}-1} = \frac{-\nu x^m (a + b x^n)^{\frac{\mu}{\nu}}}{n\mu a} + \frac{m\nu + n\mu}{n\mu a} \int x^{m-1} dx (a + b x^n)^{\frac{\mu}{\nu}}.$$

Coroll. 2.

116. Concesso ergo integrali $\int x^{m-1} dx (a + b x^n)^{\frac{\mu}{\nu}}$, etiam harum formularum $\int x^{m-1} dx (a + b x^n)^{\frac{\mu}{\nu} \pm 1}$, et vlterius progrediendo harum $\int x^{m-1} dx (a + b x^n)^{\frac{\mu}{\nu} \pm \beta}$ integralia exhiberi possunt, denotante β numerum integrum quemcunque.

Coroll. 3.

117. His cum praecedentibus coniunctis ad integrationem $\int x^{m-1} dx (a + b x^n)^{\frac{\mu}{\nu}}$ omnia haec integralia

gralia $\int x^{m\pm\alpha n-1} dx (a+bx^n)^{\frac{\mu}{\nu}\pm\beta}$ reuocari possunt, quae ergo omnia ab eadem functione transcendente pendent.

Scholion I.

118. Ex formae $x^m (a+bx^n)^{\frac{\mu}{\nu}}$ differentiali ita disposito

$$m x^{m-1} dx (a+bx^n)^{\frac{\mu}{\nu}} + \frac{n\mu}{\nu} b x^{m+n-1} dx (a+bx^n)^{\frac{\mu}{\nu}-1}$$

deducimus hanc reductionem

$$\int x^{m+n-1} dx (a+bx^n)^{\frac{\mu}{\nu}-1} = \frac{\nu x^m (a+bx^n)^{\frac{\mu}{\nu}}}{n\mu b} - \frac{m\nu}{n\mu b}\int x^{m-1} dx (a+bx^n)^{\frac{\mu}{\nu}}$$

ac praeterea hanc inuersam pro m et μ scribendo $m-n$ et $\mu+\nu$

$$\int x^{m-n-1} dx (a+bx^n)^{\frac{\mu}{\nu}+1} = \frac{x^{m-n}(a+bx^n)^{\frac{\mu}{\nu}+1}}{m-n} - \frac{n(\mu+\nu)b}{\nu(m-n)}\int x^{m-1} dx (a+bx^n)^{\frac{\mu}{\nu}}.$$

Hinc scilicet vna operatione absoluitur reductio, cum superiores formulae duplicem reductionem exigant; ex quo sex reductiones sumus nacti, omnino memorabiles, quas idcirco coniunctim conspectui exponamus.

$$\text{I. } \int x^{m+n-1}dx(a+bx^n)^{\frac{\mu}{\nu}} = \frac{\nu x^m (a+bx^n)^{\frac{\mu}{\nu}+1}}{(m\nu+n(\mu+\nu))b} - \frac{m\nu a}{(m\nu+n(\mu+\nu))b}\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$$

$$\text{II. } \int x^{m-n-1}dx\,(a+bx^n)^{\frac{\mu}{\nu}} = \frac{x^{m-n}(a+bx^n)^{\frac{\mu}{\nu}+1}}{(m-n)a} - \frac{(m\nu+n\mu)b}{(m-n)\nu a}\int x^{m-1}dx\,(a+bx^n)^{\frac{\mu}{\nu}}$$

$$\text{III. } \int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}+1} = \frac{-\nu x^m (a+bx^n)^{\frac{\mu}{\nu}+1}}{m\nu+n(\mu+\nu)} + \frac{n(\mu+\nu)a}{m\nu+n(\mu+\nu)}\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$$

$$\text{IV. } \int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}-1} = \frac{-\nu x^m (a+bx^n)^{\frac{\mu}{\nu}}}{n\mu a} + \frac{m\nu+n\mu}{n\mu a}\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$$

$$\text{V. } \int x^{m+n-1}dx(a+bx^n)^{\frac{\mu}{\nu}-1} = \frac{\nu x^m (a+bx^n)^{\frac{\mu}{\nu}}}{n\mu b} - \frac{m\nu}{n\mu b}\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$$

$$\text{VI. } \int x^{m-n-1}dx\,(a+bx^n)^{\frac{\mu}{\nu}+1} = \frac{x^{m-n}(a+bx^n)^{\frac{\mu}{\nu}+1}}{m-n} - \frac{n(\mu+\nu)b}{\nu(m-n)}\int x^{m-1}dx\,(a+bx^n)^{\frac{\mu}{\nu}}.$$

Scho-

Scholion 2.

119. Circa has reductiones primo obseruandum est, formulam priorem algebraice esse integrabilem, si coefficiens posterioris euanescat. Ita fit

pro I. si $m=0 \ldots \int x^{n-1}dx(a+bx^n)^{\frac{\mu}{\nu}} = \frac{\nu(a+bx^n)^{\frac{\mu}{\nu}+1}}{n(\mu+\nu)b}$

pro II. si $\frac{\mu}{\nu}=\frac{-m}{n} \ldots \int x^{m-n-1}dx(a+bx^n)^{\frac{-m}{n}} = \frac{x^{m-n}(a+bx^n)^{\frac{-m}{n}+1}}{(m-n)a}$

pro IV. si $\frac{\mu}{\nu}=\frac{-m}{n} \ldots \int x^{m-1}dx(a+bx^n)^{\frac{-m}{n}-1} = \frac{x^m(a+bx^n)^{\frac{-m}{n}}}{ma}$

pro V. si $m=0 \ldots \int x^{n-1}dx(a+bx^n)^{\frac{\mu}{\nu}-1} = \frac{\nu(a+bx^n)^{\frac{\mu}{\nu}}}{n\mu b}$.

Deinde etiam casus notari merentur, quibus coefficiens postremae formulae fit infinitus; tum enim reductio cessat, et prior formula peculiare habet integrale seorsim euoluendum.

In prma hoc euenit si $\frac{\mu-\nu}{\nu}=\frac{-m}{n}$, et formula $\int x^{m-n-1}dx(a+bx^n)^{\frac{-m}{n}-1}$ posito $a+bx^n=x^nz^n$, seu $x^n=\frac{a}{z^n-b}$ abit in $-\frac{z^{-m-1}dz}{z^n-b}$ cuius integrale per caput primum definiri debet.

In secunda euenit si $m=n$ et formula $\int \frac{dx}{x}(a+bx^n)^{\frac{\mu}{\nu}}$, posito $a+bx^n=z^\nu$ seu $x^n=\frac{z^\nu-a}{b}$ abit in $\frac{\nu z^{\mu+\nu-1}dz}{n(z^\nu-a)}$.

In tertia euenit si $\frac{\mu}{\nu} = \frac{-m}{n} - 1$ et formula $\int x^{m-1} dx (a + b x^n)^{\frac{-m}{n}}$ posito $a + b x^n = x^n z^n$ seu $x^n = \frac{a}{z^n - b}$ abit in $\int \frac{-z^{-m-n-1} dz}{z^n} \frac{}{b}$ seu posito $z = \frac{1}{u}$ in $\int \frac{u^{m+2n-1} du}{1 - b u^n} = \frac{-u^{m+n}}{(m+n)b} - \frac{u^m}{mbb} + \frac{1}{bb} \int \frac{u^{m-1} du}{a - b u^n}$.

In quarta euenit si $\mu = 0$ et formula $\int \frac{x^{m-1} dx}{a + b x^n}$, per se est rationalis.

In quinta idem euenit si $\mu = 0$.

In sexta autem si $m = n$ et formula $\int \frac{dx}{x} (a + b x^n)^{\frac{\mu}{\nu} + 1}$, posito $a + b x^n = z^\nu$ abit in $\frac{\nu}{n} \int \frac{z^{\mu + \nu - 1} dz}{z^\nu - a}$.

Exemplum 1.

120. *Inuenire integrale huius formulae* $\int \frac{x^{m-1} dx}{\sqrt{(1 - xx)}}$, *pro numeris positiuis exponenti* m *datis.*

Hic ob $a = 1$, $b = -1$, $n = 2$, $\mu = -1$, $\nu = 2$, prima reductio dat:

$$\int \frac{x^{m+1} dx}{\sqrt{(1 - xx)}} = \frac{-x^m \sqrt{(1 - xx)}}{m + 1} + \frac{m}{m + 1} \int \frac{x^{m-1} dx}{\sqrt{(1 - xx)}}$$

hinc prout pro m sumantur numeri vel impares vel pares, obtinebimus.

Pro

Pro numeris imparibus:

$$\int\frac{xx\,dx}{\sqrt{(1-xx)}}=-\tfrac{1}{2}x\sqrt{(1-xx)}+\tfrac{1}{2}\int\frac{dx}{\sqrt{(1-xx)}}$$

$$\int\frac{x^4\,dx}{\sqrt{(1-xx)}}=-\tfrac{1}{4}x^3\sqrt{(1-xx)}+\tfrac{3}{4}\int\frac{x^2\,dx}{\sqrt{(1-xx)}}$$

$$\int\frac{x^6\,dx}{\sqrt{(1-xx)}}=-\tfrac{1}{6}x^5\sqrt{(1-xx)}+\tfrac{5}{6}\int\frac{x^4\,dx}{\sqrt{(1-xx)}}.$$

Pro numeris paribus:

$$\int\frac{x^3\,dx}{\sqrt{(1-xx)}}=-\tfrac{1}{3}x^2\sqrt{(1-xx)}+\tfrac{2}{3}\int\frac{x\,dx}{\sqrt{(1-xx)}}$$

$$\int\frac{x^5\,dx}{\sqrt{(1-xx)}}=-\tfrac{1}{5}x^4\sqrt{(1-xx)}+\tfrac{4}{5}\int\frac{x^3\,dx}{\sqrt{(1-xx)}}$$

$$\int\frac{x^7\,dx}{\sqrt{(1-xx)}}=-\tfrac{1}{7}x^6\sqrt{(1-xx)}+\tfrac{6}{7}\int\frac{x^5\,dx}{\sqrt{(1-xx)}}$$

etc.

Cum nunc sit $\int\frac{dx}{\sqrt{(1-xx)}}=$ Arc. sin. x et $\int\frac{x\,dx}{\sqrt{(1-xx)}}=-\sqrt{(1-xx)}$ habebimus sequentia integralia.

Pro ordine priore:

$$\int\frac{dx}{\sqrt{(1-xx)}}=\text{Arc. sin. }x$$

$$\int\frac{xx\,dx}{\sqrt{(1-xx)}}=-\tfrac{1}{2}x\sqrt{(1-xx)}+\tfrac{1}{2}\text{Arc. sin. }x$$

$$\int\frac{x^4\,dx}{\sqrt{(1-xx)}}=-\left(\tfrac{1}{4}x^3+\tfrac{1.3}{2.4}x\right)\sqrt{(1-xx)}+\tfrac{1.3}{2.4}\text{Arc. sin. }x$$

$$\int\frac{x^6\,dx}{\sqrt{(1-xx)}}=-\left(\tfrac{1}{6}x^5+\tfrac{1.5}{4.6}x^3+\tfrac{1.3.5}{2.4.6}x\right)\sqrt{(1-xx)}+\tfrac{1.3.5}{2.4.6}\text{Arc sin. }x$$

$$\int\frac{x^8\,dx}{\sqrt{(1-xx)}}=-\left(\tfrac{1}{8}x^7+\tfrac{1.7}{6.8}x^5+\tfrac{1.5.7}{4.6.8}x^3+\tfrac{1.3.5.7}{2.4.6.8}x\right)\sqrt{(1-xx)}+\tfrac{1.3.5.7}{2.4.6.8}\text{Arc. sin. }x.$$

Pro

Pro ordine posteriore:

$$\int \frac{x\,dx}{\sqrt{(1-xx)}} = -\sqrt{(1-xx)}$$

$$\int \frac{x^3\,dx}{\sqrt{(1-xx)}} = -(\tfrac{1}{3}x^2 + \tfrac{2}{3})\sqrt{(1-xx)}$$

$$\int \frac{x^5\,dx}{\sqrt{(1-xx)}} = -(\tfrac{1}{5}x^4 + \tfrac{1.4}{3.5}x^2 + \tfrac{2.4}{3.5})\sqrt{(1-xx)}$$

$$\int \frac{x^7\,dx}{\sqrt{(1-xx)}} = -(\tfrac{1}{7}x^6 + \tfrac{1.6}{5.7}x^4 + \tfrac{1.4.6}{3.5.7}x^2 + \tfrac{2.4.6}{3.5.7})\sqrt{(1-xx)}.$$

Coroll. 1.

121. In genere ergo pro formula $\int \frac{x^{2i}\,dx}{\sqrt{(1-xx)}}$, si ponamus breuitatis gratia $\frac{1.3.5\ldots\ldots(2i-1)}{2.4.6\ldots\ldots 2i} = J$, habebimus hoc integrale:

$$\int \frac{x^{2i}\,dx}{\sqrt{(1-xx)}} = J\,\text{Arc. fin. } x - J(x + \tfrac{2}{3}x^3 + \tfrac{2.4}{3.5}x^5 + \tfrac{2.4.6}{3.5.7}x^7 \ldots + \tfrac{2.4.6\ldots(2i-2)}{3.5.7\ldots(2i-1)}x^{2i-1})\sqrt{(1-xx)}$$

Coroll. 2.

122. Simili modo pro formula $\int \frac{x^{2i+1}\,dx}{\sqrt{(1-xx)}}$, si ponamus breuitatis ergo $\frac{2.4.6\ldots\ldots 2i}{3.5.7\ldots\ldots(2i+1)} = K$, habebimus hoc integrale:

$$\int \frac{x^{2i+1}\,dx}{\sqrt{(1-xx)}} = K - K(1 + \tfrac{1}{2}x^2 + \tfrac{1.3}{2.4}x^4 + \tfrac{1.3.5}{2.4.6}x^6 + \ldots \tfrac{1.3.5\ldots(2i-1)}{2.4.6\ldots 2i}x^{2i})\sqrt{(1-xx)}$$

vt integrale euanescat posito $x = 0$.

Exem-

Exemplum 2.

123. *Inuenire integrale formulae* $\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}}$, *casibus quibus pro* m *numeri negatiui assumuntur.*

Hic vtendum est secunda reductione quae dat:

$$\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}} = \frac{x^{m-2}\sqrt{(1-xx)}}{m-2} + \frac{m-1}{m-2}\int \frac{x^{m-3}dx}{\sqrt{(1-xx)}}$$

vnde patet si $m=1$ fore $\int \frac{dx}{xx\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{x}$; deinde si $m=2$ formula $\int \frac{dx}{x\sqrt{(1-xx)}}$ facta substitutione $1-xx=zz$ abit in $\frac{-dz}{1-zz}$; cuius integrale est $-\frac{1}{2}l\frac{1+z}{1-z} = -\frac{1}{2}l\frac{1+\sqrt{(1-xx)}}{1-\sqrt{(1-xx)}} = -l\frac{1+\sqrt{(1-xx)}}{x}$, vnde duplicem seriem integrationum elicimus:

$$\int \frac{dx}{x\sqrt{(1-xx)}} = -l\frac{1+\sqrt{(1-xx)}}{x} = l\frac{1-\sqrt{(1-xx)}}{x}$$

$$\int \frac{dx}{x^3\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{2xx} + \frac{1}{2}\int \frac{dx}{x\sqrt{(1-xx)}}$$

$$\int \frac{dx}{x^5\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{4x^4} + \frac{3}{4}\int \frac{dx}{x^3\sqrt{(1-xx)}}$$

$$\int \frac{dx}{x^7\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{6x^6} + \frac{5}{6}\int \frac{dx}{x^5\sqrt{(1-xx)}}$$

etc.

$$\int \frac{dx}{xx\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{x}$$

$$\int \frac{dx}{x^4\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{3x^3} + \frac{2}{3}\int \frac{dx}{xx\sqrt{(1-xx)}}$$

$$\int \frac{dx}{x^6\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{5x^5} + \frac{4}{5}\int \frac{dx}{x^4\sqrt{(1-xx)}}$$

etc.

Hinc erit vt in binis praecedentibus corollariis

$$\int \frac{dx}{x^{2i+1}\sqrt{(1-xx)}} = \mathfrak{J}l\frac{1-\sqrt{(1-xx)}}{x} - \mathfrak{J}\left(\frac{1}{xx} + \frac{2}{3x^4} + \frac{2.4}{3.5x^6} + \dots + \frac{2.4\dots(2i-2)}{3.5\dots(2i-1)x^{2i}}\right)\sqrt{(1-xx)}$$

$$\int \frac{dx}{x^{2i}\sqrt{(1-xx)}} = C - K\left(\frac{1}{x} + \frac{1}{2x^3} + \frac{1.3}{2.4x^5} + \dots + \frac{1.3\dots(2i-1)}{2.4\dots 2ix^{2i+1}}\right)\sqrt{(1-xx)}.$$

 Scho-

Scholion 1.

124. Hinc iam facile integralia formularum $\int x^{m-1} dx (1-xx)^{\frac{\mu}{2}}$ tam pro omnibus numeris m, quam pro imparibus μ assignari poterunt. Reductiones autem nostrae generales ad hunc casum accommodatae sunt:

$$\text{I. } \int x^{m+1} dx (1-xx)^{\frac{\mu}{2}} = \frac{-x^m (1-xx)^{\frac{\mu}{2}+1}}{m+\mu+2} + \frac{m}{m+\mu+2} \int x^{m-1} dx (1-xx)^{\frac{\mu}{2}}$$

$$\text{II. } \int x^{m-3} dx (1-xx)^{\frac{\mu}{2}} = \frac{x^{m-2} (1-xx)^{\frac{\mu}{2}+1}}{m-2} + \frac{m+\mu}{m-2} \int x^{m-1} dx (1-xx)^{\frac{\mu}{2}}$$

$$\text{III. } \int x^{m-1} dx (1-xx)^{\frac{\mu}{2}+1} = \frac{x^m (1-xx)^{\frac{\mu}{2}+1}}{m+\mu+2} + \frac{\mu+2}{m+\mu+2} \int x^{m-1} dx (1-xx)^{\frac{\mu}{2}}$$

$$\text{IV. } \int x^{m-1} dx (1-xx)^{\frac{\mu}{2}-1} = \frac{-x^m (1-xx)^{\frac{\mu}{2}}}{\mu} + \frac{m+\mu}{\mu} \int x^{m-1} dx (1-xx)^{\frac{\mu}{2}}$$

$$\text{V. } \int x^{m+1} dx (1-xx)^{\frac{\mu}{2}-1} = \frac{-x^m (1-xx)^{\frac{\mu}{2}}}{\mu} + \frac{m}{\mu} \int x^{m-1} dx (1-xx)^{\frac{\mu}{2}}$$

VI.

$$\text{VI. } \int x^{m-1} dx (1-xx)^{\frac{\mu}{2}+1} = \frac{x^{m-1}(1-xx)^{\frac{\mu}{2}+1}}{m-2} + \frac{\mu+2}{m-2}\int x^{m-1} dx (1-xx)^{\frac{\mu}{2}}.$$

Posito enim $\mu = -1$ quatuor posteriores dant:

$$\int x^{m-1} dx \sqrt{(1-xx)} = \frac{x^m \sqrt{(1-xx)}}{m+1} + \frac{1}{m+1}\int \frac{x^{m-1} dx}{\sqrt{(1-xx)}}$$

$$\int \frac{x^{m-1} dx}{\sqrt{(1-xx)^3}} = \frac{x^m}{\sqrt{(1-xx)}} - (m-1)\int \frac{x^{m-1} dx}{\sqrt{(1-xx)}}$$

$$\int \frac{x^{m+1} dx}{\sqrt{(1-xx)^3}} = \frac{x^m}{\sqrt{(1-xx)}} - m\int \frac{x^{m-1} dx}{\sqrt{(1-xx)}}$$

$$\int x^{m-1} dx \sqrt{(1-xx)} = \frac{x^{m-1}\sqrt{(1-xx)}}{m-2} + \frac{1}{m-2}\int \frac{x^{m-1} dx}{\sqrt{(1-xx)}}$$

vnde integrationes pro casibus $\mu = 1$; et $\mu = -3$ eliciuntur, indeque porro reliqui.

Scholion 2.

125. Pro aliis formulis irrationalibus magis complicatis vix regulas dare licet, quibus ad formam simpliciorem reduci queant: et quoties eiusmodi formulae occurrant, reductio, si quam admittunt plerumque sponte se offert. Veluti si formula fuerit huiusmodi $\int \frac{P dx}{Q^{n+1}}$ siue n sit numerus integer siue fractus, semper ad aliam huius formae $\int \frac{S dx}{Q^n}$, quae vtique simplicior aestimatur reduci potest. Cum

 enim

enim ſit $d.\frac{R}{Q^n}=\frac{QdR-nRdQ}{Q^{n+1}}$, poſito $\int\frac{Pdx}{Q^{n+1}}=y$; erit $y+\frac{R}{Q^n}=\int\frac{Pdx+QdR-nRdQ}{Q^{n+1}}$. Iam definiatur R ita vt $Pdx+QdR-nRdQ$ per Q fiat diuiſibile, vel quia QdR iam factorem habet Q, vt fiat $Pdx-nRdQ=QTdx$, prodibitque

$$y+\frac{R}{Q^n}=\int\frac{dR+Tdx}{Q^n}, \text{ ſeu}$$

$$\int\frac{Pdx}{Q^{n+1}}=-\frac{R}{Q^n}+\int\frac{dR+Tdx}{Q^n}$$

at ſemper functionem R ita definire licet, vt $Pdx-nRdQ$ factorem Q obtineat, quod etſi in genere praeſtari nequit, tamen rem in exemplis tentando mox perſpicietur negotium ſemper ſuccedere. Aſſumo autem hic P et Q eſſe functiones integras, ac talis quoque ſemper pro R erui poterit. Si forte eueniat, vt $dR+Tdx=0$, formula propoſita algebraicum habebit integrale, quod hoc modo reperietur; contra autem haec forma vlterius reduci poterit in alias, vbi denominatoris exponens continuo vnitate diminuatur; ac ſi n ſit numerus integer negotium tandem reducitur ad huiusmodi formam $\frac{Vdx}{Q}$, quae ſine dubio eſt ſimpliciſſima. Quamobrem cum in hoc capite vix quicquam amplius proferri poſſit ad integrationem formularum irrationalium iuuandam, methodum easdem integrationes per ſeries infinitas perficiendi exponamus.

Ad-

ADDITAMENTUM.

Problema.

Propofita formula $dy = (x + \sqrt{(1 + xx)})^n dx$ *inuenire eius integrale.*

Solutio.

Pofito $x + \sqrt{(1 + xx)} = u$, fit $x = \frac{uu - 1}{2u}$, et $dx = \frac{du(uu + 1)}{2uu}$ vnde formula noftra $dy = \frac{1}{2}u^{n-2}du(uu+1)$, ideoque eius integrale $y = \frac{u^{n+1}}{2(n+1)} + \frac{u^{n-1}}{2(n-1)} +$ Conft. quod ergo femper eft algebraicum nifi fit vel $n = 1$ vel $n = -1$.

Coroll. 1.

Patet etiam hanc formam latius patentem $dy = (x + \sqrt{(1 + xx)})^n X dx$ hoc modo integrari poffe, dummodo X fuerit functio rationalis ipfius x. Pofito enim $x = \frac{uu - 1}{2u}$ pro X prodit functio rationalis ipfius u quae fit = U, hincque fit $dy = \frac{1}{2}Uu^{n-2}du(uu + 1)$, quae formula vel eft rationalis, fi n fit numerus integer, vel ad rationalitatem facile reducitur, fi n fit numerus fractus.

Coroll. 2.

Cum fit $\sqrt{(1 + xx)} = \frac{uu+1}{2u}$; pofito $\sqrt{(1+xx)} = v$, etiam haec formula $dy = (x + \sqrt{(1 + xx)})^n X dx$ integrabitur, fi X fuerit functio rationalis quaecunque quantitatum x et v. Facto enim $x = \frac{uu - 1}{2u}$,

 functio

functio X abit in functionem rationalem ipsius u, qua posita $=\mathrm{U}$ habebitur vt ante $dy=\frac{1}{2}\mathrm{U}u^{n-2}du(uu+1)$.

Exemplum.

Proposita sit formula

$$dy=(ax+b\sqrt{(1+xx)})(x+\sqrt{(1+xx)})^{n}dx.$$

Posito $x=\frac{uu-1}{2u}$ fit

$$dy=\left(\frac{a(uu-1)+b(uu+1)}{2u}\right)\times\tfrac{1}{2}u^{n-2}du(uu+1)$$

seu $dy=\frac{1}{4}u^{n-3}du(a(u^{4}-1)+b(u^{4}+2uu+1))$, cuius integrale est:

$$y=\frac{a+b}{4(n+2)}u^{n+2}+\frac{b}{2n}u^{n}+\frac{b-a}{4(n-2)}u^{n-2}+\mathrm{Const.}$$

quae est algebraica nisi sit vel $n=2$, vel $n=-2$, vel etiam $n=0$.

CAPVT III.

DE INTEGRATIONE FORMVLARVM DIFFERENTIALIVM PER SERIES INFINITAS.

Problema 12.

126.

Si X fuerit functio rationalis fracta ipsius x, formulae differentialis $dy = X dx$ integrale per seriem infinitam exhibere.

Solutio.

Cum X sit functio rationalis fracta, eius valor semper ita euolui potest, vt fiat

$$X = Ax^m + Bx^{m+n} + Cx^{m+2n} + Dx^{m+3n} + Ex^{m+4n} + \text{etc.}$$

vbi coefficientes A, B, C etc. seriem recurrentem constituent, ex denominatore fractionis determinandam. Multiplicentur ergo singuli termini per dx, et integrentur, quo facto integrale y per sequentem seriem exprimetur

$$y = \frac{Ax^{m+1}}{m+1} + \frac{Bx^{m+n+1}}{m+n+1} + \frac{Cx^{m+2n+1}}{m+2n+1} + \text{etc.} + \text{Const.}$$

vbi si in serie pro X occurrat huiusmodi terminus $\frac{M}{x}$ inde in integrale ingredietur terminus Mlx.

Scholion.

127. Cum integrale $\int X dx$, nisi sit algebraicum, per logarithmos et angulos exprimatur, hinc valores logarithmorum et angulorum per series infinitas exhiberi possunt. Cuiusmodi series cum iam in Introductione plures sint traditae, non solum eaedem, sed etiam infinitae aliae hic per integrationem erui possunt. Hoc exemplis declarasse iuuabit, vbi potissimum eiusmodi formulas euoluemus, in quibus denominator est binomium, tum vero etiam casus aliquot denominatore trinomio vel multinomio praeditos contemplabimur. Imprimis autem eiusmodi eligemus, quibus fractio in aliam, cuius denominator est binomius, transmutari potest.

Exemplum 1.

128. *Formulam differentialem* $\frac{dx}{a+x}$ *per seriem integrare.*

Sit $y=\int\frac{dx}{a+x}$ erit $y=l(a+x)+$Const. vnde integrali ita determinato, vt euanescat posito $x=0$, erit $y=l(a+x)-la$. Iam cum sit

$$\frac{1}{a+x}=\frac{1}{a}-\frac{x}{a^2}+\frac{xx}{a^3}-\frac{x^3}{a^4}+\frac{x^4}{a^5}-\text{etc.}$$

erit eadem lege integrale definiendo:

$$y=\frac{x}{a}-\frac{x^2}{2a^2}+\frac{x^3}{3a^3}-\frac{x^4}{4a^4}+\frac{x^5}{5a^5}-\text{etc.}$$

vnde colligimus, vti quidem iam constat:

$$l(a+x)=la+\frac{x}{a}-\frac{x^2}{2a^2}+\frac{x^3}{3a^3}-\frac{x^4}{4a^4}+\text{etc.}$$

Coroll. 1.

Coroll. 1.

129. Si capiamus x negatiuum, vt fit $dy = \frac{-a\,x}{a-x}$, eodem modo patebit esse:

$$l(a-x) = la - \frac{x}{a} - \frac{x^2}{2a^2} - \frac{x^3}{3a^3} - \frac{x^4}{4a^4} - \text{etc.}$$

hisque combinandis:

$$l(aa-xx) = 2la - \frac{xx}{aa} - \frac{x^4}{2a^4} - \frac{x^6}{3a^6} - \frac{x^8}{4a^8} - \text{etc. et}$$

$$l\frac{a+x}{a-x} = \frac{2x}{a} + \frac{2x^3}{3a^3} + \frac{2x^5}{5a^5} + \frac{2x^7}{7a^7} + \text{etc.}$$

Coroll. 2.

130. Hae posteriores series eruuntur per integrationem formularum

$$\frac{-2x\,dx}{aa-xx} = -2x\,dx\left(\frac{1}{aa} + \frac{xx}{a^4} + \frac{x^4}{a^6} + \text{etc.}\right) \text{ et}$$

$$\frac{2a\,dx}{aa-xx} = 2a\,dx\left(\frac{1}{aa} + \frac{xx}{a^4} + \frac{x^4}{a^6} + \text{etc.}\right)$$

Est autem $\int \frac{-2x\,dx}{aa-xx} = l(aa-xx) - laa$ et $\int \frac{2a\,dx}{aa-xx} = l\frac{a+x}{a-x}$, ita vt iam his formulis per series integrandis supersedere possimus.

Exemplum 2.

131. *Formulam differentialem* $\frac{a\,dx}{aa+xx}$ *per seriem integrare.*

Sit $dy = \frac{a\,dx}{aa+xx}$, et cum sit $y = \text{Arc tang.}\frac{x}{a}$, idem angulus serie infinita exprimetur. Quia enim habemus:

$$\frac{a}{aa+xx} = \frac{1}{a} - \frac{xx}{a^3} + \frac{x^4}{a^5} - \frac{x^6}{a^7} + \frac{x^8}{a^9} - \text{etc.}$$

erit integrando:

$$y = \text{Arc. tang.}\frac{x}{a} = \frac{x}{a} - \frac{x^3}{3a^3} + \frac{x^5}{5a^5} - \frac{x^7}{7a^7} + \text{etc.}$$

Exemplum 3.

132. *Integralia harum formularum* $\frac{dx}{1+x^3}$ *et* $\frac{x\,dx}{1+x^3}$, *per series exprimere.*

Cum sit $\frac{1}{1+x^3} = 1 - x^3 + x^6 - x^9 + x^{12} - \text{etc.}$ erit

$$\int \frac{dx}{1+x^3} = x - \tfrac{1}{4}x^4 + \tfrac{1}{7}x^7 - \tfrac{1}{10}x^{10} + \tfrac{1}{13}x^{13} - \text{etc. et}$$

$$\int \frac{x\,dx}{1+x^3} = \tfrac{1}{2}x^2 - \tfrac{1}{5}x^5 + \tfrac{1}{8}x^8 - \tfrac{1}{11}x^{11} + \tfrac{1}{14}x^{14} - \text{etc.}$$

Verum per §. 77. habemus per logarithmos et angulos:

$$\int \frac{dx}{1+x^3} = \tfrac{1}{3}l(1+x) - \tfrac{2}{3}\text{cof.}\tfrac{\pi}{3}\, l\sqrt{(1 - 2x\,\text{cof.}\tfrac{\pi}{3} + xx)} + \tfrac{2}{3}\text{fin.}\tfrac{\pi}{3}\,\text{Arc. tang.}\frac{x\,\text{fin.}\frac{\pi}{3}}{1 - x\,\text{cof.}\frac{\pi}{3}}$$

$$\int \frac{x\,dx}{1+x^3} = -\tfrac{1}{3}l(1+x) - \tfrac{2}{3}\text{cof.}\tfrac{2\pi}{3}\, l\sqrt{(1 - 2x\,\text{cof.}\tfrac{\pi}{3} + xx)} + \tfrac{2}{3}\text{fin.}\tfrac{2\pi}{3}\,\text{Arc. taug.}\frac{x\,\text{fin.}\frac{\pi}{3}}{1 - x\,\text{cof.}\frac{\pi}{3}}.$$

At eft $\text{cof.}\frac{\pi}{3} = \frac{1}{2}$; $\text{cof}\frac{2\pi}{3} = -\frac{1}{2}$; $\text{fin.}\frac{\pi}{3} = \frac{\sqrt{3}}{2}$; $\text{fin.}\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$, vnde fit

$$\int \frac{dx}{1+x^3} = \tfrac{1}{3}l(1+x) - \tfrac{1}{3}l\sqrt{(1 - x + xx)} + \tfrac{1}{\sqrt{3}}\,\text{Arc. tang.}\frac{x\sqrt{3}}{2-x}$$

$$\int \frac{x\,dx}{1+x^3} = -\tfrac{1}{3}l(1+x) + \tfrac{1}{3}l\sqrt{(1 - x + xx)} + \tfrac{1}{\sqrt{3}}\,\text{Arc. tang.}\frac{x\sqrt{3}}{2-x}$$

integralibus vt feriebus ita fumtis, vt euanefcant pofito $x = 0$.

Coroll. 1.

133. His igitur feriebus additis prodit

$$\tfrac{2}{\sqrt{3}}\,\text{Arc. tang.}\frac{x\sqrt{3}}{2-x} = x + \tfrac{1}{2}xx - \tfrac{1}{4}x^4 - \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 + \tfrac{1}{8}x^8 - \tfrac{1}{10}x^{10} - \tfrac{1}{11}x^{11} + \text{etc.}$$

fub-

subtracta autem posteriori a priori fit

$$\tfrac{2}{3}\, l\, \frac{1+x}{\sqrt{(1-x+xx)}} = x - \tfrac{1}{2}x^2 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 - \tfrac{1}{8}x^8 - \tfrac{1}{10}x^{10} + \tfrac{1}{11}x^{11} \text{ etc.}$$

cuius valor etiam est $= \tfrac{1}{3}\, l\, \frac{(1+x)^3}{1-x+xx} = \tfrac{1}{3}\, l\, \frac{(1+x)^4}{1+x^3}$.

Coroll. 2.

134. Cum sit $\int \frac{xx\,dx}{1+x^3} = \tfrac{1}{3} l(1+x^3)$ erit eodem modo

$$\tfrac{1}{3} l(1+x^3) = \tfrac{1}{3}x^3 - \tfrac{1}{6}x^6 + \tfrac{1}{9}x^9 - \tfrac{1}{12}x^{12} + \text{etc.}$$

qua serie illis adiecta omnes potestates ipsius x occurrent.

Exemplum 4.

135. *Integrale hoc* $y = \int \frac{(1+xx)\,dx}{1+x^4}$ *per seriem exprimere.*

Cum sit $\frac{1}{1+x^4} = 1 - x^4 + x^8 - x^{12} + x^{16} - \text{etc.}$ erit

$$y = x + \tfrac{1}{3}x^3 - \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \tfrac{1}{9}x^9 + \tfrac{1}{11}x^{11} - \tfrac{1}{13}x^{13} - \tfrac{1}{15}x^{15} + \text{etc.}$$

Verum per §. 82. vbi $m = 1$ et $n = 4$, posito $\frac{\pi}{4} = \omega$ fit integrale idem:

$$y = \sin.\omega \text{ Arc. tang.} \frac{x \sin.\omega}{1 - x \cos.\omega} + \sin. 3\omega \text{ Arc tang.} \frac{x \sin. 3\omega}{1 - x \cos. 3\omega}$$

at ob $\frac{\pi}{4} = \omega = 45^\circ$, est $\sin.\omega = \frac{1}{\sqrt{2}}$; $\cos.\omega = \frac{1}{\sqrt{2}}$; $\sin. 3\omega = \frac{1}{\sqrt{2}}$; $\cos. 3\omega = \frac{-1}{\sqrt{2}}$ habebimus:

$$y = \frac{1}{\sqrt{2}} \text{ Arc. tang.} \frac{x}{\sqrt{2} - x} + \frac{1}{\sqrt{2}} \text{ Arc. tang.} \frac{x}{\sqrt{2} + x}$$
$$= \frac{1}{\sqrt{2}} \text{ Arc. tang.} \frac{x\sqrt{2}}{1 - xx}$$

Exemplum 5.

136. *Integrale hoc* $y=\int\frac{(1+x^4)dx}{1+x^6}$ *per seriem exprimere.*

Cum sit $\frac{1}{1+x^6}=1-x^6+x^{12}-x^{18}+x^{24}-$ etc. erit

$$y=x+\tfrac{1}{5}x^5-\tfrac{1}{7}x^7-\tfrac{1}{11}x^{11}+\tfrac{1}{13}x^{13}+\tfrac{1}{17}x^{17}-\text{etc.}$$

At per §. 82. vbi $m=1$, $n=6$, et $\omega=\frac{\pi}{6}=30^\circ$ est

$$y=\tfrac{2}{3}\,\text{sin.}\,\omega\,\text{Arc. tang.}\frac{x\,\text{sin.}\,\omega}{1-x\,\text{cos.}\,\omega}+\tfrac{2}{3}\,\text{sin.}\,3\omega\,\text{Arc. tang.}\frac{x\,\text{sin.}\,3\omega}{1-x\,\text{cos.}\,3\omega}$$
$$+\tfrac{2}{3}\,\text{sin.}\,5\omega\,\text{Arc. tang.}\frac{x\,\text{sin.}\,5\omega}{1-x\,\text{cos.}\,5\omega}$$

est vero $\text{sin.}\,\omega=\frac{1}{2}$; $\text{cos.}\,\omega=\frac{\surd 3}{2}$; $\text{sin.}\,3\omega=1$; $\text{cos.}\,3\omega=0$; $\text{sin.}\,5\omega=\frac{1}{2}$; $\text{cos.}\,5\omega=-\frac{\surd 3}{2}$ ergo

$$y=\tfrac{1}{3}\,\text{Arc. tang.}\frac{x}{2-x\surd 3}+\tfrac{2}{3}\,\text{Arc. tang.}\,x+\tfrac{1}{3}\,\text{Arc. tang.}\frac{x}{2+x\surd 3}$$

seu

$$y=\tfrac{1}{3}\,\text{Arc. tang.}\frac{x}{1-xx}+\tfrac{2}{3}\,\text{Arc. tang.}\,x=\tfrac{1}{3}\,\text{Arc. tang.}\frac{3x(1-xx)}{1-4xx+x^4}.$$

Coroll. 1.

137. Sit $z=\int\frac{xxdx}{1+x^6}=\frac{1}{3}x^3-\frac{1}{9}x^9+\frac{1}{15}x^{15}-\frac{1}{21}x^{21}+$ etc. at facto $x^3=u$ est $z=\frac{1}{3}\int\frac{du}{1+uu}=\frac{1}{3}$ Arc. tang. u $=\frac{1}{3}$ Arc. tang. x^3.

Hinc series huiusmodi mixta formatur:

$$x+\tfrac{n}{3}x^3+\tfrac{1}{5}x^5-\tfrac{1}{7}x^7-\tfrac{n}{9}x^9-\tfrac{1}{11}x^{11}+\tfrac{1}{13}x^{13}+\tfrac{n}{15}x^{15}+\tfrac{1}{17}x^{17}-\text{etc.}$$

cuius summa est $\frac{1}{3}$ Arc. tang. $\frac{3x(1-xx)}{1-4xx+x^4}+\frac{n}{3}$ Arc. tang. x^3.

Coroll. 2.

138. Si hic capiatur $n=-1$, binos angulos in vnum colligendo fit $\frac{1}{3}$ Arc. tang. $\frac{3x(1-xx)}{1-4xx+x^4}-\frac{1}{3}$ Arc. tang. x^3

$=\frac{1}{3}$

$= \frac{1}{3}$ Arc. tang. $\frac{3x - 4x^3 + 4x^5 - x^7}{1 - 4xx + 4x^4 - 3x^6}$ quae fractio per $1 - xx + x^4$ diuidendo reducitur ad $\frac{3x - x^3}{1 - 3xx}$, quae est tang. tripli anguli x pro tangente habentis, ita vt sit $\frac{1}{3}$ Arc. tang $\frac{3x - x^3}{1 - 3xx} =$ Arc. tang. x, quod idem series inuenta manifesto indicat.

Exemplum 6.

139. *Hanc formulam* $dy = \frac{(x^{m-1} + x^{n-m-1})\,dx}{1 + x^n}$, *per seriem integrare.*

Ob $\frac{1}{1 + x^n} = 1 - x^n + x^{2n} - x^{3n} + x^{4n} -$ etc. habebitur

$$y = \frac{x^m}{m} + \frac{x^{n-m}}{n-m} - \frac{x^{m+n}}{n+m} - \frac{x^{2n-m}}{2n-m} + \frac{x^{2n+m}}{2n+m} + \frac{x^{3n-m}}{3n-m} - \text{etc.}$$

Haec ergo series per §. 82. aggregatum aliquot arcuum circularium exprimit, quos ibi videre licet.

Corollarium.

140. Eodem proposita formula $dz = \frac{(x^{m-1} - x^{n-m-1})dx}{1 - x^n}$

ob $\frac{1}{1 - x^n} = 1 + x^n + x^{2n} + x^{3n} +$ etc. inuenitur:

$$z = \frac{x^m}{m} - \frac{x^{n-m}}{n-m} + \frac{x^{n+m}}{n+m} - \frac{x^{2n-m}}{2n-m} + \frac{x^{2n+m}}{2n+m} - \frac{x^{3n-m}}{3n-m} + \text{etc.}$$

cuius valor §. 84. est exhibitus.

Exemplum 7.

141. *Hanc formulam* $dy = \frac{(1+2x)\,dx}{1+x+xx}$, *per seriem integrare,*

Primo integrale est manifesto $y = l(1+x+xx)$; vt autem in seriem conuertatur, multiplicetur numerator et denominator per $1-x$, vt fiat $dy = \frac{(1+x-2xx)\,dx}{1-x^3}$. Cum nunc sit $\frac{1}{1-x^3} = 1 + x^3 + x^6 + x^9 + x^{12} +$ etc. erit integrando:

$$y = x + \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} - \frac{2x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} - \frac{2x^9}{9} + \text{etc.}$$

Coroll. 1.

142. Eodem modo inueniri potest $y = l(1+x+xx+x^3)$ per seriem. Cum enim fiat $y + l(1-x) = l(1-x^4)$, erit

$$y = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \frac{x^9}{9} + \frac{x^{10}}{10} + \text{etc.}$$
$$\qquad\qquad - x^4 \qquad\qquad\qquad - \frac{x^8}{2}$$

siue

$$y = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{3x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{3x^8}{8} + \frac{x^9}{9} + \text{etc.}$$

Coroll. 2.

143. At fractio $\frac{1+2x}{1+x+xx}$ per seriem recurrentem euoluta dat

$$1 + x - 2xx + x^3 + x^4 - 2x^5 + x^6 + x^7 - 2x^8 + \text{etc.}$$

vnde per integrationem eadem series obtinetur, quae ante.

Exem

Exemplum 8.

144. *Hanc formulam* $dy = \frac{dx}{1 - 2x\cos\zeta + xx}$ *per seriem integrare.*

Per §. 64. vbi $A = 1$, $B = 0$, $a = 1$, et $b = 1$ est huius formulae integrale $y = \frac{1}{\sin.\zeta}$ Arc. tang. $\frac{x \sin.\zeta}{1 - x\cos.\zeta}$. At per seriem recurrentem reperimus

$$\frac{1}{1 - 2x\cos.\zeta + xx} = 1 + 2x\cos\zeta + (4\cos\zeta^2 - 1)xx + (8\cos\zeta^3 - 4\cos\zeta)x^3 + (16\cos.\zeta^4 - 12\cos.\zeta^2 + 1)x^4 + (32\cos.\zeta^5 - 32\cos.\zeta^3 + 6\cos.\zeta)x^5 + \text{etc.}$$

qua serie per dx multiplicata et integrata obtinetur quaesitum. Potestatibus autem ipsius $\cos.\zeta$ in cosinus angulorum multiplorum conuersis reperitur:

$$y = x + \tfrac{1}{2}xx(2\cos.\zeta) + \tfrac{1}{3}x^3(2\cos.2\zeta + 1) + \tfrac{1}{4}x^4(2\cos 3\zeta + 2\cos.\zeta) + \tfrac{1}{5}x^5(2\cos 4\zeta + 2\cos.2\zeta + 1) + \tfrac{1}{6}x^6(2\cos 5\zeta + 2\cos.3\zeta + 2\cos.\zeta) \text{ etc.}$$

Coroll. 1.

145. Si ponatur $dz = \frac{(1 - x\cos.\zeta)dx}{1 - 2x\cos.\zeta + xx}$ erit per (§.63.) $A = 1$, $B = -\cos.\zeta$, $a = 1$ et $b = 1$ ideoque $z = -\cos.\zeta\, l\sqrt{(1 - 2x\cos.\zeta + xx)} + \sin.\zeta \text{Arc.tang.} \frac{x \sin.\zeta}{1 - x\cos.}$ at per seriem ob $\frac{1 - x\cos.\zeta}{1 - 2x\cos.\zeta + xx} = 1 + x\cos.\zeta + x^2\cos.2\zeta + x^3\cos.3\zeta + x^4\cos.4\zeta + \text{etc.}$ fit $z = x + \tfrac{1}{2}xx\cos.\zeta + \tfrac{1}{3}x^3\cos.2\zeta + \tfrac{1}{4}x^4\cos.3\zeta + \tfrac{1}{5}x^5\cos.4\zeta + \text{etc.}$

Coroll. 2.

146. At quia $dz = \frac{dx(-x\cos.\zeta + \cos.\zeta^2 + \sin.\zeta^2)}{1 - 2x\cos.\zeta + xx}$ erit $z = -\cos.\zeta\, l\sqrt{(1 - 2x\cos.\zeta + xx)} + \sin.\zeta^2 \int \frac{dx}{1 - 2x\cos.\zeta + xx}$.

Hinc

Hinc ergo pro $y = \int \frac{dx}{1 - 2x\,\cos.\zeta + xx}$ alia reperitur series infinita cum logarithmo connexa, scilicet

$$y = \frac{\cos.\zeta}{\sin.\zeta^2}\, l\sqrt{(1 - 2x\cos.\zeta + xx)} + \frac{1}{\sin.\zeta^2}\left(x + \tfrac{1}{2}xx\cos.\zeta + \tfrac{1}{3}x^3\cos.2\zeta + \tfrac{1}{4}x^4\cos.3\zeta + \text{etc.}\right)$$

Problema 12.

147. Formulam differentialem irrationalem $dy = x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$ per seriem infinitam integrare.

Solutio.

Sit $a^{\frac{\mu}{\nu}} = c$, erit $dy = cx^{m-1}dx(1 + \frac{b}{a}x^n)^{\frac{\mu}{\nu}}$, vbi quidem assumimus c non esse quantitatem imaginariam. Cum igitur sit

$$(1 + \tfrac{b}{a}x^n)^{\frac{\mu}{\nu}} = 1 + \frac{\mu b}{1\nu.a}x^n + \frac{\mu(\mu-\nu)bb}{1\nu.2\nu.aa}x^{2n} + \frac{\mu(\mu-\nu)(\mu-2\nu)b^3}{1\nu.2\nu.3\nu.a^3}x^{3n} + \text{etc.}$$

erit integrando:

$$y = c\left(\frac{x^m}{m} + \frac{\mu b}{\nu.a}\cdot\frac{x^{m+n}}{m+n} + \frac{\mu(\mu-\nu)bb}{1\nu.2\nu.aa}\cdot\frac{x^{m+2n}}{m+2n} + \frac{\mu(\mu-\nu)(\mu-2\nu)b^3}{1\nu.2\nu.3\nu.a^3}\cdot\frac{x^{m+3n}}{m+3n} + \text{etc.}\right)$$

quae series in infinitum excurrit, nisi $\frac{\mu}{\nu}$ sit numerus integer positiuus.

Sin autem casu, quo ν numerus par, a fuerit quantitas negatiua, expressio nostra ita est repraesentanda,

$$dy = x^{m-1}dx(bx^n - a)^{\frac{\mu}{\nu}} = b^{\frac{\mu}{\nu}}x^{m+\frac{\mu n}{\nu}-1}dx(1 - \tfrac{a}{b}x^{-n})^{\frac{\mu}{\nu}}.$$

Cum

Cum igitur fit

$$(1-\frac{a}{b}x^{-n})^{\frac{\mu}{\nu}} = 1 - \frac{\mu a}{1\nu . b}x^{-n} + \frac{\mu(\mu-\nu)a^2}{1\nu . 2\nu . b^2}x^{-2n} - \frac{\mu(\mu-\nu)(\mu-2\nu)a^3}{1\nu . 2\nu . 3\nu . b^3}x^{-3n} + \text{etc.}$$

erit integrando

$$y = b^{\frac{\mu}{\nu}}\Big(\frac{\nu x^{m+\frac{\mu n}{\nu}}}{m\nu+\mu n} - \frac{\mu a}{1\nu . b} \cdot \frac{\nu x^{m+\frac{(\mu-\nu)n}{\nu}}}{m\nu+(\mu-\nu)n}$$

$$+ \frac{\mu(\mu-\nu)a^2}{1\nu . 2\nu . b^2} \cdot \frac{\nu x^{m+\frac{(\mu-2\nu)n}{\nu}}}{m\nu+(\mu-2\nu)n} - \text{etc.}\Big)$$

Si a et b fint numeri pofitiui, vtraque euolutione vti licet.

Exemplum 1.

148. *Formulam* $dy = \frac{dx}{\sqrt{(1-xx)}}$, *per feriem integrare.*

Primo ex fuperioribus patet effe $y = \text{Arc.fin.}\, x$ qui ergo angulus etiam per feriem infinitam exprimetur. Cum enim fit

$$\frac{1}{\sqrt{(1-xx)}} = 1 + \frac{1}{2}x^2 + \frac{1.3}{2.4}x^4 + \frac{1.3.5}{2.4.6}x^6 + \frac{1.3.5.7}{2.4.6.8}x^8 + \text{etc.}$$

erit

$$y = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7} + \frac{1.3.5.7}{2.4.6.8}\cdot\frac{x^9}{9} + \text{etc.}$$

vtroque valore ita definito, vt euanefcat pofito $x=0$.

Coroll. 1.

149. Si ergo fit $x=1$, ob $\text{Arc.fin.}\, 1 = \frac{\pi}{2}$ erit

$$\frac{\pi}{2} = 1 + \frac{1}{2.3} + \frac{1.3}{2.4.5} + \frac{1.3.5}{2.4.6.7} + \frac{1.3.5.7}{2.4.6.8.9} + \text{etc.}$$

At si ponatur $x = \frac{1}{2}$ ob Arc. sin. $\frac{1}{2} = 30^\circ = \frac{\pi}{6}$ erit

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2 \cdot 2^3 \cdot 3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 2^5 \cdot 5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 2^7 \cdot 7} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 2^9 \cdot 9} + \text{etc.}$$

cuius seriei decem termini additi dant 0,52359877 cuius sextuplum 3,14159262 tantum in octaua figura a veritate discrepat.

Coroll. 2.

150. Proposita hac formula $dy = \frac{dx}{\sqrt{(x - xx)}}$, posito $x = uu$ fit $dy = \frac{2udu}{\sqrt{(uu - u^4)}} = \frac{2du}{\sqrt{(1 - uu)}}$ ergo $y = 2\,\text{Arc. sin.}\,u = 2\,\text{Arc. sin.}\,\sqrt{x}$, tum vero per seriem erit

$$y = 2\left(u + \frac{1}{2} \cdot \frac{u^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{u^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{u^7}{7} + \text{etc.}\right) \text{ seu}$$

$$y = 2\left(1 + \frac{1}{2} \cdot \frac{x}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{xx}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^3}{7} + \text{etc.}\right)\sqrt{x}.$$

Exemplum 2.

151. *Formulam* $dy = dx\sqrt{(2ax - xx)}$ *per seriem integrare.*

Posito $x = uu$ fit $dy = 2uudu\sqrt{(2a - uu)}$: at per reductionem I. (§. 118.) est $n = 2$; $m = 1$; $a = 2a$; $b = -1$; $\mu = 1$; $\nu = 2$ vnde

$$\int uudu\sqrt{(2a - uu)} = -\frac{1}{4}u(2a - uu)^{\frac{3}{2}} + \frac{1}{2}a\int du\sqrt{(2a - uu)}$$

et per tertiam, sumendo $m = 1$, $a = 2a$, $b = -1$, $n = 2$, $\mu = -1$, $\nu = 2$ fit

$$\int du\sqrt{(2a - uu)} = \frac{1}{2}u\sqrt{(2a - uu)} + a\int\frac{du}{\sqrt{(2a - uu)}},$$

at est

$$\int\frac{du}{\sqrt{(2a - uu)}} = \text{Arc. sin.}\,\frac{u}{\sqrt{2a}} = \text{Arc. sin.}\,\frac{\sqrt{x}}{\sqrt{2a}},$$

ideoque

ideoque

$$\int uudu\sqrt{(2a-uu)} = -\tfrac{1}{4}u(2a-uu)^{\frac{3}{2}} + \tfrac{1}{4}au\sqrt{(2a-uu)} + \tfrac{1}{2}aa\,\text{Arc.fin.}\tfrac{\sqrt{x}}{\sqrt{2a}}$$
$$= \tfrac{1}{4}u(uu-a)\sqrt{(2a-uu)} + \tfrac{1}{2}aa\,\text{Arc.fin.}\tfrac{\sqrt{x}}{\sqrt{2a}}.$$

Ergo $y = \tfrac{1}{2}(x-a)\sqrt{(2ax-xx)} + aa\,\text{Arc.fin.}\tfrac{\sqrt{x}}{\sqrt{2a}}.$

Pro ferie autem inuenienda eft $dy = dx\sqrt{2ax}\left(1-\tfrac{x}{2a}\right)^{\frac{1}{2}}$

$$= x^{\frac{1}{2}}dx\left(1 - \tfrac{1}{2}\cdot\tfrac{x}{2a} - \tfrac{1.1}{2.4}\cdot\tfrac{xx}{4aa} - \tfrac{1.1.3}{2.4.6}\cdot\tfrac{x^3}{8a^3} - \text{etc.}\right)\sqrt{2a}$$

hincque integrando:

$$y = \left(\tfrac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}\cdot\frac{2x^{\frac{5}{2}}}{5.2a} - \frac{1.1}{2.4}\cdot\frac{2x^{\frac{7}{2}}}{7.4aa} - \frac{1.1.3}{2.4.6}\cdot\frac{2x^{\frac{9}{2}}}{9.8a^3} - \text{etc.}\right)\sqrt{2a}$$

feu

$$y = \left(\tfrac{x}{3} - \tfrac{1}{2}\cdot\tfrac{x^2}{5.2a} - \tfrac{1.1}{2.4}\cdot\tfrac{x^3}{7.4aa} - \tfrac{1.1.3}{2.4.6}\cdot\tfrac{x^4}{9.8a^3} - \text{etc.}\right)2\sqrt{2ax}.$$

Coroll. 1.

152. Integrale facilius inueniri poteft ponendo $x = a - v$ vnde fit $dy = -dv\sqrt{(aa-vv)}$, et per reductionem tertiam

$$\int dv\sqrt{(aa-vv)} = \tfrac{1}{2}v\sqrt{(aa-vv)} + \tfrac{1}{2}aa\int\frac{dv}{\sqrt{(aa-vv)}},$$ hinc

$$y = C - \tfrac{1}{2}v\sqrt{(aa-vv)} - \tfrac{1}{2}aa\,\text{Arc.fin.}\tfrac{v}{a}$$ feu

$$y = C - \tfrac{1}{2}(a-x)\sqrt{(2ax-xx)} - \tfrac{1}{2}aa\,\text{Arc.fin.}\tfrac{a-x}{a}$$

vt igitur fiat $y = 0$ pofito $x = 0$, capi debet $C = \tfrac{1}{2}aa\,\text{Arc.fin.}1$, ita vt fit

$$y = -\tfrac{1}{2}(a-x)\sqrt{(2ax-xx)} + \tfrac{1}{2}aa\,\text{Arc.cof.}\tfrac{a-x}{a}.$$

Eft vero $\text{Arc.fin.}\tfrac{\sqrt{x}}{\sqrt{2a}} = \tfrac{1}{2}\,\text{Arc.cof.}\tfrac{a-x}{a}.$

Coroll. 2.

153. Si ponamus $x = \frac{a}{2}$ fit $y = \frac{-aa\sqrt{3}}{4} + \frac{\pi aa}{6}$, feries autem dat

$$y = 2aa\left(\frac{1}{2\cdot 3} - \frac{1}{2\cdot 5\cdot 2^3} - \frac{1\cdot 3}{2\cdot 4\cdot 7\cdot 2^5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 9\cdot 2^7} - \text{etc.}\right)$$

vnde colligitur

$$\pi = \frac{3\sqrt{3}}{4} + 6\left(\frac{1}{3} - \frac{1}{2\cdot 5\cdot 2^2} - \frac{1\cdot 3}{2\cdot 4\cdot 7\cdot 2^4} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 9\cdot 2^6} - \text{etc.}\right)$$

at per fuperiorem eft

$$\pi = 3\left(1 + \frac{1}{2\cdot 3\cdot 2^2} + \frac{1\cdot 3}{2\cdot 4\cdot 5\cdot 2^4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7\cdot 2^6}\right.$$

ex quarum combinatione plures aliae formari poffunt.

Exemplum 3.

154. *Formulam* $dy = \frac{dx}{\sqrt{(1+xx)}}$, *per feriem integrare.*

Integrale eft $y = l(x + \sqrt{(1+xx)})$, ita fumtum vt euanefcat pofito $x = 0$. At ob $\frac{1}{\sqrt{(1+xx)}} = 1 - \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \text{etc.}$ erit idem integrale per feriem expreffum:

$$y = x - \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7} + \text{etc.}$$

Exemplum 4.

155. *Formulam* $dy = \frac{dx}{\sqrt{(xx-1)}}$ *per feriem integrare.*

Integratio dat $y = l(x + \sqrt{(xx-1)})$ quod euanefcit pofito $x = 1$. Iam ob $\frac{1}{\sqrt{(xx-1)}} = \frac{1}{x} + \frac{1}{2x^3} + \frac{1\cdot 3}{2\cdot 4x^5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6x^7}$ etc. erit idem integrale:

$$y = C + lx - \frac{1}{2\cdot 2x^2} - \frac{1\cdot 3}{2\cdot 4\cdot 4x^4} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6x^6} - \text{etc.}$$

quod

quod vt euanescat posito $x=1$, constans ita definitur, vt fiat:

$$y=lx+\frac{1}{2\cdot2}\left(1-\frac{1}{xx}\right)+\frac{1\cdot3}{2\cdot4\cdot4}\left(1-\frac{1}{x^4}\right)+\frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot6}\left(1-\frac{1}{x^6}\right)+\text{etc.}$$

Coroll.

156 Posito $x=1+u$ fit $dy=\frac{du}{\sqrt{(2u+uu)}}$
$$=\frac{du}{\sqrt{2u}}\left(1+\frac{u}{2}\right)^{-\frac{1}{2}}=\frac{du}{\sqrt{2u}}\left(1-\frac{1}{2}\cdot\frac{u}{2}+\frac{1\cdot3}{2\cdot4}\cdot\frac{uu}{4}-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{u^3}{8}-\text{etc.}\right)$$
vnde integrando habebitur:

$$y=\frac{1}{\sqrt{2}}\left(2\sqrt{u}-\frac{1}{2}\cdot\frac{2u^{\frac{3}{2}}}{2.3}+\frac{1.3}{2.4}\cdot\frac{2u^{\frac{5}{2}}}{5.4}-\frac{1.3.5}{2.4.6}\cdot\frac{2u^{\frac{7}{2}}}{7.8}+\text{etc.}\right) \text{ seu}$$

$$y=\left(1-\frac{1u}{2\cdot3\cdot2}+\frac{1\cdot3\cdot uu}{2\cdot4\cdot5\cdot4}-\frac{1\cdot3\cdot5\cdot u^3}{2\cdot4\cdot6\cdot7\cdot8}+\text{etc.}\right)\sqrt{2u}.$$

Exemplum 5.

157. *Formulam* $dy=\frac{dx}{(1-x)^n}$ *per seriem integrare.*

Per integrationem fit $y=\frac{1}{(n-1)(1-x)^{n-1}}-\frac{1}{n-1}$, facto $y=0$ si $x=0$, seu $y=\frac{(1-x)^{-n+1}-1}{n-1}$. Iam vero per seriem est

$$dy=dx\left(1+nx+\frac{n(n+1)}{1\cdot2}x^2+\frac{n(n+1)(n+2)}{1\cdot2\cdot3}x^3+\text{etc.}\right)$$
vnde idem integrale ita exprimetur:
$$y=x+\frac{nx^2}{2}+\frac{n(n+1)x^3}{1\cdot2\cdot3}+\frac{n(n+1)(n+2)x^4}{1\cdot2\cdot3\cdot4}+\text{etc.}$$
Hinc autem quoque manifesto fit $(n-1)y+1=\frac{1}{(1-x)^{n-1}}$.

 Scholion.

Scholion.

158. Haec autem cum sint nimis obuia, quam vt iis fusius inhaerere sit opus, aliam methodum series eliciendi exponam magis absconditam, quae saepe in Analysi eximium vsum afferre potest.

Problema 13.

159. Proposita formula differentiali $dy = x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}-1}$ eius integrale, altera methodo in seriem conuertere.

Solutio.

Ponatur $y=(a+bx^n)^{\frac{\mu}{\nu}}z$, erit

$$dy=(a+bx^n)^{\frac{\mu}{\nu}-1}(dz(a+bx^n)+\tfrac{n\mu}{\nu}bx^{n-1}zdx)$$

vnde fit

$$x^{m-1}dx=dz(a+bx^n)+\tfrac{n\mu}{\nu}bx^{n-1}zdx$$

seu $\nu x^{m-1}dx=\nu dz(a+bx^n)+n\mu bx^{n-1}zdx.$

Iam antequam seriem, qua valor ipsius z definiatur, iuuestigemus, notandum est casu, quo x euanescit, fieri $dy=a^{\frac{\mu}{\nu}-1}x^{m-1}dx=a^{\frac{\mu}{\nu}}dz$, vt sit $dz=\frac{1}{a}x^{m-1}dx$. Statuamus ergo:

$$z=Ax^m+Bx^{m+n}+Cx^{m+2n}+Dx^{m+3n}+\text{etc.}$$

eritque

$$\frac{dz}{dx}=mAx^{m-1}+(m+n)Bx^{m+n-1}+(m+2n)Cx^{m+2n-1}+\text{etc.}$$

Substituantur hae series loco z et $\frac{dz}{dx}$ in aequatione

$$\frac{\nu dz}{dx}(a+bx^n)+n\mu bx^{n-1}z-\nu x^{m-1}=0$$

sin-

ſingulisque terminis ſecundum poteſtates ipſius x diſpoſitis orietur iſta aequatio:

$$\left.\begin{array}{llll} m\nu a A x^{m-1} & +(m+n)\nu a B x^{m+n-1} & +(m+2n)\nu a C x^{m+2n-1} & +\text{etc.} \\ -\nu & +m\nu b A & +(m+n)\nu b B & \\ & +n\mu b A & +n\mu b B & \end{array}\right\} = 0$$

vnde ſingulis terminis nihilo aequalibus poſitis, coefficientes ficti per ſequentes formulas definientur:

$$m\nu a A - \nu = 0 \qquad \text{hinc} \qquad A = \frac{1}{m a}$$

$$(m+n)\nu a B + (m\nu + n\mu) b A = 0; \quad B = -\frac{(m\nu + n\mu) b}{(m+n)\nu a} A$$

$$(m+2n)\nu a C + ((m+n)\nu + n\mu) b B = 0; C = -\frac{((m+n)\nu + n\mu) b}{(m+2n)\nu a} B$$

$$(m+3n)\nu a D + ((m+2n)\nu + n\mu) b C = 0; D = -\frac{((m+2n)\nu + n\mu) b}{(m+3n)\nu a} C$$

ſicque quilibet coefficiens facile ex praecedente reperitur. Tum vero erit

$$y = (a + b x^n)^{\frac{\mu}{\nu}} (A x^m + B x^{m+n} + C x^{m+2n} + D x^{m+3n} + \text{etc.})$$

Solutio 2.

Quemadmodum hic ſeriem ſecundum poteſtates ipſius x aſcendentem aſſumſimus, ita etiam deſcendentem conſtituere licet:

$$z = A x^{m-n} + B x^{m-2n} + C x^{m-3n} + D x^{m-4n} \text{ etc.}$$

vt ſit

$$\frac{dz}{dx} = (m-n) A x^{m-n-1} + (m-2n) B x^{m-2n-1} + (m-3n) C x^{m-3n-1} \text{ etc.}$$

quibus ſeriebus ſubſtitutis prodit

$$\left.\begin{array}{lllll} +(m-n)\nu b A x^{m-1} & +(m-n)\nu a A x^{m-n-1} & +(m-2n)\nu a B x^{m-2n-1} & +(m-3n)\nu a C x^{m-3n-1} & \\ +n\mu b A & +(m-2n)\nu b B & +(m-3n)\nu b C & +(m-4n)\nu b D & \\ -\nu & +n\mu b B & +n\mu b C & +n\mu b D & \end{array}\right\} = 0.$$

Hinc

Hinc ergo ſequenti modo litterae A, B, C, etc. determinantur:

$$(m-n)\nu b A + n\mu b A - \nu = 0 \text{ ergo } A = \frac{\nu}{(m-n)\nu + n\mu} \cdot \frac{1}{b}$$

$$(m-n)\nu a A + (m-2n)\nu b B + n\mu b B = 0, \quad B = \frac{-(m-n)\nu}{(m-2n)\nu + n\mu} \cdot \frac{a}{b} A$$

$$(m-2n)\nu a B + (m-3n)\nu b C + n\mu b C = 0, \quad C = \frac{-(m-2n)\nu}{(m-3n)\nu + n\mu} \cdot \frac{a}{b} B$$

$$(m-3n)\nu a C + (m-4n)\nu b D + n\mu b D = 0, \quad D = \frac{-(m-3n)\nu}{(m-4n)\nu + n\mu} \cdot \frac{a}{b} C$$

vbi iterum lex progreſſionis harum litterarum eſt manifeſta.

Coroll. 1.

160. Prior ſeries ideo eſt memorabilis, quod caſibus, quibus $(m+in)\nu + n\mu = 0$, ſeu $-\frac{m}{n} - \frac{\mu}{\nu} = i$ abrumpitur, atque ipſum integrale algebraicum exhibet. Poſterior vero abrumpitur, quoties $m - in = 0$ ſeu $\frac{m}{n} = i$, denotante i numerum integrum poſitiuum.

Coroll. 2.

161. Vtraque vero ſeries etiam incommodo quodam laborat, quod non ſemper in vſum vocari poteſt. Quando enim vel $m = 0$, vel $m + in = 0$, priori vti non licet, quando vero $(m - in)\nu + n\mu = 0$, ſeu $\frac{m}{n} + \frac{\mu}{\nu} = i$ vſus poſterioris tollitur, quia termini fierent infiniti.

Coroll. 3.

162. Hoc vero commode vſu venit, vt quoties altera applicari nequit, altera certo in vſum vocari poſſit, iis tantum caſibus exceptis, quibus et $-\frac{m}{n}$ et $\frac{\mu}{\nu} + \frac{m}{n}$ ſunt numeri integri poſitiui. Quia autem

autem tum est $\nu = 1$, hi casus sunt rationales integri, nihilque difficultatis habent.

Coroll. 4.

163. Possunt etiam ambae series simul pro z coniungi hoc modo: Sit prior series $= P$, posterior vero $= Q$, vt capi possit tam $z = P$, quam $z = Q$. Binis autem coniungendis erit $z = \alpha P + \beta Q$ dummodo sit $\alpha + \beta = 1$.

Scholion.

164. Inde autem, quod duas series pro z exhibemus, minime sequitur, has duas series inter se esse aequales, neque enim necesse est, vt valores ipsius y inde orti fiant aequales, dummodo quantitate constante a se inuicem differant. Ita si prior series inuenta per P, posterior per Q indicetur, quia ex illa fit $y = (a + bx^n)^{\frac{\mu}{\nu}} P$, ex hac vero $y = (a + bx^n)^{\frac{\mu}{\nu}} Q$, certo erit $(a + bx^n)^{\frac{\mu}{\nu}} (P - Q)$ quantitas constans, ideoque $P - Q = C(a + bx^n)^{-\frac{\mu}{\nu}}$. Vtraque scilicet series tantum integrale particulare praebet, quoniam nullam constantem inuoluit, quae non iam in formula differentiali contineatur. Interim tamen eadem methodo etiam valor completus pro z erui potest: praeter seriem enim assumtam P vel Q statui potest:

$$z = P + \alpha + \beta x^n + \gamma x^{2n} + \delta x^{3n} + \varepsilon x^{4n} + \text{etc.}$$

ac substitutione facta series P vt ante definitur, pro altera vero noua serie efficiendum est, vt sit

$$\left.\begin{array}{llll} n\nu a\beta x^{n-1} + 2n\nu a\gamma x^{2n-1} & + 3n\nu a\delta x^{3n-1} & + 4n\nu a\varepsilon x^{4n-1} \\ + n\mu b\alpha + n\nu b\beta & + 2n\nu b\gamma & + 3n\nu b\delta \\ \quad\quad + n\mu b\beta & + n\mu b\gamma & + n\mu b\delta \end{array}\right\} = 0$$

vnde ducuntur hae determinationes:

$$\beta = \frac{-\mu}{\nu}\cdot\frac{b}{a}\alpha;\quad \gamma = \frac{-(\mu+\nu)b}{2\nu a}\cdot\beta;\quad \delta = \frac{-(\mu+2\nu)b}{3\nu a}\cdot\gamma;$$
$$\varepsilon = \frac{-(\mu+3\nu)b}{4\nu a}\cdot\delta \text{ etc.}$$

ita vt prodeat

$$z = P + \alpha\left(1 - \frac{\mu}{\nu}\cdot\frac{b}{a}x^n + \frac{\mu(\mu+\nu)}{\nu\cdot 2\nu}\cdot\frac{b^2}{a^2}x^{2n} - \frac{\mu(\mu+\nu)(\mu+2\nu)}{\nu\cdot 2\nu\cdot 3\nu}\cdot\frac{b^3}{a^3}x^{3n} + \text{etc.}\right)$$

seu $z = P + \alpha\left(1 + \frac{b}{a}x^n\right)^{-\frac{\mu}{\nu}}$, hincque

$$y = P(a + bx^n)^{\frac{\mu}{\nu}} + \alpha a^{\frac{\mu}{\nu}}$$

quod est integrale completum quia constans α manfit arbitraria.

Exemplum I.

165. *Formulam* $dy = \frac{dx}{\sqrt{(1-xx)}}$ *hoc modo per seriem integrare.*

Comparatione cum forma generali instituta, fit $a = 1$, $b = -1$, $m = 1$, $n = 2$, $\mu = 1$, $\nu = 2$, vnde posito $y = z\sqrt{(1-xx)}$ prima solutio

$$z = Ax + Bx^3 + Cx^5 + Dx^7 + \text{etc. praebet;}$$
$$A = 1;\ B = \tfrac{2}{3}A;\ C = \tfrac{4}{5}B;\ D = \tfrac{6}{7}C;\ E = \tfrac{8}{9}D \text{ etc.}$$

vnde colligimus:

$$y = \left(x + \tfrac{2}{3}x^3 + \tfrac{2\cdot 4}{3\cdot 5}x^5 + \tfrac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}x^7 + \text{etc.}\right)\sqrt{(1-xx)}$$

quod

quod integrale euanefcit pofito $x=0$, eft ergo $y=\text{Arc.fin.}x$. Altera methodus hic fruftra tentatur, ob $\frac{m}{n}+\frac{\mu}{\nu}=1$.

Coroll. 1.

166. Pofito $x=1$ videtur hinc fieri $y=0$, ob $\sqrt{(1-xx)}=0$ at perpendendum eft, fieri hoc cafu feriei infinitae fummam infinitam, ita vt nihil obftet, quo minus fit $y=\frac{\pi}{2}$. Si ponamus $x=\frac{1}{2}$, fit $y=30^\circ=\frac{\pi}{6}$, ideoque

$$\frac{\pi}{6}=\left(1+\frac{2}{3\cdot4}+\frac{2\cdot4}{3\cdot5\cdot4^2}+\frac{2\cdot4\cdot6}{3\cdot5\cdot7\cdot4^3}+\text{etc.}\right)\frac{\sqrt{3}}{4}.$$

Coroll. 2.

167. Simili modo propofita formula $dy=\frac{dx}{\sqrt{(1+xx)}}$ reperitur:

$$y=\left(x-\tfrac{2}{3}x^3+\tfrac{2\cdot4}{3\cdot5}x^5-\tfrac{2\cdot4\cdot6}{3\cdot5\cdot7}x^7+\text{etc.}\right)\sqrt{(1+xx)}$$

eftque $y=l(x+\sqrt{(1+xx)})$.

Exemplum 2.

168. *Formulam* $dy=\frac{dx}{x\sqrt{(1-xx)}}$ *hoc modo per feriem integrare.*

Eft ergo $m=0$, $n=2$, $\mu=1$, $\nu=2$, $a=1$, et $b=-1$, vtendum igitur eft altera ferie fumendo

$$z=\frac{y}{\sqrt{(1-xx)}}=Ax^{-2}+Bx^{-4}+Cx^{-6}Dx^{-8}+\text{etc.}$$

fitque

$$A=1;\ B=\tfrac{2}{3}A;\ C=\tfrac{4}{5}B;\ D=\tfrac{6}{7}C;\ \text{etc.}$$

Hinc ergo colligimus:

$$y=\left(\frac{1}{xx}+\frac{2}{3x^4}+\frac{2\cdot4}{3\cdot5x^6}+\frac{2\cdot4\cdot6}{3\cdot5\cdot7x^8}+\text{etc.}\right)\sqrt{(1-xx)}.$$

 At

At integratio praebet $y = l\frac{1-\sqrt{(1-xx)}}{x}$, qui valores conueniunt, quia vterque euanescit posito $x = 1$.

Coroll. 1.

169. Cum autem haec series non conuergat nisi capiatur $x > 1$, hoc autem casu formula $\sqrt{(1-xx)}$ fiat imaginaria, haec series nullius est vsus.

Coroll. 2.

170. Si proponatur $dy = \frac{dx}{x\sqrt{(xx-1)}}$, eadem pro y series emergit per $\sqrt{-1}$ multiplicata, eritque

$$y = -\left(\frac{1}{xx} + \frac{2}{3x^4} + \frac{2.4}{3.5x^6} + \frac{2.4.6}{3.5.7x^8} + \text{etc.}\right)\sqrt{(xx-1)}$$

Posito autem $x = \frac{1}{u}$ erit $dy = \frac{-du}{\sqrt{(1-uu)}}$ et $y = C - \text{Arc.sin.}u$ seu $y = C - \text{Arc.sin.}\frac{1}{x}$: vbi sumi oportet $C = 0$, quia series illa euanescit posito $x = \infty$ ita vt sit $y = -\text{Arc.sin.}\frac{1}{x}$, quae cum superiori conuenit statuendo $\frac{1}{x} = v$.

Exemplum 3.

171. *Formulam* $dy = \frac{dx}{\sqrt{(a+bx^4)}}$ *hoc modo per seriem integrare.*

Est hic $m = 1$, $n = 4$, $\mu = 1$, $\nu = 2$, ideoque posito $y = z\sqrt{(a+bx^4)}$ prior resolutio dat

$$z = Ax + Bx^5 + Cx^9 + Dx^{13} + \text{etc.}$$

existente

$$A = \frac{1}{a};\ B = \frac{-3b}{5a}A;\ C = \frac{-7b}{9a}B;\ D = \frac{-11b}{13a}C \text{ etc.}$$

ita vt sit

$$y = \left(\frac{x}{a} - \frac{3bx^5}{5aa} + \frac{3.7b^2x^9}{5.9a^3} - \frac{3.7.11b^3x^{13}}{5.9.13a^4} + \text{etc.}\right)\sqrt{(a+bx^4)}.$$

Hic

Hic autem quoque altera resolutio locum habet, ponendo

$$z = Ax^{-3} + Bx^{-7} + Cx^{-11} + Dx^{-15} + \text{etc.}$$

existente

$$A = \frac{-1}{b};\; B = \frac{-3a}{5b}A;\; C = \frac{-7a}{9b}B;\; D = \frac{-11a}{13b}C \text{ etc.}$$

vnde colligitur:

$$y = -\left(\frac{1}{bx^3} - \frac{3a}{5b^2x^7} + \frac{3.7aa}{5.9b^3x^{11}} - \frac{3.7.11a^3}{5.9.13b^4x^{15}} + \text{etc.}\right)\sqrt{(a+bx^4)}$$

quarum serierum illa euanescit posito $x=0$, haec vero posito $x=\infty$.

Coroll. 1.

172. Differentia ergo harum duarum serierum est constans, scilicet:

$$\left\{\begin{array}{l} +\frac{x}{a} - \frac{3bx^5}{5aa} + \frac{3.7b^2x^9}{5.9a^3} - \frac{3.7.11b^3x^{13}}{5.9.13a^4} + \text{etc.} \\ +\frac{1}{bx^3} - \frac{3a}{5bbx^7} + \frac{3.7a^2}{5.9b^3x^{11}} - \frac{3.7.11a^3}{5.9.13b^4x^{15}} + \text{etc.} \end{array}\right\}\sqrt{(a+bx^4)} = \text{Const}$$

Coroll. 2.

173. Has ergo binas series colligendo habebimus

$$\frac{a+bx^4}{abx^3} - \frac{3}{5}\cdot\frac{a^3+b^3x^{12}}{a^2b^2x^7} + \frac{3.7}{5.9}\cdot\frac{a^5+b^5x^{20}}{a^3b^3x^{11}} - \text{etc.} = \frac{C}{\sqrt{(a+bx^4)}}$$

vbi quicunque valor ipsi x tribuatur pro C semper eadem quantitas obtinetur.

Coroll. 3.

174. Ita si $a=1$ et $b=1$, erit haec series in $\sqrt{(1+x^4)}$ ducta semper constans, scilicet

$$\left(\frac{1+x^4}{x^3} - \frac{3}{5}\cdot\frac{1+x^{12}}{x^7} + \frac{3.7}{5.7}\cdot\frac{1+x^{20}}{x^{11}} - \text{etc.}\right)\sqrt{(1+x^4)} = C.$$

 Cum

Cum igitur posito $x=1$ fiat

$$C=\left(1-\tfrac{3}{5}+\tfrac{3\cdot 7}{5\cdot 9}-\tfrac{3\cdot 7\cdot 11}{5\cdot 9\cdot 13}+\text{etc.}\right)2\sqrt{2}$$

huicque valori etiam illa series, quicunque valor ipsi x tribuatur, est aequalis.

Coroll. 4.

175. Haec postrema series signis alternantibus procedens, per differentias facile in aliam iisdem signis praeditam transformatur, vnde eadem constans concluditur

$$C=\left(1+\tfrac{1}{5}+\tfrac{1\cdot 3}{5\cdot 9}+\tfrac{1\cdot 3\cdot 5}{5\cdot 9\cdot 13}+\tfrac{1\cdot 3\cdot 5\cdot 7}{5\cdot 9\cdot 13\cdot 17}+\text{etc.}\right)\sqrt{2}$$

quae series satis cito conuergit, eritque proxime $C=\frac{13}{7}$.

Scholion.

176. Ista methodus in hoc consistit, vt series quaedam indefinita fingatur, eiusque determinatio ex natura rei deriuetur. Eius vsus autem potissimum cernitur in aequationibus differentialibus resoluendis; verum etiam in praesenti instituto saepe vtiliter adhibetur. Eiusdem quoque methodi ope quantitates transcendentes reciprocae, veluti exponentiales et sinus cosinusue angulorum, per series exprimuntur, quae etsi iam aliunde sint cognitae, tamen earum inuestigationem per integrationem exposuisse iuuabit, cum simili modo alia praeclara erui queant.

Problema 14.

177. Quantitatem exponentialem $y=a^x$ in seriem conuertere.

Solutio.

Solutio.

Sumtis logarithmis habemus $ly = xla$ et differentiando $\frac{dy}{y} = dxla$ seu $\frac{dy}{dx} = yla$, vnde valorem ipsius y per seriem quaeri oportet. Cum autem integrale completum lat us pateat, notetur nostro casu posito $x = 0$ fieri debere $y = 1$, quare fingatur haec pro y series:

$$y = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \text{etc.}$$

vnde fit

$$\frac{dy}{dx} = A + 2Bx + 3Cx^2 + 4Dx^3 + \text{etc.}$$

quibus substitutis in aequatione $\frac{dy}{dx} - yla = 0$ erit

$$\left.\begin{array}{l} A + 2Bx + 3Cx^2 + 4Dx^3 + 5Ex^4 + \text{etc.} \\ -la - Ala - Bla - Cla - Dla - \text{etc.} \end{array}\right\} = 0$$

hincque coefficientes ita determinantur:

$$A = la;\ B = \tfrac{1}{2}Ala;\ C = \tfrac{1}{3}Bla;\ D = \tfrac{1}{4}Cla \text{ etc.}$$

sicque consequimur:

$$y = a^x = 1 + \frac{xla}{1} + \frac{x^2(la)^2}{1\cdot 2} + \frac{x^3(la)^3}{1\cdot 2\cdot 3} + \frac{x^4(la)^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}$$

quae est ipsa series notissima in introductione data.

Scholion.

178. Pro sinibus et cosinibus angulorum ad differentialia secundi gradus est descendendum, ex quibus deinceps series integrale referens elici debet. Cum autem gemina integratio duplicem determinationem requirat, series ita est fingenda, vt duabus conditionibus ex natura rei petitis satisfaciat. Verum haec

haec methodus etiam ad alias inueſtigationes extenditur, quae adeo in quantitatibus algebraicis verſantur, a cuiusmodi exemplo hic inchoemus.

Problema 15.

179. Hanc expreſſionem $y=(x+\surd(1+xx))^n$ in ſeriem ſecundum poteſtates ipſius x progredientem conuertere.

Solutio.

Quia eſt $ly=nl(x+\surd(1+xx))$ erit $\frac{dy}{y}=\frac{ndx}{\surd(1+xx)}$; iam ad ſignum radicale tollendum ſumantur quadrata, erit $(1+xx)dy^2=nnyydx^2$. Aequatio ſumto dx conſtante denuo differentietur, vt per $2dy$ diuiſo prodeat

$$ddy(1+xx)+xdxdy-nnydx^2=0$$

vnde y per ſeriem elici debet. Primo autem patet ſi ſit $x=0$ fore $y=1$, ac ſi x infinite paruum, $y=(1+x)^n=1+nx$. Fingatur ergo talis ſeries:

$$y=1+nx+Ax^2+Bx^3+Cx^4+Dx^5+Ex^6+\text{etc.}$$

ex qua colligitur:

$$\frac{dy}{dx}=n+2Ax+3Bxx+4Cx^3+5Dx^4+6Ex^5\text{etc. et}$$

$$\frac{ddy}{dx^2}=2A+6Bx+12Cxx+20Dx^3+30Ex^4\text{etc.}$$

Facta ergo ſubſtitutione adipiſcimur:

$$\left.\begin{array}{rrrrrr} 2A+6Bx+12Cxx+20Dx^3+30Ex^4+42Fx^5+\text{etc.} \\ +2A & +6B & +12C & +20D & +\text{etc.} \\ +nx+2A & +3B & +4C & +5D & +\text{etc.} \\ -nn-n^2-An^2 & -Bn^2 & -Cn^2 & -Dn^2 & +\text{etc.} \end{array}\right\}=0$$

hinc-

hincque deriuantur sequentes determinationes

$$A=\frac{nn}{2};\ B=\frac{n(nn-1)}{2.3};\ C=\frac{A(nn-4)}{3.4};\ D=\frac{B(nn-9)}{4.5}\ \text{etc.}$$

ita vt sit

$$y=1+nx+\frac{nn}{1.2}x^2+\frac{n(nn-1)}{1.2.3}x^3+\frac{nn(nn-4)}{1.2.3.4}x^4+\frac{n(nn-1)(nn-9)}{1.2.3.4.5}x^5$$
$$+\frac{nn(nn-4)(nn-16)}{1.2.3.4.5.6}x^6+\frac{n(nn-1)(nn-9)(nn-25)}{1.2.3.4.5.6.7}x^7+\text{etc.}$$

Coroll. 1.

180. Vti est $y=(x+\sqrt{(1+xx)})^n$ si statuamus $z=(-x+\sqrt{(1+xx)})^n$ pro z similis series prodit, in qua x tantum negatiue capitur, hinc ergo concluditur:

$$\frac{y+z}{2}=1+\frac{nn}{1.2}x^2+\frac{nn(nn-4)}{1.2.3.4}x^4+\frac{nn(nn-4)(nn-16)}{1.2.3.4.5.6}x^6+\text{etc. et}$$

$$\frac{y-z}{2}=nx+\frac{n(nn-1)}{1.2.3}x^3+\frac{n(nn-1)(nn-9)}{1.2.3.4.5}x^5$$
$$+\frac{n(nn-1)(nn-9)(nn-25)}{1.2.3.4.5.6.7}x^7.$$

Coroll. 2.

181. Si ponatur $x=\sqrt{-1}\,\text{sin.}\,\Phi$ erit $\sqrt{(1+xx)}=\text{cos.}\,\Phi$; hincque

$$y=(\text{cos.}\,\Phi+\sqrt{-1}.\,\text{sin.}\,\Phi)^n=\text{cos.}\,n\Phi+\sqrt{-1}.\,\text{sin.}\,n\Phi$$

et

$$z=(\text{cos.}\,\Phi-\sqrt{-1}.\,\text{sin.}\,\Phi)^n=\text{cos.}\,n\Phi-\sqrt{-1}.\,\text{sin.}\,n\Phi$$

vnde deducimus:

$$\text{cos.}\,n\Phi=1-\frac{nn}{1.2}\text{sin.}\,\Phi^2+\frac{nn(nn-4)}{1.2.3.4}\text{sin.}\,\Phi^4-\frac{nn(nn-4)(nn-16)}{1.2.3.4.5.6}\text{sin}\,\Phi^6+\text{etc.}$$

$$\text{sin.}\,n\Phi=n\,\text{sin.}\,\Phi-\frac{n(nn-1)}{1.2.3}\text{sin.}\,\Phi^3+\frac{n(nn-1)(nn-9)}{1.2.3.4.5}\text{sin.}\,\Phi^5$$
$$-\frac{n(nn-1)(nn-9)(nn-25)}{1.2.3.4.5.6.7}\text{sin.}\,\Phi^7+\text{etc.}$$

Coroll. 3.

182. Hae feries ad multiplicationem angulorum pertinent, atque hoc habent fingulare, quod prior tantum cafibus, quibus n eft numerus par, pofterior vero, quibus eft numerus impar, abrumpatur.

Problema 16.

183. Propofito angulo Φ, tam eius finum quam cofinum per feriem infinitam exprimere.

Solutio.

Sit $y = \text{fin.}\,\Phi$ et $z = \text{cof.}\,\Phi$, erit $dy = d\Phi\sqrt{(1-yy)}$ et $dz = -d\Phi\sqrt{(1-zz)}$. Sumantur quadrata

$$dy^2 = d\Phi^2(1-yy) \quad \text{et} \quad dz^2 = d\Phi^2(1-zz)$$

differentietur fumto $d\Phi$ conftante, fietque

$$ddy = -yd\Phi^2 \quad \text{et} \quad ddz = -zd\Phi^2$$

ficque y et z ex eadem aequatione difiniri oportet. Sed pro $y = \text{fin.}\,\Phi$ obferuandum eft, fi Φ euanefcat, fieri $y = \Phi$; pro $z = \text{cof.}\,\Phi$ fi Φ euanefcat, fieri $z = 1 - \frac{1}{2}\Phi\Phi$ feu $z = 1 + 0\,\Phi$. Fingatur ergo

$$y = \Phi + A\Phi^3 + B\Phi^5 + C\Phi^7 \text{ etc.}$$
$$z = 1 + \alpha\Phi^2 + \beta\Phi^4 + \gamma\Phi^6 + \delta\Phi^8 \text{ etc.}$$

fietque fubftitutione facta:

$$\left.\begin{array}{l} 2.3\,A\Phi + 4.5\,C\Phi^3 + 6.7\,C\Phi^5 \text{ etc.} \\ +1 \quad\; + A \qquad\quad\; + B \end{array}\right\} 0 \quad \text{et}$$

$$\left.\begin{array}{l} 1.2\,\alpha + 3.4\,\beta\Phi^2 + 5.6\,\gamma\Phi^4 + \text{etc.} \\ +1 + \alpha \qquad\quad + \beta \end{array}\right\} 0$$

vnde

vnde colligimus:

$$A=\frac{-1}{2.3};\ B=\frac{-A}{4.5};\ C=\frac{-B}{6.7};\ D=\frac{-C}{8.9}\text{ etc.}$$

$$\alpha=\frac{-1}{1.2};\ \beta=\frac{-\alpha}{3.4};\ \gamma=\frac{-\beta}{5.6};\ \delta=\frac{-\gamma}{7.8}\text{ etc.}$$

vnde feries iam notiffimae obtinentur:

$$\text{fin.}\Phi=\frac{\Phi}{1}-\frac{\Phi^3}{1.2.3}+\frac{\Phi^5}{1.2.3.4.5}-\frac{\Phi^7}{1.2.\ldots.7}+\text{etc.}$$

$$\text{cof.}\Phi=1-\frac{\Phi^2}{1.2}+\frac{\Phi^4}{1.2.3.4}-\frac{\Phi^6}{1.2.\ldots.6}+\text{etc.}$$

Scholion.

184. Non opus erat ad differentialia fecundi gradus defcendere: fed ex formularum $y=\text{fin.}\Phi$ et $z=\text{cof.}\Phi$ differentialibus, quae funt $dy=zd\Phi$ et $dz=-yd\Phi$, eaedem feries facile reperiuntur. Fictis enim feriebus vt ante $y=\Phi+A\Phi^3+B\Phi^5+C\Phi^7$ etc. et $z=1+\alpha\Phi^2+\beta\Phi^4+\gamma\Phi^6$ etc. fubftitutione facta obtinebitur

ex priore

$$\left.\begin{matrix}1+3A\Phi^2+5B\Phi^4+7C\Phi^6\text{ etc.}\\-1-\alpha\quad-\beta\quad-\gamma\end{matrix}\right\}=0$$

ex pofteriore

$$\left.\begin{matrix}2\alpha\Phi+4\beta\Phi^3+6\gamma\Phi^5\text{ etc.}\\+1+A\quad+B\end{matrix}\right\}=0$$

vnde colliguntur hae determinationes:

$$\alpha=-\tfrac{1}{2};\ A=\tfrac{\alpha}{3};\ \beta=\tfrac{-A}{4};\ B=\tfrac{\beta}{5};\ \gamma=\tfrac{-B}{6};\ C=\tfrac{\gamma}{7}\text{ etc.}$$

ideoque

$$\alpha=-\tfrac{1}{2};\ \beta=+\tfrac{1}{2.3.4};\ \gamma=-\tfrac{1}{2.3.4.5.6}\text{ etc.}$$

$$A=-\tfrac{1}{2.3};\ B=+\tfrac{1}{2.3.4.5};\ C=-\tfrac{1}{2.3.4.5.6.7}\text{ etc.}$$

 qui

qui valores cum praecedentibus conueniunt. Hinc intelligitur, quomodo saepe duae aequationes simul facilius per series euoluuntur, quam si alteram seorsim tractare velimus.

Problema 17.

185. Per seriem exprimere valorem quantitatis y, qui satisfaciat huic aequationi $\frac{m\,dy}{\sqrt{(a+byy)}} = \frac{n\,dx}{\sqrt{(f+gxx)}}$.

Solutio.

Integratio huius aequationis suppeditat:

$$\frac{m}{\sqrt{b}} l(\sqrt{(a+byy)}+\sqrt{b}) = \frac{n}{\sqrt{g}} l(\sqrt{(f+gxx)}+x\sqrt{g}) + C$$

vnde deducimus:

$$y = \frac{1}{2\sqrt{b}}\left(\frac{\sqrt{(f+gxx)}+x\sqrt{g}}{b}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} - \frac{a}{2\sqrt{b}}\left(\frac{\sqrt{(f+gxx)}-x\sqrt{g}}{k}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}$$

constantes b et k ita capiendo vt sit $bk = f$. Hinc discimus, si x sumatur euanescens, fore

$$y = \frac{1}{2\sqrt{b}}\left(\frac{\sqrt{f}+x\sqrt{g}}{b}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} - \frac{a}{2\sqrt{b}}\left(\frac{\sqrt{f}-x\sqrt{g}}{k}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} \text{ seu}$$

$$y = \frac{1}{2\sqrt{b}}\left(\left(\frac{\sqrt{k}}{\sqrt{b}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} - a\left(\frac{\sqrt{b}}{\sqrt{k}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}\right) + \frac{nx}{2m\sqrt{f}}\left(\left(\frac{\sqrt{k}}{\sqrt{b}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} + a\left(\frac{\sqrt{b}}{\sqrt{k}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}\right)$$

vel posito $y = A + Bx$ erit $B = \frac{n\sqrt{(AAb+a)}}{m\sqrt{f}}$, ita vt constans B definiatur ex constante

$$A = \frac{1}{2\sqrt{b}}\left(\left(\frac{\sqrt{k}}{\sqrt{b}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} - a\left(\frac{\sqrt{b}}{\sqrt{k}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}\right)$$

et vicissim $\left(\frac{\sqrt{k}}{\sqrt{b}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} = A\sqrt{b} + \sqrt{(a+bAA)}$, atque

$$a\left(\frac{\sqrt{b}}{\sqrt{k}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} = -A\sqrt{b} + \sqrt{(a+bAA)}.$$

Nunc ad seriem inueniendam, aequatio proposita, sumtis quadratis $mm(f+gxx)dy^2 = nn(a+byy)dx^2$, denuo

denuo differentietur, capto dx constante, vt facta divisione per $2dy$ prodeat:

$$mmddy(f+gxx)+mmgxdxdy-nnbdx^2=0.$$

Iam pro y fingatur series:

$$y=A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+\text{etc.}$$

qua substituta habebitur:

$$\left.\begin{array}{l} 2mmfC+6mmfDx+12mmfEx^2+20mmfFx^3+\text{etc.} \\ \qquad\qquad +2mmgC+6mmgD+\text{etc.} \\ \qquad +mmgB+2mmgC+3mmgD+\text{etc.} \\ -nnbA-nnbB-nnbC-nnbD+\text{etc.} \end{array}\right\}=0.$$

Cum ergo A et B dentur, reliquae litterae ita determinantur:

$$C=\frac{nnb}{2mmf}A$$

$$D=\frac{nnb-mmg}{2.3\,mmf}B;\quad E=\frac{nnb-4mmg}{3.4\,mmf}C$$

$$F=\frac{nnb-9mmg}{4.5\,mmg}D;\quad G=\frac{nnb-16mmg}{5.6\,mmf}E$$

$$H=\frac{nnb-25mmg}{6.7\,mmf}F;\quad J=\frac{nnb-36mmg}{7.8\,mmf}G$$

sicque series pro y erit cognita.

Exemplum 1.

186. *Functionem transcendentem* $c^{\text{Arc.sin.}x}$ *per seriem secundum potestates ipsius* x *progredientem exprimere.*

Ponatur $y=c^{\text{Arc.sin.}x}$ erit $ly=lc.\text{Arc.sin.}x$ et $\frac{dy}{y}=\frac{dxlc}{\surd(1-xx)}$, hinc $dy^2(1-xx)=yydx^2(lc)^2$, et differentiando $ddy(1-xx)-xdxdy-ydx^2(lc)^2=0$. Obseruetur iam posito x euanescente, fore $y=c^x=1+xlc$, hinc

hinc fingatur feries $y = 1 + xlc + Ax^2 + Bx^3 + Cx^4 + Dx^5$ etc. qua fubftituta habebitur:

$$\left.\begin{array}{llllll} 1.2A & +2.3Bx & +3.4Cx^2 & +4.5Dx^3 & +5.6Ex^4 & \\ & & -1.2A & -2.3B & -3.4C & \\ & -lc & -2A & -3B & -4C & \text{etc.} \\ -(lc)^2 & -(lc)^3 & -A(lc)^2 & -B(lc)^2 & -C(lc)^2 & \end{array}\right\} = 0$$

vnde reliqui coefficientes ita definiuntur:

$$A = \frac{(lc)^2}{1.2}; \qquad B = \frac{(1+(lc)^2)lc}{2.3}$$

$$C = \frac{4+(lc)^2}{3.4}A; \qquad D = \frac{9+(lc)^2}{4.5}B$$

$$E = \frac{16+(lc)^2}{5.6}C; \qquad F = \frac{25+(lc)^2}{6.7}D$$

etc.

Sit breuitatis gratia $lc = \gamma$ eritque

$$c^{\text{Arc. fin. } x} = 1 + \gamma x + \frac{\gamma\gamma}{1.2}x^2 + \frac{\gamma(1+\gamma\gamma)}{1.2.3}x^3 + \frac{\gamma\gamma(4+\gamma\gamma)}{1.2.3.4}x^4 + \frac{\gamma(1+\gamma\gamma)(9+\gamma\gamma)}{1.2.3.4.5}x^5 + \frac{\gamma\gamma(4+\gamma\gamma)(16+\gamma\gamma)}{1.2.3.4.5.6}x^6 + \text{etc.}$$

Exemplum 2.

187. *Pofito* $x = \text{fin.}\,\Phi$, *inuenire feriem fecundum poteftates ipfius* x *progredientem, quae finum anguli* $n\Phi$ *exprimat.*

Ponatur $y = \text{fin.}\,n\Phi$, ac notetur euanefcente Φ fieri $x = \Phi$ et $y = n\Phi = nx$, hoc eft $y = 0 + nx$, quod eft feriei quaefitae initium. Nunc autem eft $d\Phi = \frac{dx}{\surd(1-xx)}$ et $nd\Phi = \frac{dy}{\surd(1-yy)}$. Ergo $\frac{dy}{\surd(1-yy)} = \frac{ndx}{\surd(1-xx)}$, et fumtis quadratis $(1-xx)dy^2 = nndx^2(1-yy)$ hinc

hinc $ddy(1-xx)-xdxdy+nnydx^2=0$. Quare fingatur haec feries

$$y=nx+Ax^3+Bx^5+Cx^7+Dx^9+\text{etc.}$$

qua fubftituta habebitur:

$$\left.\begin{array}{lllll} 2.3\,Ax & +4.5\,Bx^3 & +6.7\,Cx^5 & +8.9\,Dx^7 & \\ & -2.3\,A & -4.5\,B & -6.7\,C & \\ -n & -\;3A & -\;5B & -\;7C & \text{etc.} \\ +n^3 & +nnA & +nnB & +nnC & \end{array}\right\}=0$$

vnde hae determinationes colliguntur:

$$A=\frac{-n(nn-1)}{2.3};\ B=\frac{-(nn-9)A}{4.5};\ C=\frac{-(nn-25)B}{6.7}\ \text{etc.}$$

ita vt fit:

$$y=nx-\frac{n(nn-1)}{1.2.3}x^3+\frac{n(nn-1)(nn-9)}{1.2.3.4.5}x^5-\frac{n(nn-1)(nn-9)(nn-25)}{1.2.3.4.5.6.7}x^7+\text{etc.}$$

fiue

$$\text{fin.}\,n\Phi=n\,\text{fin.}\,\Phi-\frac{n(nn-1)}{1.2.3}\text{fin.}\,\Phi^3+\frac{n(nn-1)(nn-9)}{1.2.3.4.5}\text{fin.}\,\Phi^5-\text{etc.}$$

Scholion.

188. Quia haec feries tantum cafibus, quibus n eft numerus impar abrumpitur, pro paribus notandum eft, feriem commode exprimi poffe per productum ex fin. Φ in aliam feriem, fecundum cofinus ipfius Φ poteftates progredientem. Ad quam inueniendam ponamus cof. $\Phi=u$, fitque fin. $n\Phi=z$ fin. Φ $=z\sqrt{(1-uu)}$; vnde ob $d\Phi=-\frac{du}{\sqrt{(1-uu)}}$ erit differentiando $-\frac{ndu\,\text{cof.}\,n\Phi}{\sqrt{(1-uu)}}=dz\sqrt{(1-uu)}-\frac{zudu}{\sqrt{(1-uu)}}$ feu $-ndu\,\text{cof.}\,n\Phi=dz(1-uu)-zudu$, quae fumto du conftante denuo

denuo differentiata dat: $-\frac{nndu^2 \operatorname{fin}.n\Phi}{\sqrt{(1-uu)}} = ddz(1-uu) - 3ududz - zdu^2 = -nnzdu^2$ ob $\frac{\operatorname{fin}.n\Phi}{\sqrt{(1-uu)}} = z$. Quocirca feries quaefita pro $z = \frac{\operatorname{fin}.n\Phi}{\operatorname{fin}.\Phi}$ ex hac aequatione erui debet:

$$ddz(1-uu) - 3ududz - zdu^2 + nnzdu^2 = 0$$

vbi notandum eft, quia $u = \operatorname{cof}.\Phi$ euanefcente u, quo cafu fit $\Phi = 90°$, fore vel $z = 0$, fi n numerus par, vel $z = 1$, fi $n = 4\alpha + 1$; vel $z = -1$, fi $n = 4\alpha - 1$. Qui finguli cafus feorfim funt euoluendi: et quo principium cuiusque ferici pateat, fit $\Phi = 90° - \omega$, et euanefcente ω fit $u = \operatorname{cof}.\Phi = \omega$; $\operatorname{fin}.\Phi = 1$; $\operatorname{fin}.n\Phi = \operatorname{fin}.(90.n - n\omega) = z$.

Nunc pro cafibus fingulis:

I. fi $n = 4\alpha$; fit $z = -n\omega = -nu$

II. fi $n = 4\alpha + 1$; fit $z = \operatorname{cof}.n\omega = 1$

III. fi $n = 4\alpha + 2$; fit $z = \operatorname{fin}.n\omega = +nu$

IV. fi $n = 4\alpha + 3$; fit $z = -\operatorname{cof}.n\omega = -1$

vnde feries iam fatis notae deducuntur.

CAPVT IV.

CAPVT IV.

DE INTEGRATIONE FORMVLARVM LOGARITHMICARVM ET EXPONENTIALIVM.

Problema 18.

189.

Si X designet functionem algebraicam ipsius x, inuenire integrale formulae $X dx lx$.

Solutio.

Quaeratur integrale $\int X dx$ quod sit $= Z$, et cum quantitatis $Z lx$ differentiale sit $dZ lx + \frac{Z dx}{x}$, erit $Z lx = \int dZ lx + \int \frac{Z dx}{x}$ ideoque

$$\int dZ lx = \int X dx lx = Z lx - \int \frac{Z dx}{x}.$$

Sicque integratio formulae propositae reducta est ad integrationem huius $\frac{Z dx}{x}$, quae, si Z fuerit functio algebraica ipsius x, non amplius logarithmum inuoluit, ideoque per praecedentes regulas tractari poterit. Sin autem $\int X dx$ algebraice exhiberi nequeat, hinc nihil subsidii nascitur, expedietque indicatione integralis $\int X dx lx$ acquiescere, eiusque valorem per approximationem inuestigare.

Nisi forte sit $X = \frac{1}{x}$ quo casu manifesto dat: $\int \frac{dx}{x} lx = \frac{1}{2}(lx)^2 + C.$

Coroll. 1.

190. Eodem modo, si denotante V functionem quamcunque ipsius x, proposita sit formula $X dx lV$, erit existente $\int X dx = Z$ eius integrale $= Z lV - \int \frac{Z dV}{V}$, sicque ad formulam algebraicam reducitur, si modo Z algebraice detur.

Coroll. 2.

191. Pro casu singulari $\frac{dx}{x} lx$ notare licet, si posito $lx = u$, fuerit U functio quaecunque algebraica ipsius u integrationem huius formulae $\frac{U dx}{x}$ non fore difficilem, quia ob $\frac{dx}{x} = du$ abit in $U du$ cuius integratio ad praecedentia capita refertur.

Scholion.

192. Haec reductio innititur illi fundamento, quod cum sit $d.xy = y dx + x dy$, hinc vicissim fiat $xy = \int y dx + \int x dy$, ideoque $\int y dx = xy - \int x dy$, ita vt hoc modo in genere integratio formulae $y dx$ ad integrationem formulae $x dy$ reducatur. Quod si ergo proposita quacunque formula $V dx$, functio V in duos factores puta $V = PQ$ resolui queat, ita vt integrale $\int P dx = S$ assignari queat, ob $P dx = dS$ erit

erit $V dx = PQ dx = Q dS$, hincque $\int V dx = QS - \int S dQ$. Huiusmodi reductio insignem vsum affert, cum formula $\int S dQ$ simplicior fuerit quam proposita $\int V dx$, eaque insuper simili modo ad simpliciorem reduci queat. Interdum etiam commode euenit, vt hac methodo tandem ad formulam propositae similem perueniatur, quo casu integratio pariter obtinetur. Veluti si vlteriori reductione inueniremus $\int S dQ = T + n \int V dx$ foret vtique $\int V dx = QS - T - n \int V dx$, hincque $\int V dx = \frac{QS - T}{n + 1}$. Tum igitur talis reductio insignem praestat vsum, cum vel ad formulam simpliciorem, vel ad eandem perducit. Atque ex hoc principio praecipuos casus, quibus formula $X dx lx$ vel integrationem admittit, vel per seriem commode exhiberi potest, euoluamus.

Exemplum 1.

193. *Formulae differentialis* $x^n dx lx$ *integrale inuenire, denotante* n *numerum quemcunque.*

Cum sit $\int x^n dx = \frac{1}{n+1} x^{n+1}$ erit $\int x^n dx lx = \frac{1}{n+1} x^{n+1} lx - \int \frac{1}{n+1} x^{n+1} d.lx = \frac{1}{n+1} x^{n+1} lx - \frac{1}{n+1} \int x^n dx = \frac{1}{n+1} x^{n+1} lx - \frac{1}{(n+1)^2} x^{n+1}$; ideoque $\int x^n dx lx = \frac{1}{n+1} x^{n+1} \left(lx - \frac{1}{n+1} \right)$. Sicque haec formula absolute est integrabilis.

Coroll. 1.

194. Casus simpliciores, quibus n est numerus integer siue positiuus siue negatiuus, tenuisse iuua-

iuuabit:

$$\int dx\,lx = x\,lx - x\,; \qquad \int \frac{dx}{xx}lx = -\frac{1}{x}lx - \frac{1}{x}$$

$$\int x\,dx\,lx = \tfrac{1}{2}xx\,lx - \tfrac{1}{4}xx\,; \quad \int \frac{dx}{x^3}lx = -\frac{1}{2xx}lx - \frac{1}{4xx}$$

$$\int x^2\,dx\,lx = \tfrac{1}{3}x^3\,lx - \tfrac{1}{9}x^3\,; \quad \int \frac{dx}{x^4}lx = -\frac{1}{3x^3}lx - \frac{1}{9x^3}$$

$$\int x^3\,dx\,lx = \tfrac{1}{4}x^4\,lx - \tfrac{1}{16}x^4\,; \quad \int \frac{dx}{x^5}lx = -\frac{1}{4x^4}lx - \frac{1}{16x^4}.$$

Coroll. 2.

195. Casum $\int \frac{dx}{x}lx = \frac{1}{2}(lx)^2$, qui est omnino singularis, iam supra annotauimus, sequitur vero etiam ex reductione ad eandem formulam. Namque per superiorem reductionem habemus:

$$\int \frac{dx}{x}lx = lx.\,lx - \int lx.dlx = (lx)^2 - \int \frac{dx}{x}\,lx \text{ hincque}$$

$$2\int \frac{dx}{x}lx = (lx)^2, \text{ consequenter } \int \frac{dx}{x}lx = \tfrac{1}{2}(lx)^2.$$

Exemplum 2.

196. *Formulae* $\frac{dx}{1-x}lx$ *Integrale per seriem exprimere.*

Reductione ante adhibita parum lucramur, prodit enim:

$$\int \frac{dx}{1-x}lx = l\frac{1}{1-x}.\,lx - \int \frac{dx}{x}l\frac{1}{1-x}.$$

Cum autem sit

$$l\frac{1}{1-x} = x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \text{etc. erit}$$

$$\int \frac{dx}{x}l\frac{1}{1-x} = x + \tfrac{1}{4}x^2 + \tfrac{1}{9}x^3 + \tfrac{1}{16}x^4 + \tfrac{1}{25}x^5 + \text{etc.}$$

ideo-

ideoque

$$\int \frac{dx}{1-x} lx = l\frac{1}{1-x}.lx - x - \tfrac{1}{4}x^2 - \tfrac{1}{9}x^3 - \tfrac{1}{16}x^4 - \tfrac{1}{25}x^5 \text{ etc.}$$

quod integrale euanescit casu $x=0$, etsi enim lx tum in infinitum abit, tamen $l\frac{1}{1-x} = x + \frac{1}{2}x^2 + \frac{1}{3}x^3$ etc. ita euanescit, vt etiam si per lx multiplicetur, in nihilum abeat, est enim in genere $x^n lx = 0$ posito $x=0$, dum n numerus positiuus.

Coroll. 1.

197. Si ponamus $1-x=u$, fit $\frac{dx}{1-x}lx = -\frac{du}{u}l(1-u) = \frac{du}{u}l\frac{1}{1-u}$, ideoque

$$\int \frac{dx}{1-x} lx = C + u + \tfrac{1}{4}u^2 + \tfrac{1}{9}u^3 + \tfrac{1}{16}u^4 + \tfrac{1}{25}u^5 + \text{etc.}$$

quae vt etiam casu $x=0$ seu $u=1$ euanescat, capi debet

$$C = -1 - \tfrac{1}{4} - \tfrac{1}{9} - \tfrac{1}{16} - \tfrac{1}{25} \text{etc.} = -\tfrac{1}{6}\pi\pi.$$

Coroll. 2.

198. Sumto ergo $1-x=u$ seu $x+u=1$, aequales erunt inter se hae expressiones:

$$-lx.lu - x - \tfrac{1}{4}x^2 - \tfrac{1}{9}x^3 - \tfrac{1}{16}x^4 \text{etc.}$$
$$= -\tfrac{1}{6}\pi^2 + u + \tfrac{1}{4}u^2 + \tfrac{1}{9}u^3 + \text{etc.}$$

seu erit

$$\tfrac{1}{6}\pi^2 - lx.lu = x + u + \tfrac{1}{4}(x^2+u^2) + \tfrac{1}{9}(x^3+u^3) + \tfrac{1}{16}(x^4+u^4) + \text{etc.}$$

Coroll. 3.

199. Haec series maxime conuergit ponendo $x=u=\frac{1}{2}$ hoc ergo casu habebimus:

$$\tfrac{1}{6}\pi - (l2)^2 = 1 + \frac{1}{2\cdot 4} + \frac{1}{4\cdot 9} + \frac{1}{8\cdot 16} + \frac{1}{16\cdot 25} + \frac{1}{32\cdot 36} + \text{etc.}$$

 Huius

Huius ergo feriei

$$x + \tfrac{1}{4}x^2 + \tfrac{1}{9}x^3 + \tfrac{1}{16}x^4 + \tfrac{1}{25}x^5 + \text{etc.}$$

fumma habetur non folum cafu $x = 1$, quo eft $= \frac{\pi\pi}{6}$ fed etiam cafu $x = \frac{1}{2}$, quo eft $= \frac{1}{12}\pi^2 - \frac{1}{2}(l2)^2$.

Coroll. 4.

200. Si ponamus $x = \frac{1}{3}$ et $u = \frac{2}{3}$ erit huius feriei

$$1 + \frac{5}{3^2 \cdot 4} + \frac{9}{3^3 \cdot 9} + \frac{17}{3^4 \cdot 16} + \frac{33}{3^5 \cdot 25} + \frac{65}{3^6 \cdot 36} + \text{etc.}$$

cuius terminus generalis $= \frac{1 + 2^n}{3^n nn}$, fumma $= \frac{1}{6}\pi^2 - l3\, l\frac{3}{2}$ neque vero hinc feriei $x + \frac{1}{4}x^2 + \frac{1}{9}x^3 + \frac{1}{16}x^4$ etc. binos cafus $x = \frac{1}{3}$ et $x = \frac{2}{3}$ feorfim fummare licet.

Exemplum 3.

201. *Formulae* $\frac{dx}{(1-x)^2}\,lx$ *integrale inuenire idemque in feriem conuertere.*

Cum fit $\int \frac{dx}{(1-x)^2} = \frac{1}{1-x}$ erit $\int \frac{dx}{(1-x)^2} lx = \frac{1}{1-x} lx - \int \frac{dx}{x(1-x)}$, at ob $\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}$, fit $\int \frac{dx}{x(1-x)} = lx + l\frac{1}{1-x}$, vnde colligimus integrale $\int \frac{dx}{(1-x)^2} lx = \frac{lx}{1-x} - lx - l\frac{1}{1-x} = \frac{x\,lx}{1-x} - l\frac{1}{1-x}$ ita fumtum, vt euanefcat pofito $x = 0$.

Iam pro ferie commodiffime inuenienda ftatuatur $1 - x = u$ et noftra formula fit $= \frac{-du}{uu} l(1-u) = \frac{du}{uu} l\frac{1}{1-u} = \frac{du}{uu}(u + \frac{1}{2}u^2 + \frac{1}{3}u^3 + \frac{1}{4}u^4 + \frac{1}{5}u^5 + \text{etc.})$ Quocirca integrando nancifcimur:

$$\int \frac{dx}{(1-x)^2} lx = C + lu + \frac{u}{1 \cdot 2} + \frac{uu}{2 \cdot 3} + \frac{u^3}{3 \cdot 4} + \frac{u^4}{4 \cdot 5} + \text{etc.}$$

quae

quae expressio vt etiam euanescat, facto $x = 0$ seu $u = 1$, oporteat sit

$$C = -\frac{1}{1\cdot 2} - \frac{1}{2\cdot 3} - \frac{1}{3\cdot 4} - \frac{1}{4\cdot 5} - \text{etc.} = -1.$$

Quare ob $x = 1 - u$, obtinebimus:

$$\frac{u}{1\cdot 2} + \frac{u^2}{2\cdot 3} + \frac{u^3}{3\cdot 4} + \frac{u^4}{4\cdot 5} + \text{etc.} = 1 - lu + \frac{(1-u)l(1-u)}{u},$$

$$+ lu = 1 + \frac{(1-u)l(1-u)}{u}.$$

Coroll. 1.

202. Simili modo si $dy = \frac{du}{u\sqrt{u}} l \frac{1}{1-u}$, erit $y = -\frac{2}{\sqrt{u}} l \frac{1}{1-u} + \int \frac{2du}{(1-u)\sqrt{u}}$; at posito $u = xx$ fit $\int \frac{2du}{(1-u)\sqrt{u}} = 4\int \frac{dx}{1-xx} = 2l\frac{1+x}{1-x}$. Ergo $y = 2l\frac{1+\sqrt{u}}{1-\sqrt{u}} - \frac{2}{\sqrt{u}} l \frac{1}{1-u}$. At quia per seriem

$$dy = \frac{du}{u\sqrt{u}}(u + \tfrac{1}{2}uu + \tfrac{1}{3}u^3 + \tfrac{1}{4}u^4 + \text{etc.})$$

erit etiam

$$y = +2\sqrt{u} + \frac{2}{2\cdot 3}u\sqrt{u} + \frac{2}{3\cdot 5}u^2\sqrt{u} + \frac{2}{4\cdot 7}u^3\sqrt{u} + \text{etc.}$$

Coroll. 2.

203. Si ergo multiplicemus per $\frac{\sqrt{u}}{2}$, adipiscimur:

$$u + \frac{uu}{2\cdot 3} + \frac{u^3}{3\cdot 5} + \frac{u^4}{4\cdot 7} + \frac{u^5}{5\cdot 9} + \text{etc.} = \sqrt{u}.l\frac{1+\sqrt{u}}{1-\sqrt{u}} + l(1-u).$$

quae summa est etiam $= (1+\sqrt{u})\,l(1+\sqrt{u}) + (1-\sqrt{u})\,l(1-\sqrt{u})$. Quare sumto $u = 1$ ob $(1-\sqrt{u})l(1-\sqrt{u}) = 0$ erit

$$1 + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{4\cdot 7} + \frac{1}{5\cdot 9} + \frac{1}{6\cdot 11}\text{etc.} = 2l2.$$

Proble-

Problema 19.

204. Si P denotet functionem ipsius x, inuenire integrale huius formulae $dy = dP(lx)^n$.

Solutio.

Per reductionem supra monstratam fit $y = P(lx)^n - \int P d.(lx)^n = P(lx)^n - n\int \frac{Pdx}{x}(lx)^{n-1}$. Hinc si fit $\int \frac{Pdx}{x} = Q$ erit simili modo $\int \frac{Pdx}{x}(lx)^{n-1} = Q(lx)^{n-1} - (n-1)\int \frac{Qdx}{x}(lx)^{n-2}$. Quo modo si vlterius progredimur, haecque integralia capere liceat $\int \frac{Pdx}{x} = Q$; $\int \frac{Qdx}{x} = R$; $\int \frac{Rdx}{x} = S$; $\int \frac{Sdx}{x} = T$ etc. obtinebimus integrale quaesitum:

$$\int dP(lx)^n = P(lx)^n - nQ(lx)^{n-1} + n(n-1)R(lx)^{n-2} - n(n-1)(n-2)S(lx)^{n-3} \text{ etc.}$$

ac si exponens n fuerit numerus integer positiuus, integrale forma finita exprimetur.

Exemplum 1.

205. *Formulae* $x^m dx(lx)^2$ *integrale assignare.*

Hic est $n = 2$, et $P = \frac{x^{m+1}}{m+1}$; hinc $Q = \frac{x^{m+1}}{(m+1)^2}$, et $R = \frac{x^{m+1}}{(m+1)^3}$, vnde colligimus:

$$\int x^m dx(lx)^2 = x^{m+1}\left(\frac{(lx)^2}{m+1} - \frac{2lx}{(m+1)^2} + \frac{2}{(m+1)^3}\right)$$

quod integrale euanescit posito $x = 0$, dum sit $m+1 > 0$.

Coroll. 1.

Coroll. 1.

206. Hinc posito $x=1$ fit $\int x^m dx(lx)^2=\frac{2\cdot 1}{(m+1)^3}$ Ex praecedentibus autem patet, si formula $\int x^m dxlx$ ita integretur, vt euanescat posito $x=0$, tum facto $x=1$, fieri $\int x^m dxlx=\frac{-1}{(m+1)^2}$.

Coroll. 2.

207. At si sit $m=-1$ vt habeatur $\frac{dx}{x}(lx)^2$, erit eius integrale $\int\frac{dx}{x}(lx)^2=\frac{1}{3}(lx)^3$ qui solus casus ex formula generali est excipiendus.

Exemplum 2.

208. *Formulae* $x^{m-1}dx(lx)^3$ *integrale assignare.*

Hic est $n=3$ et $P=\frac{x^m}{m}$, hinc $Q=\frac{x^m}{m^2}$; $R=\frac{x^m}{m^3}$ et $S=\frac{x^m}{m^4}$, vnde integrale quaesitum fit:

$$\int x^{m-1}dx(lx)^3=x^m\left(\frac{(lx)^3}{m}-\frac{3(lx)^2}{m^2}+\frac{3\cdot 2\, lx}{m^3}-\frac{3\cdot 2\cdot 1}{m^4}\right)$$

quod integrale euanescit posito $x=0$, dum sit $m>0$.

Coroll. 1.

209. Quod si integrali ita sumto, vt euanescat posito $x=0$, tum ponatur $x=1$, erit: $\int x^{m-1}dx=\frac{1}{m}$; $\int x^{m-1}dxlx=-\frac{1}{m^2}$; $\int x^{m-1}dx(lx)^2=+\frac{1\cdot 2}{m^3}$ et $\int x^{m-1}dx(lx)^3=-\frac{1\cdot 2\cdot 3}{m^4}$.

Coroll. 2.

210. Casu autem $m = 0$, erit integrale $\int \frac{dx}{x}(lx)^3 = \frac{1}{4}(lx)^4$ quod ita determinari nequit, vt euanescat posito $x = 0$; oporteret enim constantem infinitam adiici. Hoc autem integrale euanescit posito $x = 1$.

Exemplum 3.

211. *Formulae* $x^{m-1}dx(lx)^n$ *integrale assignare.*

Cum hic sit $P = \frac{x^m}{m}$; erit $Q = \frac{x^m}{m^2}$; $R = \frac{x^m}{m^3}$; $S = \frac{x^m}{m^4}$, etc. Hinc integrale quaesitum prodit

$$\int x^{m-1}dx(lx)^n = x^m\left(\frac{(lx)^n}{m} - \frac{n(lx)^{n-1}}{m^2} + \frac{n(n-1)(lx)^{n-2}}{m^3} - \frac{n(n-1)(n-2)(lx)^{n-3}}{m^4} + \text{etc.}\right)$$

Casu autem $m = 0$ est $\int \frac{dx}{x}(lx)^n = \frac{1}{n+1}(lx)^{n+1}$.

Coroll. 1.

212. Si $m > 0$ integrale assignatum euanescit posito $x = 0$, deinceps ergo si sumatur $x = 1$, erit integrale

$$\int x^{m-1}dx(lx)^n = \pm\frac{1.2.3.\ldots\ldots n}{m^{n+1}}$$

vbi signum $+$ valet si n sit numerus par, inferius vero si n impar.

Coroll. 2.

Coroll. 2.

213. Haec ergo ambiguitas tollitur, si loco lx scribatur $-l\frac{1}{x}$, tum enim integratione eodem modo instituta positoque $x=1$, fiet

$$\int x^{m-1}dx(l\tfrac{1}{x})^n = +\frac{1.\,2.\,3.\ldots\ldots n}{m^{n+1}}.$$

Scholion.

214. Si exponens n sit numerus fractus, integrale inuentum per seriem infinitam exprimitur, veluti si sit $n=-\frac{1}{2}$ reperitur;

$$\int\frac{x^{m-1}dx}{\sqrt{lx}} = x^m\left(\frac{1}{m\sqrt{lx}} + \frac{1}{2m^2(lx)^{\frac{3}{2}}} + \frac{1.3}{4m^3(lx)^{\frac{5}{2}}}\right.$$
$$\left. + \frac{1.\,3.\,5}{8m^4(lx)^{\frac{7}{2}}} + \text{etc.}\right)$$

quae etiam quatenus initio x ab o ad 1 crescere sumitur, hoc modo repraesentari potest:

$$\int\frac{x^{m-1}dx}{\sqrt{l\frac{1}{x}}} = \frac{x^m}{\sqrt{l\frac{1}{x}}}\left(\frac{1}{m} + \frac{1}{2m^2lx} + \frac{1\ 3.}{4m^3(lx)^2}\right.$$
$$\left. + \frac{1.\,3.\,5}{8m^4(lx)^3} + \text{etc.}\right)$$

Si exponens n sit negatiuus, etsi integer, tamen integrale inuentum in infinitum progreditur: verum hoc casu alia ratione integrationem instituere licet, qua tandem reducitur ad huiusmodi formulam $\int\frac{Tdx}{lx}$, cuius integratio nullo modo simplicior reddi potest.

Hanc ergo reductionem sequenti problemate doceamus.

Problema 20.

215. Integrationem huius formulae $dy = \frac{X\,dx}{(lx)^n}$ continuo ad formulas simpliciores reducere.

Solutio.

Formula proposita ita repraesentetur $dy = Xx . \frac{dx}{x(lx)^n}$ et cum sit $\int \frac{dx}{x(lx)^n} = \frac{-1}{(n-1)(lx)^{n-1}}$ erit

$$y = \frac{-Xx}{(n-1)(lx)^{n-1}} + \frac{1}{n-1}\int \frac{1}{(lx)^{n-1}} . d(Xx).$$

Quare si ponamus continuo

$$d.(Xx) = P\,dx;\ d.(Px) = Q\,dx;\ d.(Qx) = R\,dx \text{ etc.}$$

erit hanc reductionem continuando:

$$y = \frac{-Xx}{(n-1)(lx)^{n-1}} - \frac{Px}{(n-1)(n-2)(lx)^{n-2}} - \frac{Qx}{(n-1)(n-2)(n-3)(lx)^{n-3}} - \text{etc.}$$

donec tandem perueniatur ad hanc integralem

$$+ \frac{1}{(n-1)(n-2)\ldots 1}\int \frac{V\,dx}{lx}$$

ita vt quoties n fuerit numerus integer positiuus, integratio tandem ad huiusmodi formulam perducatur.

Exem-

Exemplum 1.

216. *Formulae differentialis* $dy = \frac{x^{m-1}dx}{(lx)^2}$ *integrale inuestigare.*

Hic est $n = 2$ et $X = x^{m-1}$, vnde fit $P = mx^{m-1}$, hincque integrale

$$y = \int \frac{x^{m-1}dx}{(lx)^2} = -\frac{x^m}{lx} + \frac{m}{1}\int \frac{x^{m-1}dx}{lx}.$$

At formulae $\frac{x^{m-1}dx}{lx}$ integrale exhiberi nequit, nisi casu $m = 0$, quo fit $\int \frac{dx}{xlx} = llx$. Verum si $m = 0$, formulae propositae integratio ne hinc quidem pendet: fit enim absolute $y = \int \frac{dx}{x(lx)^2} = -\frac{1}{lx} + C$.

Exemplum 2.

217. *Formulae differentialis* $dy = \frac{x^{m-1}dx}{(lx)^n}$ *integrale inuestigare casibus, quibus* n *est numerus integer positiuus.*

Cum sit $X = x^{m-1}$ erit $P = \frac{d.(Xx)}{dx} = mx^{m-1}$, tum vero $Q = \frac{d.Px}{dx} = m^2x^{m-1}$; $R = m^3x^{m-1}$; $S = m^4x^{m-1}$ etc. Quare integrale hinc ita formabitur vt sit

$$y = \int \frac{x^{m-1}dx}{(lx)^n} = \frac{-x^m}{(n-1)(lx)^{n-1}} - \frac{mx^m}{(n-1)(n-2)(lx)^{n-2}}$$
$$- \frac{m^2x^m}{(n-1)(n-2)(n-3)(lx)^{n-3}} \text{ etc.}$$
$$\ldots + \frac{m^{n-1}}{(n-1)(n-2)\ldots\ldots 1}\int \frac{x^{m-1}dx}{lx}.$$

 Corol-

Corollarium.

218. Pro n ergo fucceffiue numeros 1, 2, 3, 4, etc. fubftituendo habebimus iftas reductiones:

$$\int \frac{x^{m-1}\,dx}{(lx)^2} = \frac{-x^m}{lx} + \frac{m}{1}\int \frac{x^{m-1}\,dx}{lx}$$

$$\int \frac{x^{m-1}\,dx}{(lx)^3} = \frac{-x^m}{2(lx)^2} - \frac{mx^m}{2.1\,lx} + \frac{m^2}{2.1}\int \frac{x^{m-1}\,dx}{lx}$$

$$\int \frac{x^{m-1}\,dx}{(lx)^4} = \frac{-x^m}{3(lx)^3} - \frac{mx^m}{3.2(lx)^2} - \frac{m^2 x^m}{3.2.1\,lx} + \frac{m^3}{3.2.1}\int \frac{x^{m-1}\,dx}{lx}.$$

Scholion.

219. Hae ergo integrationes pendent a formula $\int \frac{x^{m-1}\,dx}{lx}$ quae pofito $x^m = z$, ob $x^{m-1}dx = \frac{1}{m}dz$ et $lx = \frac{1}{m}lx$ reducitur ad hanc fimpliciffimam formam $\int \frac{dz}{lz}$, cuius integrale fi affignari poffet, ampliffimum vfum in Analyfi effet allaturum, verum nullis adhuc artificiis neque per logarithmos, neque angulos, exhiberi potuit: quomodo autem per feriem exprimi poffit, infra oftendemus (§. 227.) Videtur ergo haec formula $\int \frac{dz}{lz}$ fingularem fpeciem functionum tranfcendentium fuppeditare, quae vtique accuratiorem euolutionem meretur. Eadem autem quantitas tranfcendens in integrationibus formularum exponentialium frequenter occurrit, quas in hoc capite tractare inftituimus, propterea quod cum logarithmicis tam arcte cohaerent, vt alterum genus facile in alterum con-

conuerti possit: veluti ipsa formula modo considerata $\frac{dz}{lz}$ posito $lz = x$, vt sit $z = e^x$ et $dz = e^x dx$ transformatur in hanc exponentialem $e^x \frac{dx}{x}$, cuius ergo integratio aeque est abscondita. Formulas igitur tractabiles euoluamus et eiusmodi quidem, quae non obuia substitutione ad formam algebraicam reduci possunt. Veluti si V fuerit functio quaecunque ipsius v, sitque $v = a^x$, formula Vdx, ob $x = \frac{lv}{la}$ et $dx = \frac{dv}{vla}$ abit in $\frac{Vdv}{vla}$, qua ratione variabilis v est algebraica. Huiusmodi ergo formulas $\frac{a^x dx}{\sqrt{(1 + a^{nx})}}$, quippe quae posito $a^x = v$ nihil habent difficultatis, hinc excludimus.

Problema 21.

220. Formulae differentialis $a^x X dx$, denotante X functionem quamcunque ipsius x, integrale inuestigare.

Solutio 1.

Cum sit $d.a^x = a^x dx la$ erit vicissim $\int a^x dx = \frac{1}{la} . a^x$ quare si formula proposita in hos factores resoluatur, $X.a^x dx$ habebitur per reductionem:

$$\int a^x X dx = \frac{1}{la} a^x X - \frac{1}{la} \int a^x dX.$$

Quodsi vlterius ponamus $dX = P dx$, vt sit

$$\int a^x P dx = \frac{1}{la} a^x P - \frac{1}{la} \int a^x dP,$$

prodibit haec reductio

$$\int a^x X dx = \frac{1}{la} a^x X - \frac{1}{(la)^2} a^x P + \frac{1}{(la)^2} \int a^x dP.$$

Si

Si porro ponamus $dP = Q dx$, habebitur haec reductio:

$$\int a^x X dx = \frac{1}{la} a^x X - \frac{1}{(la)^2} a^x P + \frac{1}{(la)^3} a^x Q - \frac{1}{(la)^3} \int a^x dQ$$

sicque vlterius ponendo $dQ = R dx$, $dR = S dx$ etc. progredi licet, donec ad formulam vel integrabilem, vel in suo genere simplicissimam perueniatur.

Solutio 2.

Alio modo resolutio formulae in factores institui potest; ponatur $\int X dx = P$ seu $X dx = dP$, et formula ita relata $a^x . dP$ habebitur:

$$\int a^x X dx = a^x P - la \int a^x P dx$$

simili modo si ponamus $\int P dx = Q$, obtinebimus:

$$\int a^x X dx = a^x P - la . a^x Q + (la)^2 \int a^x Q dx.$$

Ponamus porro $\int Q dx = R$, et consequimur:

$$\int a^x X dx = a^x P - la . a^x Q + (la)^2 . a^x R - (la)^3 \int a^x R dx$$

hocque modo quousque lubuerit progredi licet, donec ad formulam vel integrabilem vel in suo genere simplissimam perueniamus.

Coroll. 1.

221. Priori solutione semper vti licet, quia functiones P, Q, R, etc. per differentiationem functionis X eliciuntur, dum est

$$P = \frac{dX}{dx}; \quad Q = \frac{dP}{dx}; \quad R = \frac{dQ}{dx} \text{ etc.}$$

Quare si X fuerit functio rationalis integra; tandem ad formulam peruenietur $\int a^x dx = \frac{1}{la} . a^x$, ideoque his casibus integrale absolute exhiberi potest.

Coroll. 2.

Coroll. 2.

222. Altera ſolutio locum non inuenit, niſi formulae $X dx$ integrale P aſſignari queat; neque etiam eam continuare licet, niſi quatenus ſequentes integrationes $\int P dx = Q$, $\int Q dx = R$ etc. ſuccedunt.

Exemplum 1.

223. *Formulae* $a^x x^n dx$ *integrale definire, denotante* n *numerum integrum poſitiuum.*

Cum ſit $X = x^n$ ſolutione prima vtentes habebimus

$$\int a^x x^n dx = \frac{1}{la} \cdot a^x x^n - \frac{n}{la} \int a^x x^{n-1} dx$$

hinc ponendo pro n ſucceſſiue numeros 0, 1, 2, 3, etc. quia primo caſu integratio conſtat, ſequentia integralia eruemus;

$$\int a^x dx = \frac{1}{la} \cdot a^x$$

$$\int a^x x dx = \frac{1}{la} \cdot a^x x - \frac{1}{(la)^2} a^x$$

$$\int a^x x^2 dx = \frac{1}{la} \cdot a^x x^2 - \frac{2}{(la)^2} a^x x + \frac{2 \cdot 1}{(la)^3} a^x$$

$$\int a^x x^3 dx = \frac{1}{la} \cdot a^x x^3 - \frac{3}{(la)^2} a^x x^2 + \frac{3 \cdot 2}{(la)^3} a^x x - \frac{3 \cdot 2 \cdot 1}{(la)^4} a^x$$

etc.

vnde in genere pro quouis exponente n concludimus:

$$\int a^x x^n dx = a^x \left(\frac{x^n}{la} - \frac{n x^{n-1}}{(la)^2} + \frac{n(n-1) x^{n-2}}{(la)^3} + \frac{n(n-1)(n-2) x^{n-3}}{(la)^4} + \text{etc.} \right)$$

ad quam expressionem insuper constantem arbitrariam adiici oportet, vt integrale completum obtineatur.

Corollarium.

224. Si integrale ita determinari debeat, vt euanescat posito $x = 0$, erit

$$\int a^x dx = \frac{1}{la} \cdot a^x - \frac{1}{la}$$

$$\int a^x x\, dx = a^x \left(\frac{x}{la} - \frac{1}{(la)^2} \right) + \frac{1}{(la)^2}$$

$$\int a^x x^2 dx = a^x \left(\frac{x^2}{(la)} - \frac{2x}{(la)^2} + \frac{2 \cdot 1}{(la)^3} \right) - \frac{2 \cdot 1}{(la)^3}$$

$$\int a^x x^3 dx = a^x \left(\frac{x^3}{la} - \frac{3x^2}{(la)^2} + \frac{3 \cdot 2x}{(la)^3} - \frac{3 \cdot 2 \cdot 1}{(la)^4} \right) + \frac{3 \cdot 2 \cdot 1}{(la)^4}$$

etc.

Exemplum 2.

225. *Formulae* $\frac{a^x dx}{x^n}$ *integrale inuestigare, si quidem* n *denotet numerum integrum positiuum.*

Hic commode altera solutione vtemur, vbi cum sit $X = \frac{1}{x^n}$ erit $P = \frac{-1}{(n-1)x^{n-1}}$, hincque resultat ista reductio:

$$\int \frac{a^x dx}{x^n} = \frac{-a^x}{(n-1)x^{n-1}} + \frac{la}{n-1} \int \frac{a^x dx}{x^{n-1}}.$$

Perspicuum igitur est posito $n = 1$ hinc nihil concludi posse; qui est ipse casus supra memoratus $\int \frac{a^x dx}{x}$ singula-

larem speciem transcendentium functionum complectens, qua admissa integralia sequentium casuum exhibere poterimus:

$$\int\frac{a^x dx}{x^2}=C-\frac{a^x}{1x}+\frac{la}{1}\int\frac{a^x dx}{x}$$

$$\int\frac{a^x dx}{x^3}=C-\frac{a^x}{2x^2}-\frac{a^x la}{2.1x}+\frac{(la)^2}{2.1}\int\frac{a^x dx}{x}$$

$$\int\frac{a^x dx}{x^4}=C-\frac{a^x}{3x^3}-\frac{a^x la}{3.2x^2}-\frac{a^x (la)^2}{3.2.1x}+\frac{(la)^3}{3.2.1}\int\frac{a^x dx}{x}$$

vnde in genere colligimus:

$$\int\frac{a^x dx}{x^n}=C-\frac{a^x}{(n-1)x^{n-1}}-\frac{a^x la}{(n-1)(n-2)x^{n-2}}-\frac{a^x (la)^2}{(n-1)(n-2)(n-3)x^{n-3}}-\ldots-\frac{a^x (la)^{n-2}}{(n-1)(n-2)\ldots\ldots 1.x}$$
$$+\frac{(la)^{n-1}}{(n-1)(n-2)\ldots\ldots 1}\int\frac{a^x dx}{x}.$$

Coroll. 1.

226. Admissa ergo quantitate transcendente $\int\frac{a^x dx}{x}$ hanc formulam $a^x x^m dx$ integrare poterimus, siue exponens m fuerit numerus integer positiuus, siue negatiuus. Illis quidem casibus integratio ab ista noua quantitate transcendente non pendet.

Coroll. 2.

227. At si m fuerit fractus numerus, neutra solutio negotium conficit, sed vtraque seriem infi-

nitam pro integrali exhibet. Veluti si sit $m = -\frac{1}{2}$ habebimus ex priore

$$\int \frac{a^x dx}{\sqrt{x}} = a^x\left(\frac{1}{la} + \frac{1}{2x(la)^2} + \frac{1.3}{4x^2(la)^3} + \frac{1.3.5}{8x^3(la)^4} + \text{etc.}\right):\sqrt{x} + C$$

ex posteriore autem;

$$\int \frac{a^x dx}{\sqrt{x}} = C + \frac{a^x}{\sqrt{x}}\left(\frac{2x}{1} - \frac{4x^2 la}{1.3} + \frac{8x^3(la)^2}{1.3.5} - \frac{16x^4(la)^3}{1.3.5.7} + \text{etc.}\right)$$

Scholion 1.

228. Hinc quantitas transcendens $\int \frac{a^x dx}{x}$ per seriem exprimi potest secundum potestates ipsius x progredientem. Cum enim sit

$$a^x = 1 + x la + \frac{x^2(la)^2}{1.2} + \frac{x^3(la)^3}{1.2.3} + \text{etc.}$$

erit:

$$\int \frac{a^x dx}{x} = C + lx + \frac{x la}{1} + \frac{x^2(la)^2}{1.2.2} + \frac{x^3(la)^3}{1.2.3.3} + \frac{x^4(la)^4}{1.2.3.4.4} + \text{etc.}$$

ac si pro a sumamus numerum, cuius logarithmus hyperbolicus est vnitas, quem numerum littera e indicemus, habebimus

$$\int \frac{e^x dx}{x} = C + lx + \frac{x}{1} + \frac{1}{2}\cdot\frac{x^2}{1.2} + \frac{1}{3}\cdot\frac{x^3}{1.2.3} + \frac{1}{4}\cdot\frac{x^4}{1.2.3.4} + \text{etc.}$$

Atque hinc etiam ponendo $e^x = z$, vt sit $x = lz$ formulam supra memoratam $\frac{dz}{lz}$ per seriem integrare poterimus:

$$\int \frac{dz}{lz} = C + llz + \frac{lz}{1} + \frac{1}{2}\cdot\frac{(lz)^2}{1.2} + \frac{1}{3}\cdot\frac{(lz)^3}{1.2.3} + \frac{1}{4}\cdot\frac{(lz)^4}{1.2.3.4} \text{ etc.}$$

quod

quod integrale si debeat euanescere, sumto $z=0$, constans C fit infinita, vnde pro reliquis casibus nihil concludi potest. Idem incommodum locum habet, si euanescens reddamus casu $z=1$, quia $llz=l0$ fit infinitum. Caeterum patet, si integrale sit reale, pro valoribus ipsius z vnitate minoribus, vbi lz est negatiuus, tum pro valoribus vnitate maioribus fieri imaginarium, et vicissim. Hinc ergo natura huius functionis transcendentis parum cognoscitur.

Scholion 2.

229. Quando vel integratio non succedit, vel series ante inuentae minus idoneae videntur, hinc quantitatem a^x in seriem resoluendo statim sine aliis subsidiis formulae $a^x X dx$ integrale per seriem exhiberi potest, erit enim:

$$\int a^x X\,dx = \int X\,dx + \frac{la}{1}\int X x\,dx + \frac{(la)^2}{1.2}\int X x^2\,dx + \frac{(la)^3}{1.2.3}\int X x^3\,dx + \text{etc.}$$

Ita si sit $X = x^n$ habebitur:

$$\int a^x x^n\,dx = C + \frac{x^{n+1}}{n+1} + \frac{x^{n+2}\,la}{1(n+2)} + \frac{x^{n+3}(la)^2}{1.2(n+3)} + \frac{x^{n+4}(la)^3}{1.2.3(n+4)} + \text{etc.}$$

vbi notantum, si n fuerit numerus integer negatiuus, puta $n=-i$, loco $\frac{x^{n+i}}{n+i}$ scribi debere lx.

Exemplum 3.

230. *Formulae* $\frac{a^x dx}{1-x}$ *integrale per seriem infinitam exprimere.*

Per priorem solutionem obtinemus ob $X = \frac{1}{1-x}$; $P = \frac{dX}{dx} = \frac{1}{(1-x)^2}$; $Q = \frac{dP}{dx} = \frac{1 \cdot 2}{(1-x)^3}$; $R = \frac{dQ}{dx} = \frac{1 \cdot 2 \cdot 3}{(1-x)^4}$ etc. hincque sequentem seriem:

$$\int \frac{a^x dx}{1-x} = a^x \left(\frac{1}{(1-x)la} - \frac{1}{(1-x)^2 (la)^2} + \frac{1 \cdot 2}{(1-x)^3 (la)^3} - \frac{1 \cdot 2 \cdot 3}{(1-x)^4 (la)^4} + \text{etc.} \right)$$

Aliae series reperiuntur si vel a^x vel fractio $\frac{1}{1-x}$ in seriem euoluatur. Commodissima autem videtur, quae seriem fingendo eruitur: breuitatis gratia pro a sumamus numerum e, vt $le = 1$, ac statuatur

$$dy = \frac{e^x dx}{1-x} \text{ seu } \frac{dy}{dx}(1-x) - 1 - x - \frac{x^2}{1 \cdot 2} - \frac{x^3}{1 \cdot 2 \cdot 3} - \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} \text{ etc.} = 0$$

iam pro y fingatur haec series

$$y = \int \frac{e^x dx}{1-x} = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.}$$

eritque facta substitutione:

$$\left. \begin{array}{llllll} B & + 2Cx & + 3Dx^2 & + 4Ex^3 & + 5Fx^4 & + \text{etc.} \\ & - B & - 2C & - 3D & - 4E & \\ -1 & - 1 & - \frac{1}{2} & - \frac{1}{6} & - \frac{1}{24} & \end{array} \right\} = 0$$

vnde eliciuntur istae determinationes:

$$\begin{array}{l|l} B = 1 & E = \frac{1}{4}(1 + 1 + \frac{1}{2} + \frac{1}{6}) \\ C = \frac{1}{2}(1 + 1) & F = \frac{1}{5}(1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}) \\ D = \frac{1}{3}(1 + 1 + \frac{1}{2}) & \quad \text{etc.} \end{array}$$

Proble-

Problema 22.

231. *Formulae differentialis* $dy = x^{nx}dx$ *integrale inuestigare, ac per seriem infinitam exprimere.*

Solutio.

Commodius hoc praestari nequit, quam vt formula exponentialis x^{nx} in seriem infinitam conuertatur, quae est

$$x^{nx} = 1 + nx\,lx + \frac{n^2x^2(lx)^2}{1\cdot 2} + \frac{n^3x^3(lx)^3}{1\cdot 2\cdot 3} + \frac{n^4x^4(lx)^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}$$

qua per dx multiplicata, et singulis terminis integratis, erit

$$\int dx = x$$

$$\int x\,dx\,lx = x^2\left(\frac{lx}{2} - \frac{1}{2^2}\right)$$

$$\int x^2dx(lx)^2 = x^3\left(\frac{(lx)^2}{3} - \frac{2lx}{3^2} + \frac{2\cdot 1}{3^3}\right)$$

$$\int x^3dx(lx)^3 = x^4\left(\frac{(lx)^3}{4} - \frac{3(lx)^2}{4^2} + \frac{3\cdot 2\,lx}{4^3} - \frac{3\cdot 2\cdot 1}{4^4}\right)$$

$$\int x^4dx(lx)^4 = x^5\left(\frac{(lx)^4}{5} - \frac{4(lx)^3}{5^2} + \frac{4\cdot 3(lx)^2}{5^3} - \frac{4\cdot 3\cdot 2\,lx}{5^4} + \frac{4\cdot 3\cdot 2\cdot 1}{5^5}\right)$$

etc.

Quare si haec series substituantur, et secundum potestates ipsius lx disponantur, integrale quaesitum exprimetur per has innumerabiles series infinitas:

$$y = \int x^{nx}dx = + x\left(1 - \frac{nx}{2^2} + \frac{n^2x^2}{3^3} - \frac{n^3x^3}{4^4} + \frac{n^4x^4}{5^5} - \text{etc.}\right)$$

$$+ \frac{nx^2lx}{1}\left(\frac{1}{2^2} - \frac{nx}{3^3} + \frac{n^2x^2}{4^4} - \frac{n^3x^3}{5^5} + \frac{n^4x^4}{6^6} - \text{etc.}\right)$$

$$+ \frac{n^2x^3(lx)^2}{1\cdot 2}\left(\frac{1}{3^3} - \frac{nx}{4^4} + \frac{n^2x^2}{5^5} - \frac{n^3x^3}{6^6} + \frac{n^4x^4}{7^7} - \text{etc.}\right)$$

$$+ \frac{n^3x^4(lx)^3}{1\cdot 2\cdot 3}\left(\frac{1}{4^4} - \frac{nx}{5^5} + \frac{n^2x^2}{6^6} - \frac{n^3x^3}{7^7} + \frac{n^4x^4}{8^8} - \text{etc.}\right)$$

etc.

quod

quod integrale ita eſt ſumtum, vt euaneſcat, poſito $x=0$.

Corollarium.

232. Hac ergo lege inſtituta integratione, ſi ponatur $x=1$, valor integralis $\int x^{nx}dx$ huic ſeriei aequatur:

$$1-\frac{n}{2^2}+\frac{n^2}{3^3}-\frac{n^3}{4^4}+\frac{n^4}{5^5}-\frac{n^5}{6^6}+\text{etc.}$$

quae ob concinnitatem terminorum omnino eſt notatu digna.

Scholion.

233. Eodem modo reperitur integrale huius formulae:

$$y=\int x^{nx}x^m dx=\int x^m dx\left(1+nxlx+\frac{n^2x^2(lx)^2}{1\cdot 2}+\frac{n^3x^3(lx)^3}{1\cdot 2\cdot 3}+\text{etc.}\right)$$

erit ſingulis terminis integrandis;

$$\int x^m dx=\frac{x^{m+1}}{m+1}$$

$$\int x^{m+1}dx(lx)=x^{m+2}\left(\frac{lx}{m+2}-\frac{1}{(m+2)^2}\right)$$

$$\int x^{m+2}dx(lx)^2=x^{m+3}\left(\frac{(lx)^2}{m+3}-\frac{2\cdot lx}{(m+3)^2}+\frac{2\cdot 1}{(m+3)^3}\right)$$

$$\int x^{m+3}dx(lx)^3=x^{m+4}\left(\frac{(lx)^3}{m+4}-\frac{3(lx)^2}{(m+4)^2}+\frac{3\cdot 2\,lx}{(m+4)^3}-\frac{3\cdot 2\cdot 1}{(m+4)^4}\right)$$

etc.

Quod ſi ergo integrale ita determinetur, vt euaneſcat poſito $x=0$, tum vero ſtatuatur $x=1$, pro hoc caſu valor formulae integralis $\int x^{nx}x^m dx$ exprimetur hac ſerie ſatis memorabili;

$$\frac{1}{m+1}-\frac{n}{(m+2)^2}+\frac{nn}{(m+3)^3}-\frac{n^3}{(m+4)^4}+\frac{n^4}{(m+5)^5}-\text{etc.}$$

quae vti manifeſtum eſt, locum habere nequit, quoties m eſt numerus integer negatiuus.

Alia

Alia exempla formularum exponentialium non adiungo, quia plerumque integralia nimis inconcinne exprimuntur, methodus autem eas tractandi hic sufficienter est exposita. Interim tamen singularem attentionem merentur formulae integrationem absolute admittentes, quae in hac forma continentur $e^x(dP+Pdx)$ cuius integrale manifesto est e^xP. Huiusmodi autem casibus difficile est regulas tradere integrale inueniendi, et coniecturae plerumque plurimum est tribuendum. Veluti si proponeretur haec formula $\frac{e^x x dx}{(1+x)^2}$, facile est suspicari integrale, si datur, talem formam esse habiturum $\frac{e^x z}{1+x}$. Huius ergo differentiale $\frac{e^x(dz(1+x)+xzdx)}{(1+x)^2}$ cum illo comparatum dat $dz(1+x)+xzdx=xdx$, vbi statim patet esse $z=1$, quod nisi per se pateret, ex regulis difficulter cognosceretur. Quare transeo ad alterum genus formularum transcendentium iam in Analysin receptarum, quae vel angulos vel sinus, tangentesue angulorum complectuntur.

CAPVT V.

DE

INTEGRATIONE FORMVLARVM ANGVLOS SINVSVE ANGVLORVM IMPLICANTIVM.

Problema 23.

234.

Propofita formula differentiali Xdx Ang. fin. x, eius integrale inueftigare.

Solutio.

Cum fit $d.$ Ang. fin. $x = \frac{dx}{\sqrt{(1-xx)}}$, formula propofita, ita in factores difcerpatur, Ang. fin. $x \times Xdx$. Si iam Xdx integrationem patiatur, fitque $\int Xdx = P$, erit noftrum integrale $\int Xdx$ Ang. fin. $x = P$ Ang. fin. $x - \int \frac{Pdx}{\sqrt{(1-xx)}}$; itaque opus reductum eft ad integrationem formulae algebraicae, pro qua fupra praecepta funt tradita.

Caeterum fi fuerit $X = \frac{1}{\sqrt{(1-xx)}}$, manifeftum eft integrale fore $\int \frac{dx}{\sqrt{(1-xx)}}$ Ang. fin. $x = \frac{1}{2}($Ang. fin. $x)^2$; quo folo cafu quadratum anguli in integrale ingreditur.

Exemplum 1.

235. *Hanc formulam* $dy = x^n dx$ Ang. fin. x *integrare.* Cum

Cum sit $P = \int x^n dx = \frac{x^{n+1}}{n+1}$ habebimus

$$y = \frac{x^{n+1}}{n+1} \text{Ang. sin.} x - \frac{1}{n+1} \int \frac{x^{n+1} dx}{\sqrt{(1-xx)}}.$$

Hinc pro variis valoribus ipsius n erunt integralia ope §. 120. eruta, vt sequentur:

$$\int dx \text{ Ang. sin.} x = x \text{ Ang. sin.} x + \sqrt{(1-xx)} - 1$$

$$\int x dx \text{ Ang. sin.} x = \tfrac{1}{2} xx \text{ Ang. sin.} x + \tfrac{1}{4} x \sqrt{(1-xx)} - \tfrac{1}{4} \text{ Ang. sin.} x$$

$$\int x^2 dx \text{ Ang. sin.} x = \tfrac{1}{3} x^3 \text{ Ang. sin.} x + \tfrac{1}{3} (\tfrac{1}{3} x^2 + \tfrac{2}{3}) \sqrt{(1-xx)} - \tfrac{1}{3} \cdot \tfrac{2}{3}$$

$$\int x^3 dx \text{ Ang. sin.} x = \tfrac{1}{4} x^4 \text{ Ang. sin.} x + \tfrac{1}{4} (\tfrac{1}{4} x^3 + \tfrac{1 \cdot 3}{2 \cdot 4} x) \sqrt{(1-xx)}$$
$$- \tfrac{1}{4} \cdot \tfrac{1 \cdot 3}{2 \cdot 4} \text{ Ang. sin.} x$$

quae ita sunt sumta, vt euanescant posito $x = 0$.

Exemplum 2.

236. *Hanc formulam* $dy = \frac{x dx}{\sqrt{(1-xx)}} \text{Ang. sin.} x$ *integrare.*

Cum sit $\int \frac{x dx}{\sqrt{(1-xx)}} = -\sqrt{(1-xx)} = P$ erit integrale quaesitum $y = C - \sqrt{(1-xx)} \text{ Ang. sin.} x + \int \frac{dx \sqrt{(1-xx)}}{\sqrt{(1-xx)}}$, sicque habebitur:

$$y = \int \frac{x dx}{\sqrt{(1-xx)}} \text{Ang. sin.} x = C - \sqrt{(1-xx)} . \text{Ang. sin.} x + x.$$

Exemplum 3.

237. *Hanc formulam* $dy = \frac{dx}{(1-xx)^{\frac{3}{2}}} \text{Ang. sin.} x$ *integrare.*

 Hic

Hic eſt $P = \int \frac{dx}{(1-xx)^{\frac{3}{2}}} = \frac{x}{\sqrt{(1-xx)}}$, vnde fit

$y = \frac{x}{\sqrt{(1-xx)}}$ Ang. ſin. $x - \int \frac{xdx}{1-xx}$ ſeu

$y = \int \frac{dx}{(1-xx)^{\frac{3}{2}}}$ Ang. ſin. $x = \frac{x}{\sqrt{(1-xx)}}$ Ang. ſin. $x + l\sqrt{(1-xx)}$

quod integrale euaneſcit poſito $x = 0$.

Scholion.

238. Simili modo integratur formula $dy = X dx$ Ang. coſ. x. Cum enim ſit $d.$ Ang. coſ. $x = \frac{-dx}{\sqrt{(1-xx)}}$, ſi ponamus $\int X dx = P$, erit $y = P$ Ang. coſ $x + \int \frac{Pdx}{\sqrt{(1-xx)}}$. Quin etiam ſi proponatur formula $dy = X dx$ Ang tang. x, quia eſt $d.$ Ang. tang. $x = \frac{dx}{1+xx}$, poſito $\int X dx = P$, erit hoc integrale

$$y = \int X dx \text{ Ang. tang. } x = P \text{ Ang. tang. } x - \int \frac{Pdx}{1+xx}.$$

Quoties ergo $\int X dx$ algebraice dari poteſt, toties integratio reducitur ad formulam algebraicam, ſicque negotium confectum eſt habendum. Cum igitur in his formulis angulus, cuius ſinus, coſinus, vel tangens erat $= x$, ineſſet, conſideremus etiam eiusmodi formulas, in quas quadratum huius anguli, altiorue poteſtas ingreditur.

Problema 24.

239. Denotet Φ angulum, cuius ſinus tangensue eſt functio quaedam ipſius x, vnde fiat $d\Phi$

$d\Phi = u\,dx$, propofitaque fit haec formula $dy = X\,dx.\Phi^n$ quam integrare oporteat.

Solutio.

Sit $\int X\,dx = P$, vt habeamus $dy = \Phi^n dP$, eritque integrando $y = \Phi^n P - n\int \Phi^{n-1} P u\,dx$. Iam fimili modo fit $\int P u\,dx = Q$, erit $\int \Phi^{n-1} P u\,dx = \Phi^{n-1} Q - (n-1)\int \Phi^{n-2} Q u\,dx$, tum pofito $\int Q u\,dx = R$, erit $\int \Phi^{n-2} Q u\,dx = \Phi^{n-2} R - (n-2)\int \Phi^{n-3} R u\,dx$. Hocque modo poteftas anguli Φ continuo deprimitur, donec tandem ad formulam ab angulo Φ liberam perueniatur: id quod femper eueniet, dummodo n fit numerus integer pofitiuus, et haec integralia continuo fumere liceat $\int X\,dx = P$, $\int P u\,dx = Q$, $\int Q u\,dx = R$, etc. quae integrationes, fi non fuccedant, fruftra integratio fufcipitur.

Exemplum.

240. *Sit* Φ *angulus cuius finus* $= x$, *vt fit* $d\Phi = \frac{dx}{\sqrt{(1-xx)}}$, *integrare formulam* $dy = \Phi^n dx$.

Erit ergo $X = 1$, $P = x$, $Q = \int \frac{P\,dx}{\sqrt{(1-xx)}} = -\sqrt{(1-xx)}$, $R = \int \frac{Q\,dx}{\sqrt{(1-xx)}} = -x$, $S = \int \frac{R\,dx}{\sqrt{(1-xx)}} = \sqrt{(1-xx)}$, $T = x$ etc. quibus valoribus inuentis reperietur:

$$y = \int \Phi^n dx = \Phi^n x + n\Phi^{n-1}\sqrt{(1-xx)} - n(n-1)\Phi^{n-2} x - n(n-1)(n-2)\Phi^{n-3}\sqrt{(1-xx)} + \text{etc}$$

Pro variis ergo valoribus exponentis n habebimus

$$\int \Phi\,dx = \Phi x + \sqrt{(1-xx)} - 1$$

$$\int \Phi^2 dx = \Phi^2 x + 2\Phi\sqrt{(1-xx)} - 2.1x$$

$$\int \Phi^3 dx = \Phi^3 x + 3\Phi^2\sqrt{(1-xx)} - 3.2\Phi x - 3.2.1\sqrt{(1-xx)} + 6$$

etc.

integralibus ita determinatis, vt euanescant posito $x = 0$.

Scholion.

241. Si sit $X dx = u dx = d\Phi$, formulae $\Phi^n d\Phi$ integrale est $\frac{1}{n+1}\Phi^{n+1}$, similique modo, si fuerit Φ functio quaecunque anguli Φ formulae $\Phi u dx = \Phi d\Phi$ integratio nihil habet difficultatis. Multo latius patent formulae sinus cosinusue angulorum et tangentes implicantes, quarum integratio per inuersam Analysin amplissimum habet vsum; cum praecipue Theoria Astronomiae ad huiusmodi formulas sit reducta. Prima autem fundamenta peti debent ex calculo differentiali, vnde cum sit

$$d.\sin. n\Phi = n d\Phi \cos. n\Phi;\quad d.\cos. n\Phi = -n d\Phi \sin. n\Phi;\quad d.\,\text{tang}.\, n\Phi = \frac{n d\Phi}{\cos. n\Phi^2}$$

$$d.\cot. n\Phi = \frac{-n d\Phi}{\sin. n\Phi^2};\quad d.\frac{1}{\sin. n\Phi} = \frac{-n d\Phi \cos. n\Phi}{\sin. n\Phi^2};\quad d.\frac{1}{\cos. n\Phi} = \frac{n d\Phi \sin. n\Phi}{\cos. n\Phi^2}$$

nanciscimur has integrationes elementares:

$$\int d\Phi \cos. n\Phi = \tfrac{1}{n}\sin. n\Phi;\quad \int d\Phi \sin. n\Phi = -\tfrac{1}{n}\cos. n\Phi$$

$$\int \frac{d\Phi}{\cos. n\Phi^2} = \tfrac{1}{n}\,\text{tang}.\, n\Phi;\quad \int \frac{d\Phi}{\sin. n\Phi^2} = -\tfrac{1}{n}\cot. n\Phi$$

$$\int \frac{d\Phi \cos. n\Phi}{\sin. n\Phi^2} = -\frac{1}{n \sin. n\Phi};\quad \int \frac{d\Phi \sin. n\Phi}{\cos. n\Phi^2} = \frac{1}{n \cos. n\Phi}$$

vnde statim huiusmodi formularum differentialium

$$d\Phi\,(A + B\cos.\Phi + C\cos. 2\Phi + D\cos. 3\Phi + E\cos. 4\Phi \text{ etc.})$$

consequitur, cum integrale manifesto sit

$$A\Phi + B\sin.\Phi + \tfrac{1}{2}C\sin. 2\Phi + \tfrac{1}{3}D\sin. 3\Phi + \tfrac{1}{4}E\sin. 4\Phi \text{ etc.}$$

Deinde

Deinde etiam in ſubſidium vocari conuenit, quae in elementis de angulorum compoſitione traduntur: ſcilicet

$$\sin.\alpha.\sin.\beta = \tfrac{1}{2}\cos.(\alpha-\beta) - \tfrac{1}{2}\cos.(\alpha+\beta);\quad \cos.\alpha.\cos.\beta = \tfrac{1}{2}\cos.(\alpha-\beta) + \tfrac{1}{2}\cos.(\alpha+\beta)$$

$$\sin.\alpha\cos.\beta = \tfrac{1}{2}\sin.(\alpha+\beta) + \tfrac{1}{2}\sin.(\alpha-\beta) = \tfrac{1}{2}\sin.(\alpha+\beta) - \tfrac{1}{2}\sin.(\beta-\alpha)$$

vnde producta plurium ſinuum et coſinuum in ſimplices ſinus coſinusue reſoluuntur.

Problema 25.

242. Formulae $d\Phi\sin.\Phi^n$ integrale inueſtigare.

Solutio.

Repraeſentetur in hos factores reſoluta $\sin.\Phi^{n-1}.\,d\Phi\sin.\Phi$, et quia $\int d\Phi\sin.\Phi = -\cos.\Phi$, erit

$$\int d\Phi\sin.\Phi^n = -\sin.\Phi^{n-1}\cos.\Phi + (n-1)\int d\Phi\sin.\Phi^{n-2}\cos.\Phi^2.$$

Hinc ob $\cos.\Phi^2 = 1 - \sin.\Phi^2$, habebitur

$$\int d\Phi\sin.\Phi^n = -\sin.\Phi^{n-1}\cos.\Phi + (n-1)\int d\Phi\sin.\Phi^{n-2} - (n-1)\int d\Phi\sin.\Phi^n$$

vbi cum poſtrema formula ipſi propoſitae ſit ſimilis, hinc colligitur iſta reductio

$$\int d\Phi\sin.\Phi^n = -\tfrac{1}{n}\sin.\Phi^{n-1}\cos.\Phi + \tfrac{n-1}{n}\int d\Phi\sin.\Phi^{n-2}$$

qua integratio ad hanc formulam ſimpliciorem $d\Phi\sin.\Phi^{n-2}$ reuocatur. Cum igitur caſus ſimpliſſimi conſtent, $\int d\Phi\sin.\Phi^0 = \Phi$ et $\int d\Phi\sin.\Phi = -\cos.\Phi$, hinc

hinc via ad continuo maiores exponentes n paratur:

$$\int d\Phi \sin.\Phi^0 = \Phi$$

$$\int d\Phi \sin.\Phi = -\cos.\Phi$$

$$\int d\Phi \sin.\Phi^2 = -\tfrac{1}{2}\sin.\Phi\cos.\Phi + \tfrac{1}{2}\Phi$$

$$\int d\Phi \sin.\Phi^3 = -\tfrac{1}{3}\sin.\Phi^2\cos.\Phi - \tfrac{2}{3}\cos.\Phi$$

$$\int d\Phi \sin.\Phi^4 = -\tfrac{1}{4}\sin.\Phi^3\cos.\Phi - \tfrac{1.3}{2.4}\sin.\Phi\cos.\Phi + \tfrac{1.3}{2.4}\Phi$$

$$\int d\Phi \sin.\Phi^5 = -\tfrac{1}{5}\sin.\Phi^4\cos.\Phi - \tfrac{1.4}{3.5}\sin.\Phi^2\cos.\Phi - \tfrac{2.4}{3.5}\cos.\Phi$$

$$\int d\Phi \sin.\Phi^6 = -\tfrac{1}{6}\sin.\Phi^5\cos.\Phi - \tfrac{1.5}{4.6}\sin.\Phi^3\cos.\Phi - \tfrac{1.3.5}{2.4.6}\sin.\Phi\cos.\Phi + \tfrac{1.3.5}{2.4.6}\Phi$$

etc.

Coroll. 1.

243. Quoties n est numerus impar, integrale per solum sinum et cosinum exhibetur, at si n est numerus par, integrale insuper ipsum angulum inuoluit, ideoque est functio transcendens.

Coroll. 2.

244. Casibus ergo quibus n est numerus impar, id imprimis notari conuenit; etiamsi angulus seu arcus Φ in infinitum crescat, integrale tamen nunquam vltra certum limitem excrescere posse, cum tamen si n sit numerus par, etiam in infinitum excrescat.

Scholion.

245. Simili modo formula $d\Phi \cos.\Phi^n$ tractatur, quae in hos factores resoluta $\cos.\Phi^{n-1}.d\Phi\cos.\Phi$, praebet

praebet

$$\int d\Phi \operatorname{cos}.\Phi^n = \operatorname{cos}.\Phi^{n-1} \operatorname{sin}.\Phi + (n-1)\int d\Phi \operatorname{cos}.\Phi^{n-2} \operatorname{sin}.\Phi^2$$
$$= \operatorname{cos}.\Phi^{n-1} \operatorname{sin}.\Phi + (n-1)\int d\Phi \operatorname{cos}.\Phi^{n-2} - (n-1)\int d\Phi \operatorname{cos}.\Phi^n$$

vnde cum postrema formula propositae sit similis, colligitur

$$\int d\Phi \operatorname{cos}.\Phi^n = \tfrac{1}{n} \operatorname{sin}.\Phi \operatorname{cos}.\Phi^{n-1} + \tfrac{n-1}{n}\int d\Phi \operatorname{cos}.\Phi^{n-2}.$$

Quare cum casibus $n = 0$, et $n = 1$ integratio sit in promtu, ad altiores potestates patet progressio:

$$\int d\Phi \operatorname{cos}\Phi^0 = \Phi$$

$$\int d\Phi \operatorname{cos}.\Phi = \operatorname{sin}.\Phi$$

$$\int d\Phi \operatorname{cos}.\Phi^2 = \tfrac{1}{2} \operatorname{sin}.\Phi \operatorname{cos}.\Phi + \tfrac{1}{2}\Phi$$

$$\int d\Phi \operatorname{cos}.\Phi^3 = \tfrac{1}{3} \operatorname{sin}.\Phi \operatorname{cos}.\Phi^2 + \tfrac{2}{3} \operatorname{sin}.\Phi$$

$$\int d\Phi \operatorname{cos}.\Phi^4 = \tfrac{1}{4} \operatorname{sin}.\Phi \operatorname{cos}.\Phi^3 + \tfrac{1 \cdot 3}{2 \cdot 4} \operatorname{sin}.\Phi \operatorname{cos}.\Phi + \tfrac{1 \cdot 3}{2 \cdot 4}\Phi$$

$$\int d\Phi \operatorname{cos}.\Phi^5 = \tfrac{1}{5} \operatorname{sin}.\Phi \operatorname{cos}.\Phi^4 + \tfrac{1 \cdot 4}{3 \cdot 5} \operatorname{sin}.\Phi \operatorname{cos}.\Phi^2 + \tfrac{2 \cdot 4}{3 \cdot 5} \operatorname{sin}.\Phi$$

$$\int d\Phi \operatorname{cos}.\Phi^6 = \tfrac{1}{6} \operatorname{sin}.\Phi \operatorname{cos}\Phi^5 + \tfrac{1 \cdot 5}{4 \cdot 6} \operatorname{sin}.\Phi \operatorname{cos}.\Phi^3 + \tfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \operatorname{sin}.\Phi \operatorname{cos}.\Phi + \tfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\Phi$$

etc.

Problema 26.

246. Formulae $d\Phi \operatorname{sin}.\Phi^m \operatorname{cos}.\Phi^n$ integrale invenire.

Solutio.

Quo hoc facilius praestetur, consideremus factum $\operatorname{sin}.\Phi^\mu \operatorname{cos}.\Phi^\nu$ quod differentiatum fit $\mu d\Phi \operatorname{sin}.\Phi^{\mu-1} \operatorname{cos}.\Phi^{\nu+1} - \nu d\Phi \operatorname{sin}.\Phi^{\mu+1} \operatorname{cos}.\Phi^{\nu-1}$. Iam prout vel in

parte priori $\cos.\Phi^2 = 1 - \sin.\Phi^2$, vel in posteriori $\sin.\Phi^2 = 1 - \cos.\Phi^2$ statuitur, oritur

$$\text{vel}\quad d.\sin.\Phi^\mu \cos.\Phi^\nu = \mu\, d\Phi \sin.\Phi^{\mu-1}\cos.\Phi^{\nu-1} - (\mu+\nu)\, d\Phi \sin.\Phi^{\mu+1}\cos.\Phi^{\nu-1}$$

$$\text{vel}\quad d.\sin.\Phi^\mu \cos.\Phi^\nu = -\nu d\Phi \sin.\Phi^{\mu-1}\cos.\Phi^{\nu-1} + (\mu+\nu)\, d\Phi \sin.\Phi^{\mu-1}\cos.\Phi^{\nu+1}.$$

Hinc igitur duplicem reductionem adipiscimur:

$$\text{I.}\quad \int d\Phi \sin.\Phi^{\mu+1}\cos.\Phi^{\nu-1} = -\frac{1}{\mu+\nu}\sin.\Phi^\mu \cos.\Phi^\nu + \frac{\mu}{\mu+\nu}\int d\Phi \sin.\Phi^{\mu-1}\cos.\Phi^{\nu-1}$$

$$\text{II.}\quad \int d\Phi \sin.\Phi^{\mu-1}\cos.\Phi^{\nu+1} = \frac{1}{\mu+\nu}\sin.\Phi^\mu \cos.\Phi^\nu + \frac{\nu}{\mu+\nu}\int d\Phi \sin.\Phi^{\mu-1}\cos.\Phi^{\nu-1}.$$

Quare formula proposita $\int d\Phi \sin.\Phi^m \cos.\Phi^n$ successiue continuo ad simpliciores potestates tam ipsius $\sin.\Phi$ quam ipsius $\cos.\Phi$ reducitur, donec alter vel penitus abeat, vel simpliciter adsit, quo casu integratio per se patet, cum sit $\int d\Phi \sin.\Phi^m \cos.\Phi = +\frac{1}{m+1}\sin.\Phi^{m+1}$ et $\int d\Phi \sin.\Phi \cos.\Phi^n = -\frac{1}{n+1}\cos.\Phi^{n+1}$.

Exemplum.

247. *Formulae* $d\Phi \sin.\Phi^8 \cos.\Phi^7$ *integrale invenire.*

Per priorem reductionem ob $\mu = 7$ et $\nu = 8$ impetramus

$$\int d\Phi \sin.\Phi^8 \cos.\Phi^7 = -\tfrac{1}{15}\sin\Phi^7 \cos.\Phi^8 + \tfrac{7}{15}\int d\Phi \sin.\Phi^6 \cos.\Phi^7$$

istam per posteriorem reductionem tractemus:

$$\int d\Phi \sin.\Phi^6 \cos.\Phi^7 = \tfrac{1}{13}\sin.\Phi^7 \cos.\Phi^6 + \tfrac{6}{13}\int d\Phi \sin.\Phi^6 \cos.\Phi^5$$

hoc

hoc modo vlterius progrediamur:

$$\int d\Phi \sin.\Phi^6 \cos.\Phi^5 = -\tfrac{1}{11}\sin.\Phi^5 \cos.\Phi^6 + \tfrac{5}{11}\int d\Phi \sin.\Phi^4 \cos.\Phi^5$$

$$\int d\Phi \sin.\Phi^4 \cos.\Phi^5 = \tfrac{1}{9}\sin.\Phi^5 \cos.\Phi^4 + \tfrac{4}{9}\int d\Phi \sin.\Phi^4 \cos.\Phi^3$$

$$\int d\Phi \sin.\Phi^4 \cos.\Phi^3 = -\tfrac{1}{7}\sin.\Phi^3 \cos.\Phi^4 + \tfrac{3}{7}\int d\Phi \sin.\Phi^2 \cos.\Phi^3$$

$$\int d\Phi \sin.\Phi^2 \cos.\Phi^3 = \tfrac{1}{5}\sin.\Phi^3 \cos.\Phi^2 + \tfrac{2}{5}\int d\Phi \sin.\Phi^2 \cos.\Phi$$

$$\int d\Phi \sin.\Phi^2 \cos.\Phi = -\tfrac{1}{3}\sin.\Phi \cos.\Phi^2 + \tfrac{1}{3}\int d\Phi \cos.\Phi \;(+\tfrac{1}{3}\sin\Phi).$$

Ex his colligitur formulae propositae integrale

$$\int d\Phi \sin.\Phi^8 \cos.\Phi^7$$

$$= -\tfrac{1}{15}\sin.\Phi^7 \cos.\Phi^8 + \tfrac{1.7}{15.13}\sin.\Phi^7 \cos.\Phi^6 - \tfrac{1.7.6}{15.13.11}\sin.\Phi^5 \cos.\Phi^6$$

$$+ \tfrac{1.7.6.5}{15.13.11.9}\sin.\Phi^5 \cos.\Phi^4 - \tfrac{1.7.5.6.4}{15.13.11.9.7}\sin.\Phi^3 \cos.\Phi^4$$

$$+ \tfrac{1.7.6.5.4.3}{15.13.11.9.7.5}\sin.\Phi^3 \cos.\Phi^2 - \tfrac{1.7.6.5.4.3.2}{15.13.11.9.7.5.3}\sin.\Phi \cos.\Phi^2$$

$$+ \tfrac{1.7.6.5.4.3.2}{15.13.11.9.7.5.3}\sin.\Phi.$$

Scholion.

248. Quando autem huiusmodi casus occurrunt, semper praestat productum $\sin.\Phi^m \cos.\Phi^n$ in sinus vel cosinus angulorum multiplorum resoluere, quo facto singulae partes facillime integrantur. Caeterum hic breuitatis gratia angulum simpliciter littera Φ indicaui, nihiloque res foret generalior, si per $\alpha\Phi+\beta$ exprimeretur, quemadmodum etiam ante haec expressio Ang. sin. x aeque late patet, ac sit loco x, functio quaecunque scriberetur. Contemplemur

plemur ergo eiusmodi formulas, in quibus sinus cosinusue denominatorem occupant, vbi quidem simplicissimae sunt

$$\text{I. } \frac{d\Phi}{\sin.\Phi}; \quad \text{II. } \frac{d\Phi}{\cos.\Phi}; \quad \text{III. } \frac{d\Phi \cos.\Phi}{\sin.\Phi}; \quad \text{IV. } \frac{d\Phi \sin.\Phi}{\cos.\Phi}$$

quarum integralia imprimis nosse oportet. Pro prima adhibeantur hae transformationes

$$\frac{d\Phi}{\sin.\Phi} = \frac{d\Phi \sin.\Phi}{\sin.\Phi^2} = \frac{d\Phi \sin.\Phi}{1-\cos.\Phi^2} = \frac{-dx}{1-xx} \text{ (posito } \cos.\Phi = x\text{)}$$

vnde fit

$$\int \frac{d\Phi}{\sin.\Phi} = -\tfrac{1}{2} l \frac{1+x}{1-x} = -\tfrac{1}{2} l \frac{1+\cos.\Phi}{1-\cos.\Phi}.$$

Pro secunda

$$\frac{d\Phi}{\cos.\Phi} = \frac{d\Phi \cos.\Phi}{\cos.\Phi^2} = \frac{d\Phi \cos.\Phi}{1-\sin.\Phi^2} = \frac{dx}{1-xx} \text{ (posito } \sin.\Phi = x\text{)}$$

ergo

$$\int \frac{d\Phi}{\cos.\Phi} = \tfrac{1}{2} l \frac{1+x}{1-x} = \tfrac{1}{2} l \frac{1+\sin.\Phi}{1-\sin.\Phi}.$$

Tertiae et quartae integratio manifesto logarithmis conficitur: quare haec integralia probe notasse iuvabit

$$\text{I. } \int \frac{d\Phi}{\sin.\Phi} = -\tfrac{1}{2} l \frac{1+\cos.\Phi}{1-\cos.\Phi} = l \frac{\sqrt{(1-\cos.\Phi)}}{\sqrt{(1+\cos.\Phi)}} = l \operatorname{tang}.\tfrac{1}{2}\Phi$$

$$\text{II. } \int \frac{d\Phi}{\cos.\Phi} = \tfrac{1}{2} l \frac{1+\sin.\Phi}{1-\sin.\Phi} = l \frac{\sqrt{(1+\sin.\Phi)}}{\sqrt{(1-\sin.\Phi)}} = l \operatorname{tang}.(45^\circ + \tfrac{1}{2}\Phi)$$

$$\text{III. } \int \frac{d\Phi \cos.\Phi}{\sin.\Phi} = l \sin.\Phi = \int \frac{d\Phi}{\operatorname{tang}.\Phi} = \int d\Phi \cot.\Phi$$

$$\text{IV. } \int \frac{d\Phi \sin.\Phi}{\cos.\Phi} = -l \cos.\Phi = \int d\Phi \operatorname{tang}.\Phi$$

hincque sequitur III.+IV.

$$\int \frac{d\Phi}{\sin.\Phi \cos.\Phi} = l \frac{\sin.\Phi}{\cos.\Phi} = l \operatorname{tang}.\Phi.$$

Proble-

Problema 27.

249 Formularum $\frac{d\Phi \sin.\Phi^m}{\cos.\Phi^n}$ et $\frac{d\Phi \cos.\Phi^m}{\sin.\Phi^n}$ integralia inueſtigare.

Solutio.

Primo ſtatim perſpicitur, alteram formulam in alteram transmutari poſito $\Phi = 90^\circ - \psi$, quia tum fit $\sin.\Phi = \cos.\psi$ et $\cos.\Phi = + \sin.\psi$, dummodo notetur fore $d\Phi = -d\psi$. Quare ſufficit priorem tantum tractaſſe. Reductio autem prior §. 246. data ſumto $\mu + 1 = m$ et $\nu - 1 = -n$ praebet

$$\int \frac{d\Phi \sin.\Phi^m}{\cos.\Phi^n} = -\frac{1}{m-n} \cdot \frac{\sin.\Phi^{m-1}}{\cos.\Phi^{n-1}} + \frac{m-1}{m-n} \int \frac{d\Phi \sin.\Phi^{m-2}}{\cos.\Phi^n}$$

quo pacto in numeratore exponens ipſius $\sin.\Phi$ continuo binario deprimitur, ita vt tandem perueniatur vel ad $\int \frac{d\Phi}{\cos.\Phi^n}$ vel ad $\int \frac{d\Phi \sin.\Phi}{\cos.\Phi^n} = \frac{1}{(n-1)\cos.\Phi^{n-1}}$ ideoque ſola formula $\int \frac{d\Phi}{\cos.\Phi^n}$ tractanda ſuperſit. Altera autem reductio ibidem tradita (246.) ſumto $\mu - 1 = m$ et $\nu - 1 = -n$ dat

$$\int \frac{d\Phi \sin.\Phi^m}{\cos.\Phi^{n-2}} = \frac{1}{m-n+2} \cdot \frac{\sin.\Phi^{m+1}}{\cos.\Phi^{n-1}} - \frac{(n-1)}{m-n+2} \int \frac{d\Phi \sin.\Phi^m}{\cos.\Phi^n}$$

vnde colligitur

$$\int \frac{d\Phi \sin.\Phi^m}{\cos.\Phi^n} = \frac{1}{n-1} \cdot \frac{\sin.\Phi^{m+1}}{\cos.\Phi^{n-1}} - \frac{(m-n+2)}{n-1} \int \frac{d\Phi \sin.\Phi^m}{\cos.\Phi^{n-2}}$$

 cuius

cuius reductionis ope exponens ipsius cof. Φ in denominatore continuo binario deprimitur, ita vt tandem vel ad $\int d\Phi \sin.\Phi^m$ vel ad $\int \frac{d\Phi \sin \Phi^m}{\cos.\Phi}$ perueniatur. Illius integratio iam supra est monstrata, huius vero si $m > 1$ per priorem reductionem forma tandem vel ad $\int \frac{d\Phi}{\cos.\Phi}$ vel ad $\int \frac{d\Phi \sin.\Phi}{\cos.\Phi}$ reuocatur, illius autem integrale est $l\,\mathrm{tang.}(45° + \frac{1}{2}\Phi)$ huius vero $-l\cos.\Phi$.

Coroll. 1.

250. Prior reductio non habet locum, quoties est $m = n$, hoc scilicet casu formula $\int \frac{d\Phi \sin.\Phi^n}{\cos.\Phi^n}$ non reduci potest ad formulam $\int \frac{d\Phi \sin \Phi^{n-2}}{\cos.\Phi^n}$. Altera autem reductione semper vti licet, etsi enim casus $n = 1$ inde excluditur, eius tamen integratio per priorem effici potest.

Coroll. 2.

251. Ratio autem illius exclusionis in hoc est posita, quod formula $\int \frac{d\Phi \sin.\Phi^{n-2}}{\cos.\Phi^n}$ est absolute integrabilis, habens integrale $= \frac{1}{n-1} \cdot \frac{\sin.\Phi^{n-1}}{\cos.\Phi^{n-1}}$. Erit

ergo

ergo pro his casibus:

$$\int\frac{d\Phi}{\cos.\Phi^2}=\frac{\sin.\Phi}{\cos.\Phi}=\operatorname{tang}.\Phi;\quad \int\frac{d\Phi\sin.\Phi}{\cos.\Phi^3}=\frac{1}{2}\cdot\frac{\sin.\Phi^2}{\cos.\Phi^2}=\frac{1}{2}\operatorname{tang}.\Phi^2$$

$$\int\frac{d\Phi\sin.\Phi^2}{\cos.\Phi^4}=\frac{1}{3}\cdot\frac{\sin.\Phi^3}{\cos.\Phi^3}=\frac{1}{3}\operatorname{tang}.\Phi^3;\quad \int\frac{d\Phi\sin.\Phi^3}{\cos.\Phi^5}=\frac{1}{4}\cdot\frac{\sin.\Phi^4}{\cos.\Phi^4}=\frac{1}{4}\operatorname{tang}.\Phi^4.$$

Exemplum 1.

252. *Formulae* $\dfrac{d\Phi\sin.\Phi^m}{\cos.\Phi}$ *integrale assignare.*

Prior reductio dat:

$$\int\frac{d\Phi\sin.\Phi^m}{\cos\Phi}=\frac{-1}{m-1}\sin.\Phi^{m-1}+\int\frac{d\Phi\sin.\Phi^{m-2}}{\cos.\Phi}.$$

Hinc a casibus per se notis incipiendo habebimus:

$$\int\frac{d\Phi}{\cos.\Phi}=l\operatorname{tang}.(45^\circ+\tfrac{1}{2}\Phi)$$

$$\int\frac{d\Phi\sin.\Phi}{\cos.\Phi}=-l\cos.\Phi=l\sec.\Phi$$

$$\int\frac{d\Phi\sin.\Phi^2}{\cos.\Phi}=-\sin.\Phi+\int\frac{d\Phi}{\cos.\Phi}$$

$$\int\frac{d\Phi\sin.\Phi^3}{\cos.\Phi}=-\tfrac{1}{2}\sin.\Phi^2+l\sec.\Phi$$

$$\int\frac{d\Phi\sin.\Phi^4}{\cos.\Phi}=-\tfrac{1}{3}\sin.\Phi^3-\sin\Phi+\int\frac{d\Phi}{\cos.\Phi}$$

$$\int\frac{d\Phi\sin.\Phi^5}{\cos.\Phi}=-\tfrac{1}{4}\sin.\Phi^4-\tfrac{1}{2}\sin.\Phi^2+l\sec.\Phi$$

$$\int\frac{d\Phi\sin.\Phi^6}{\cos.\Phi}=-\tfrac{1}{5}\sin.\Phi^5-\tfrac{1}{3}\sin.\Phi^3-\sin.\Phi+\int\frac{d\Phi}{\cos.\Phi}$$

$$\int\frac{d\Phi\sin.\Phi^7}{\cos.\Phi}=-\tfrac{1}{6}\sin.\Phi^6-\tfrac{1}{4}\sin.\Phi^4-\tfrac{1}{2}\sin.\Phi^2+l\sec.\Phi$$

etc.

Scholion.

Scholion.

253. Pro reliquis casibus denominatoris totum negotium conficietur his reductionibus:

$$\int \frac{d\Phi \,\text{fin.}\,\Phi^m}{\text{cof.}\,\Phi^2} = \frac{\text{fin.}\,\Phi^{m+1}}{\text{cof.}\,\Phi} - m\int d\Phi \,\text{fin.}\,\Phi^m$$

$$\int \frac{d\Phi \,\text{fin.}\,\Phi^m}{\text{cof.}\,\Phi^3} = \frac{1}{2}\cdot\frac{\text{fin.}\,\Phi^{m+1}}{\text{cof.}\,\Phi^2} - \frac{(m-1)}{2}\int \frac{d\Phi \,\text{fin.}\,\Phi^m}{\text{cof.}\,\Phi}$$

$$\int \frac{d\Phi \,\text{fin.}\,\Phi^m}{\text{cof.}\,\Phi^4} = \frac{1}{3}\cdot\frac{\text{fin}\,\Phi^{m+1}}{\text{cof.}\,\Phi^3} - \frac{(m-2)}{3}\int \frac{d\Phi \,\text{fin}\,\Phi^m}{\text{cof.}\,\Phi^2}$$

$$\int \frac{d\Phi \,\text{fin.}\,\Phi^m}{\text{cof.}\,\Phi^5} = \frac{1}{4}\cdot\frac{\text{fin.}\,\Phi^{m+1}}{\text{cof.}\,\Phi^4} - \frac{(m-3)}{4}\int \frac{d\Phi \,\text{fin}\,\Phi^m}{\text{cof.}\,\Phi^3}$$

etc.

Exemplum 2.

254. *Formulae* $\frac{d\Phi}{\text{cof.}\,\Phi^n}$ *integrale assignare.*

Altera reductio ob $m=0$ fit

$$\int \frac{d\Phi}{\text{cof.}\,\Phi^n} = \frac{1}{n-1}\cdot\frac{\text{fin.}\,\Phi}{\text{cof.}\,\Phi^{n-1}} + \frac{n-2}{n-1}\int \frac{d\Phi}{\text{cof.}\,\Phi^{n-2}}$$

quia iam casus simplicissimi $\int d\Phi = \Phi$ et $\int \frac{d\Phi}{\text{cof.}\,\Phi} = l\,\text{tang.}(45^\circ + \frac{1}{2}\Phi)$ sunt cogniti, ad eos sequentes omnes reuocabuntur:

$$\int \frac{d\Phi}{\text{cof.}\,\Phi^2} = \frac{\text{fin.}\,\Phi}{\text{cof.}\,\Phi}$$

$$\int \frac{d\Phi}{\text{cof.}\,\Phi^3} = \frac{1}{2}\cdot\frac{\text{fin.}\,\Phi}{\text{cof.}\,\Phi^2} + \frac{1}{2}\int \frac{d\Phi}{\text{cof.}\,\Phi}$$

$$\int \frac{d\Phi}{\text{cof.}\,\Phi^4} = \frac{1}{3}\cdot\frac{\text{fin.}\,\Phi}{\text{cof.}\,\Phi^3} + \frac{2}{3}\cdot\frac{\text{fin.}\,\Phi}{\text{cof.}\,\Phi}$$

$$\int \frac{d\Phi}{\text{cof.}\,\Phi^5} = \frac{1}{4}\cdot\frac{\text{fin.}\,\Phi}{\text{cof.}\,\Phi^4} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{\text{fin.}\,\Phi}{\text{cof.}\,\Phi^2} + \frac{1\cdot 3}{2\cdot 4}\int \frac{d\Phi}{\text{cof}\,\Phi}$$

$$\int \frac{d\Phi}{\text{cof.}\,\Phi^6} = \frac{1}{5}\cdot\frac{\text{fin.}\,\Phi}{\text{cof.}\,\Phi^5} + \frac{1\cdot 4}{3\cdot 5}\cdot\frac{\text{fin.}\,\Phi}{\text{cof.}\,\Phi^3} + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{\text{fin.}\,\Phi}{\text{cof.}\,\Phi}.$$

Coroll. 1.

Coroll. 1.

255. Simili modo habebimus has integrationes:

$$\int \frac{d\Phi}{\sin.\Phi} = l\,\mathrm{tang}.\tfrac{1}{2}\Phi\,;\quad \int \frac{d\Phi}{\sin.\Phi^2} = -\frac{\cos.\Phi}{\sin.\Phi}\,;$$

$$\int \frac{d\Phi}{\sin.\Phi^3} = -\frac{1}{2}\cdot\frac{\cos.\Phi}{\sin.\Phi^2} + \frac{1}{2}\int \frac{d\Phi}{\sin.\Phi}$$

$$\int \frac{d\Phi}{\sin.\Phi^4} = -\frac{1}{3}\cdot\frac{\cos.\Phi}{\sin.\Phi^3} - \frac{2}{3}\cdot\frac{\cos.\Phi}{\sin.\Phi}$$

$$\int \frac{d\Phi}{\sin.\Phi^5} = -\frac{1}{4}\cdot\frac{\cos.\Phi}{\sin.\Phi^4} - \frac{1.3}{2.4}\cdot\frac{\cos.\Phi}{\sin.\Phi^2} + \frac{1.3}{2.4}\int \frac{d\Phi}{\sin.\Phi} \text{ etc.}$$

Coroll. 2.

256. Deinde est

$$\int \frac{d\Phi \sin.\Phi}{\cos.\Phi^n} = \frac{1}{n-1}\cdot\frac{1}{\cos.\Phi^{n-1}}\,;\ \text{et}\ \int \frac{d\Phi \cos.\Phi}{\sin.\Phi^n} = \frac{-1}{n-1}\cdot\frac{1}{\sin.\Phi^{n-1}}$$

Porro

$$\int \frac{d\Phi \sin.\Phi^2}{\cos.\Phi^n} = \int \frac{d\Phi}{\cos.\Phi^n} - \int \frac{d\Phi}{\cos.\Phi^{n-2}}\,;$$

$$\int \frac{d\Phi \cos.\Phi^2}{\sin.\Phi^n} = \int \frac{d\Phi}{\sin.\Phi^n} - \int \frac{d\Phi}{\sin.\Phi^{n-2}}$$

$$\text{et}\ \int \frac{d\Phi \sin.\Phi^3}{\cos.\Phi^n} = \int \frac{d\Phi \sin.\Phi}{\cos.\Phi^n} - \int \frac{d\Phi \sin.\Phi}{\cos.\Phi^{n-2}}\,;$$

$$\int \frac{d\Phi \cos.\Phi^3}{\sin.\Phi^n} = \int \frac{d\Phi \cos.\Phi}{\sin.\Phi^n} - \int \frac{d\Phi \cos.\Phi}{\sin.\Phi^{n-2}}$$

quibus reductionibus continuo vlterius progredi licet.

Problema 28.

257. Formulae $\frac{d\Phi}{\sin.\Phi^m \cos.\Phi^n}$ integrale inuestigare.

Solutio.

Reductiones supra adhibitas huc accommodare licet, sumendo in praecendente problemate m negatiue: ita erit

$$\int \frac{d\Phi}{\sin.\Phi^m \cos.\Phi^n} = + \frac{1}{m+n} \cdot \frac{1}{\sin.\Phi^{m+1} \cos.\Phi^{n-1}} + \frac{m+1}{m+n} \int \frac{d\Phi}{\sin.\Phi^{m+2} \cos.\Phi^n}$$

vnde loco m scribendo $m-2$ per conuersionem fit

$$\int \frac{d\Phi}{\sin.\Phi^m \cos.\Phi^n} = - \frac{1}{m-1} \cdot \frac{1}{\sin.\Phi^{m-1} \cos.\Phi^{n-1}} + \frac{m+n-2}{m-1} \int \frac{d\Phi}{\sin.\Phi^{m-2} \cos.\Phi^n}$$

altera huic similis est

$$\int \frac{d\Phi}{\sin.\Phi^m \cos.\Phi^n} = \frac{1}{n-1} \cdot \frac{1}{\sin.\Phi^{m-1} \cos.\Phi^{n-1}} + \frac{m+n-2}{n-1} \int \frac{d\Phi}{\sin.\Phi^m \cos.\Phi^{n-2}}.$$

Cum iam in hoc genere formae simplicissimae sint:

$$\int \frac{d\Phi}{\sin.\Phi} = l\,\text{tang}\,\tfrac{1}{2}\Phi;\ \int \frac{d\Phi}{\cos.\Phi} = l\,\text{tang.}(45^\circ + \tfrac{1}{2}\Phi);\ \int \frac{d\Phi}{\sin.\Phi \cos.\Phi} = l\,\text{tan.}\Phi;$$

$$\int \frac{d\Phi}{\sin.\Phi^2} = -\cot.\Phi;\ \int \frac{d\Phi}{\cos.\Phi^2} = \text{tang.}\Phi$$

hinc magis compositas eliciemus:

$$\int \frac{d\Phi}{\sin.\Phi \cos.\Phi^2} = \frac{1}{\cos.\Phi} + \int \frac{d\Phi}{\sin.\Phi};\ \int \frac{d\Phi}{\sin.\Phi^2 \cos.\Phi} = -\frac{1}{\sin.\Phi} + \int \frac{d\Phi}{\cos.\Phi}$$

$$\int \frac{d\Phi}{\sin.\Phi \cos.\Phi^4} = \frac{1}{3} \cdot \frac{1}{\cos.\Phi^3} + \int \frac{d\Phi}{\sin.\Phi \cos.\Phi^2};\ \int \frac{d\Phi}{\sin.\Phi^4 \cos.\Phi} = -\frac{1}{3} \cdot \frac{1}{\sin.\Phi^3} + \int \frac{d\Phi}{\sin.\Phi^2 \cos.\Phi}$$

$\int$

$$\int\frac{d\Phi}{\sin.\Phi\cos.\Phi^6}=\frac{1}{5}\cdot\frac{1}{\cos.\Phi^5}+\int\frac{d\Phi}{\sin.\Phi\cos.\Phi^4};\quad\int\frac{d\Phi}{\sin.\Phi^6\cos.\Phi}=-\frac{1}{5}\cdot\frac{1}{\sin.\Phi^5}+\frac{d\Phi}{\sin.\Phi^4\cos.\Phi}$$

$$\int\frac{d\Phi}{\sin.\Phi\cos.\Phi^3}=\frac{1}{2}\cdot\frac{1}{\cos.\Phi^2}+\int\frac{d\Phi}{\sin.\Phi\cos.\Phi};\quad\int\frac{d\Phi}{\sin.\Phi^3\cos.\Phi}=-\frac{1}{2}\cdot\frac{1}{\sin.\Phi^2}+\int\frac{d\Phi}{\sin.\Phi\cos.\Phi}$$

$$\int\frac{d\Phi}{\sin.\Phi\cos.\Phi^5}=\frac{1}{4}\cdot\frac{1}{\cos.\Phi^4}+\int\frac{d\Phi}{\sin.\Phi\cos.\Phi^3};\quad\int\frac{d\Phi}{\sin.\Phi^5\cos.\Phi}=-\frac{1}{4}\cdot\frac{1}{\sin.\Phi^4}+\int\frac{d\Phi}{\sin.\Phi^5\cos.\Phi}$$

$$\int\frac{d\Phi}{\sin.\Phi\cos.\Phi^7}=\frac{1}{6}\cdot\frac{1}{\cos.\Phi^6}+\int\frac{d\Phi}{\sin.\Phi\cos.\Phi^5};\quad\int\frac{d\Phi}{\sin.\Phi^7\cos.\Phi}=-\frac{1}{6}\cdot\frac{1}{\sin.\Phi^6}+\int\frac{d\Phi}{\sin.\Phi^5\cos.\Phi}$$

etc.

$$\int\frac{d\Phi}{\sin.\Phi^2\cos.\Phi^2}=\frac{1}{\sin.\Phi\cos.\Phi}+2\int\frac{d\Phi}{\sin.\Phi^2}=-\frac{1}{\sin.\Phi\cos.\Phi}+2\int\frac{d\Phi}{\cos.\Phi^2}$$

$$\int\frac{d\Phi}{\sin.\Phi^2\cos.\Phi^4}=\frac{1}{3}\cdot\frac{1}{\sin.\Phi\cos.\Phi^3}+\frac{4}{3}\int\frac{d\Phi}{\sin.\Phi^2\cos.\Phi^2}$$

$$\int\frac{d\Phi}{\sin.\Phi^4\cos.\Phi^2}=-\frac{1}{3}\cdot\frac{1}{\sin.\Phi^3\cos.\Phi}+\frac{4}{3}\int\frac{d\Phi}{\sin.\Phi^2\cos.\Phi^2}.$$

Sicque formulae quantumuis compofitae ad fimpliciores, quarum integratio eft in promtu, reducuntur.

Coroll. 1.

258. Ambo exponentes ipfius fin. Φ et cof. Φ fimul binario minui poffunt: erit enim per priorem reductionem

$$\int\frac{d\Phi}{\sin.\Phi^{\mu}\cos.\Phi^{\nu}}=-\frac{1}{\mu-1}\cdot\frac{1}{\sin\Phi^{\mu-1}\cos.\Phi^{\nu-1}}+\frac{\mu+\nu-2}{\mu-1}\int\frac{d\Phi}{\sin.\Phi^{\mu-2}\cos.\Phi^{\nu}}$$

X 2 nunc

nunc haec formula per posteriorem ob $m = \mu - 2$ et $n = \nu$ dat

$$\int \frac{d\Phi}{\text{sin}\ \Phi^{\mu-2}\ \text{cos}.\ \Phi^{\nu}} = \frac{1}{\nu - 1} \cdot \frac{1}{\text{sin}.\ \Phi^{\mu-3}\ \text{cos}.\ \Phi^{\nu-1}} + \frac{\mu+\nu-4}{\nu-1} \int \frac{d\Phi}{\text{sin}.\ \Phi^{\mu-2}\ \text{cos}.\ \Phi^{\nu-2}}$$

vnde concluditur:

$$\int \frac{d\Phi}{\text{sin}.\Phi^{\mu}\text{cos}.\Phi^{\nu}} = -\frac{1}{\mu-1} \cdot \frac{1}{\text{sin}.\Phi^{\mu-1}\text{cos}.\Phi^{\nu-1}} + \frac{\mu+\nu-2}{(\mu-1)(\nu-1)} \cdot \frac{1}{\text{sin}\ \Phi^{\mu-3}\text{cos}.\Phi^{\nu-1}} + \frac{(\mu+\nu-2)(\mu+\nu-4)}{(\mu-1)(\nu-1)} \int \frac{d\Phi}{\text{sin}.\Phi^{\mu-2}\text{cos}.\Phi^{\nu-2}}.$$

Coroll. 2.

259. Prioribus membris ad communem denominatorem reductis obtinebitur

$$\int \frac{d\Phi}{\text{sin}.\ \Phi^{\mu}\ \text{cos}.\ \Phi^{\nu}} = \frac{(\mu - 1)\ \text{sin}.\ \Phi^2 - (\nu - 1)\ \text{cos}.\ \Phi^2}{(\mu - 1)(\nu - 1)\ \text{sin}.\ \Phi^{\mu-1}\ \text{cos}.\ \Phi^{\nu-1}} + \frac{(\mu+\nu-2)(\mu+\nu-4)}{(\mu-1)(\nu-1)} \int \frac{d\Phi}{\text{sin}.\Phi^{\mu-2}\text{cos}.\Phi^{\nu-2}}$$

qua reductione semper ad calculum contrahendum vti licet nisi vel $\mu = 1$ vel $\nu = 1$.

Scholion.

260. Huiusmodi formulae $\frac{d\Phi}{\text{sin}.\ \Phi^{m}\ \text{cos}\ \Phi^{n}}$ etiam hoc modo maxime obuio ad simpliciores reduci possunt; dum numerator per $\text{sin}.\Phi^2 + \text{cos}.\Phi^2 = 1$ multiplicatur, vnde fit

$$\int \frac{d\Phi}{\text{sin}.\ \Phi^{m}\ \text{cos}.\ \Phi^{n}} = \int \frac{d\Phi}{\text{sin}.\ \Phi^{m-2}\ \text{cos}.\ \Phi^{n}} + \int \frac{d\Phi}{\text{sin}.\ \Phi^{m}\ \text{cos}\ \Phi^{n-2}}$$

quae

quae eousque continuari poteſt, donec in denominatore vnica tantum poteſtas relinquatur. Ita erit

$$\int \frac{d\Phi}{\sin.\Phi \cos.\Phi} = \int \frac{d\Phi \sin.\Phi}{\cos.\Phi} + \int \frac{d\Phi \cos.\Phi}{\sin.\Phi} = l\frac{\sin.\Phi}{\cos.\Phi}$$

$$\int \frac{d\Phi}{\sin.\Phi^2 \cos.\Phi^2} = \int \frac{d\Phi}{\sin.\Phi^2} + \int \frac{d\Phi}{\cos.\Phi^2} = \frac{\sin.\Phi}{\cos.\Phi} - \frac{\cos.\Phi}{\sin.\Phi}.$$

Quodſi propoſita ſit haec formula $\int \frac{d\Phi}{\sin \Phi^n \cos.\Phi^n}$, in ſubſidium vocari poteſt, eſſe $\sin.\Phi\cos.\Phi = \frac{1}{2}\sin.2\Phi$, vnde habetur $\int \frac{2^n d\Phi}{\sin.2\Phi^n} = 2^{n-1}\int \frac{d\omega}{\sin.\omega^n}$ poſito $\omega = 2\Phi$, quae formula per ſuperiora praecepta reſoluitur. His igitur adminiculis obſeruatis circa formulam $d\Phi \sin.\Phi^m \cos.\Phi^n$, ſi quidem m et n fuerint numeri integri ſiue poſitiui ſiue negatiui, nihil amplius deſideratur: ſin autem fuerint numeri fracti, nihil admodum praecipiendum occurrit, quandoquidem caſus, quibus integratio ſuccedit, quaſi ſponte ſe produnt. Quemadmodum autem integralia, quae exhiberi nequeunt, per ſeries exprimi conueniat, in capite ſequente accuratius exponamus. Nunc vero formulas fractas conſideremus, quarum denominator eſt $a + b\cos.\Phi$ eiusque poteſtas, tales enim formulae in Theoria Aſtronomiae frequentiſſime occurrunt.

Problema 29.

261. Formulae differentialis $\frac{d\Phi}{a + b\cos.\Phi}$ integrale inueſtigare.

Solutio.

Haec inueſtigatio commodius inſtitui nequit, quam vt formula propoſita ad formam ordinariam reducatur, ponendo $\cos.\Phi = \frac{1-xx}{1+xx}$ vt rationaliter fiat $\sin.\Phi = \frac{2x}{1+xx}$, hincque $d\Phi \cos.\Phi = \frac{2dx(1-xx)}{(1+xx)^2}$, ſicque $d\Phi = \frac{2dx}{1+xx}$. Quia igitur $a+b\cos.\Phi = \frac{a+b+(a-b)xx}{1+xx}$ erit formula noſtra $\frac{d\Phi}{a+b\cos.\Phi} = \frac{2dx}{a+b+(a-b)xx}$ quae prout fuerit $a>b$ vel $a<b$, vel angulum vel logarithmum praebet.

Caſu $a<b$ reperitur

$$\int \frac{d\Phi}{a+b\cos.\Phi} = \frac{2}{\sqrt{(aa-bb)}} \text{Arc. tang.} \frac{(a-b)x}{\sqrt{(aa-bb)}}$$

caſu $a>b$ vero eſt

$$\int \frac{d\Phi}{a+b\cos.\Phi} = \frac{1}{\sqrt{(bb-aa)}}\, l \frac{\sqrt{(bb-aa)}+x(b-a)}{\sqrt{(bb-aa)}-x(b-a)}.$$

Nunc vero eſt $x = \sqrt{\frac{1-\cos.\Phi}{1+\cos.\Phi}} = \text{tang.}\tfrac{1}{2}\Phi = \frac{\sin\Phi}{1+\cos.\Phi}$; qua reſtitutione facta, cum ſit $2\,\text{Ang. tang.} \frac{(a-b)x}{\sqrt{(aa-bb)}}$ $= \text{Ang. tang.} \frac{2x\sqrt{(aa-bb)}}{a+b-(a-b)xx} = \text{Ang. tang.} \frac{2\sin.\Phi.\sqrt{(aa-bb)}}{(a+b)(1+\cos.\Phi)-(a-b)(1-\cos.\Phi)}$ $= \text{Ang. tang.} \frac{\sin.\Phi.\sqrt{(aa-bb)}}{a\cos.\Phi+b}$.

Quocirca pro caſu $a>b$ adipiſcimur:

$$\int \frac{d\Phi}{a+b\cos.\Phi} = \frac{1}{\sqrt{(aa-bb)}} \text{Ang. tang.} \frac{\sin.\Phi.\sqrt{(aa-bb)}}{a\cos.\Phi+b} \text{ ſeu}$$

$$\int \frac{d\Phi}{a+b\cos.\Phi} = \frac{1}{\sqrt{(aa-bb)}} \text{Ang. ſin.} \frac{\sin.\Phi.\sqrt{(aa-bb)}}{a+b\cos.\Phi} \text{ ſiue}$$

$$\int \frac{d\Phi}{a+b\cos.\Phi} = \frac{1}{\sqrt{(aa-bb)}} \text{Ang. ſin.} \frac{a\cos.\Phi+b}{a+b\cos.\Phi}.$$

Pro caſu autem $a<b$:

$$\int \frac{d\Phi}{a+b\cos.\Phi} = \frac{1}{\sqrt{(bb-aa)}}\, l \frac{\sqrt{(b+a)(1+\cos.\Phi)}+\sqrt{(b-a)(1-\cos.\Phi)}}{\sqrt{(b+a)(1+\cos.\Phi)}-\sqrt{(b-a)(1-\cos.\Phi)}}$$

ſeu

ſeu

$$\int \frac{d\Phi}{c + b\,\text{cof.}\,\Phi} = \frac{1}{\sqrt{(bb - aa)}}\, l\, \frac{a\,\text{cof.}\,\Phi + b + \text{ſin.}\,\Phi \cdot \sqrt{(bb - aa)}}{a + b\,\text{cof.}\,\Phi}.$$

At caſu $b = a$, integrale eſt $= \frac{x}{a} = \frac{1}{a}\text{tang.}\frac{1}{2}\Phi$, vnde fit

$$\int \frac{d\Phi}{1 + \text{cof.}\,\Phi} = \text{tang.}\,\tfrac{1}{2}\Phi = \frac{\text{ſin.}\,\Phi}{1 + \text{cof.}\,\Phi}$$

quae integralia euaneſcunt facto $\Phi = 0$.

Coroll. 1.

162. Formulae autem $\frac{d\Phi\,\text{ſin.}\,\Phi}{a + b\,\text{cof.}\,\Phi} = \frac{-d.\,\text{cof.}\,\Phi}{a + b\,\text{cof.}\,\Phi}$ integrale eſt $\frac{1}{b} l \frac{a + b}{a + b\,\text{cof.}\,\Phi}$, ita ſumtum, vt euaneſcat poſito $\Phi = 0$; ſicque habebimus:

$$\int \frac{d\Phi\,\text{ſin.}\,\Phi}{a + b\,\text{cof.}\,\Phi} = \frac{1}{b} l \frac{a + b}{a + b\,\text{cof.}\,\Phi}.$$

Coroll. 2.

263. Formula autem $\frac{d\Phi\,\text{cof.}\,\Phi}{a + b\,\text{cof.}\,\Phi}$ transformatur in $\frac{d\Phi}{b} - \frac{a\,d\Phi}{b(a + b\,\text{cof.}\,\Phi)}$, vnde integrale per ſolutionem problematis exhiberi poteſt:

$$\int \frac{d\Phi\,\text{cof.}\,\Phi}{a + b\,\text{cof.}\,\Phi} = \frac{\Phi}{b} - \frac{a}{b} \int \frac{d\Phi}{a + b\,\text{cof.}\,\Phi}.$$

Scholion 1.

264. Integratione hac inuenta, etiam huius formulae $\frac{d\Phi}{(a + b\,\text{coſ.}\,\Phi)^n}$ integrale inueniri poteſt, exiſtente n numero integro; quod fingendo integralis forma commodiſſime praeſtari videtur:

$$\int \frac{d\Phi}{(a + b\,\text{cof.}\,\Phi)^2} = \frac{A\,\text{ſin.}\,\Phi}{a + b\,\text{cof.}\,\Phi} + m \int \frac{d\Phi}{a + b\,\text{cof.}\,\Phi}.$$

2C

ac reperitur $A=\frac{-b}{aa-bb}$; et $m=\frac{a}{aa+bb}$

$$\int\frac{d\Phi}{(a+b\,\mathrm{cof.}\Phi)^3}=\frac{(A+B\,\mathrm{cof.}\Phi)\,\mathrm{fin.}\Phi}{(a+b\,\mathrm{cof.}\Phi)^2}+m\int\frac{d\Phi}{(a+b\,\mathrm{cof.}\Phi)}$$

reperiturque

$$A=\frac{-b}{aa-bb};\quad B=\frac{-ab}{2a(aa-bb)};\quad m=\frac{2aa+bb}{2a(aa-bb)}$$

fimilique modo inueftigatio ad maiores poteftates continuari poteft, labore quidem non parum taediofo. Sequenti autem modo negotium facillime expediri videtur.

Confideretur fcilicet formula generalior $\frac{d\Phi(f+g\,\mathrm{cof.}\Phi)}{(a+b\,\mathrm{cof.}\Phi)^{n+1}}$

ac ponatur:

$$\int\frac{d\Phi(f+g\,\mathrm{cof.}\Phi)}{(a+b\,\mathrm{cof.}\Phi)^{n+1}}=\frac{A\,\mathrm{fin.}\Phi}{(a+b\,\mathrm{cof.}\Phi)^n}+\int\frac{d\Phi(B+C\,\mathrm{cof.}\Phi)}{(a+b\,\mathrm{cof.}\Phi)^n}$$

fumtisque differentialibus, ifta prodibit aequatio:

$$f+g\,\mathrm{cof.}\Phi=A\,\mathrm{cof.}\Phi(a+b\,\mathrm{cof}\,\Phi)+nAb\,\mathrm{fin}\,\Phi^2$$
$$+(B+C\,\mathrm{cof.}\Phi)(a+b\,\mathrm{cof.}\Phi)$$

quae ob $\mathrm{fin.}\Phi^2=1-\mathrm{cof.}\Phi^2$ hanc formam induit

$$\left.\begin{array}{lll}-f & -g\,\mathrm{cof.}\Phi & +Ab\,\mathrm{cof.}\Phi^2\\ +nAb & +Aa\,\mathrm{cof.}\Phi & -nAb\,\mathrm{cof.}\Phi^2\\ +Ba & +Bb\,\mathrm{cof}\,\Phi & +Cb\,\mathrm{cof.}\Phi^2\\ & +Ca\,\mathrm{cof.}\Phi & \end{array}\right\}=0$$

vnde fingulis membris nihilo aequatis, elicitur:

$$A=\frac{ag-bf}{n(aa-bb)};\quad B=\frac{af+bg}{aa-bb}\ \text{et}\ C=\frac{(n-1)(ag-bf)}{n(aa-bb)}$$

ita vt haec obtineatur reductio:

$$\int\frac{d\Phi(f+g\,\mathrm{cof.}\Phi)}{(a+b\,\mathrm{cof.}\Phi)^{n+1}}=\frac{(ag-bf)\,\mathrm{fin.}\Phi}{n(aa-bb)(a+b\,\mathrm{cof}\,\Phi)^n}$$
$$+\frac{1}{n(aa-bb)}\int\frac{d\Phi(n(af-bg)+(n-1)(ag-bf)\mathrm{cof.}\Phi)}{(a+b\,\mathrm{cof.}\Phi)^n}$$

cuius

cuius ope tandem ad formulam $\int \frac{d\Phi(b+k\,\text{cof.}\,\Phi)}{a+b\,\text{cof.}\,\Phi}$ peruenitur, cuius integrale $= \frac{k}{b}\Phi + \frac{bb-ak}{b}\int \frac{d\Phi}{a+b\,\text{cof.}\,\Phi}$ ex superioribus constat. Perspicuum autem est semper fore $k = 0$.

Scholion 2.

265. Occurrunt etiam eiusmodi formulae, in quas insuper quantitas exponentialis $e^{\alpha\Phi}$ angulum ipsum Φ in exponente gerens, ingreditur, quas quomodo tractari oporteat, ostendendum videtur, cum hinc methodus reductionum supra exposita maxime illustretur. Hic enim per illam reductionem ad formulam propositae similem peruenitur, vnde ipsum integrale colligi poterit. In hunc finem notetur esse $\int e^{\alpha\Phi} d\Phi = \frac{1}{\alpha} e^{\alpha\Phi}$.

Problema 30.

266. Formulae differentialis $dy = e^{\alpha\Phi} d\Phi\,\text{fin.}\,\Phi^n$ integrale inuestigare.

Solutio.

Sumto $e^{\alpha\Phi} d\Phi$ pro factore differentiali, erit

$$y = \frac{1}{\alpha} e^{\alpha\Phi}\,\text{fin.}\,\Phi^n - \frac{n}{\alpha}\int e^{\alpha\Phi} d\Phi\,\text{fin.}\,\Phi^{n-1}\,\text{cof.}\,\Phi$$

simili modo reperitur:

$$\int e^{\alpha\Phi} d\Phi\,\text{fin.}\,\Phi^{n-1}\,\text{cof.}\,\Phi = \frac{1}{\alpha} e^{\alpha\Phi}\,\text{fin}\,\Phi^{n-1}\,\text{cof.}\,\Phi - \frac{1}{\alpha}\int e^{\alpha\Phi} d\Phi$$
$$((n-1)\,\text{fin.}\,\Phi^{n-2}\,\text{cof.}\,\Phi^2 - \text{fin.}\,\Phi^n)$$

quae postrema formula ob $\text{cof.}\,\Phi^2 = 1 - \text{fin.}\,\Phi^2$ reducitur ad has:

$$(n-1)\int e^{\alpha\Phi} d\Phi\,\text{fin.}\,\Phi^{n-2} - n\int e^{\alpha\Phi} d\Phi\,\text{fin.}\,\Phi^n$$

vnde habebitur:

$$\int e^{\alpha\Phi}d\Phi\,\text{fin.}\,\Phi^n = \tfrac{1}{\alpha}e^{\alpha\Phi}\text{fin.}\,\Phi^n - \tfrac{n}{\alpha\alpha}e^{\alpha\Phi}\text{fin.}\,\Phi^{n-1}\text{cof.}\,\Phi + \tfrac{n(n-1)}{\alpha\alpha}\int e^{\alpha\Phi}d\Phi\,\text{fin.}\,\Phi^{n-2} - \tfrac{n\,n}{\alpha\,\alpha}\int e^{\alpha\Phi}d\Phi\,\text{fin.}\,\Phi^n.$$

Quare hanc poftremam formulam cum prima coniungendo, elicitur

$$\int e^{\alpha\Phi}d\Phi\,\text{fin.}\,\Phi^n = \frac{e^{\alpha\Phi}\,\text{fin.}\,\Phi^{n-1}(\alpha\,\text{fin.}\,\Phi - n\,\text{cof.}\,\Phi)}{\alpha\alpha+nn} + \frac{n(n-1)}{\alpha\alpha+nn}\int e^{\alpha\Phi}d\Phi\,\text{fin.}\,\Phi^{n-2}.$$

Duobus ergo cafibus integrale abfolute datur, fcilicet $n=0$ et $n=1$, eritque

$$\int e^{\alpha\Phi}d\Phi = \tfrac{1}{\alpha}e^{\alpha\Phi} - \tfrac{1}{\alpha} \text{ et } \int e^{\alpha\Phi}d\Phi\,\text{fin.}\,\Phi = \frac{e^{\alpha\Phi}(\alpha\,\text{fin.}\,\Phi - \text{cof.}\,\Phi)}{\alpha\alpha+1} + \frac{1}{\alpha\alpha+1}$$

atque ad hos fequentes omnes, vbi n eft numerus integer vnitate maior, reducuntur.

Coroll. 1.

267. Ita fi $n=2$ acquirimus hanc integrationem

$$\int e^{\alpha\Phi}d\Phi\,\text{fin.}\,\Phi^2 = \frac{e^{\alpha\Phi}\text{fin}\,\Phi(\alpha\,\text{fin.}\,\Phi - 2\,\text{cof.}\,\Phi)}{\alpha\alpha+4} + \frac{1.2}{\alpha(\alpha\alpha+4)}e^{\alpha\Phi} - \frac{1.2}{\alpha(\alpha\alpha+4)}$$

at

at si sit $n = 3$ istam:

$$\int e^{\alpha\Phi} d\Phi \sin.\Phi^3 = \frac{e^{\alpha\Phi}\sin.\Phi^2(\alpha\sin.\Phi - 3\cos.\Phi)}{\alpha\alpha+9} + \frac{2.3e^{\alpha\Phi}(\alpha\sin.\Phi - \cos.\Phi)}{(\alpha\alpha+1)(\alpha\alpha+9)} + \frac{2.3}{(\alpha\alpha+1)(\alpha\alpha+9)}.$$

Integralibus ita sumtis, vt euanescant, posito $\Phi = 0$.

Coroll. 2.

268. Si igitur determinatis hoc modo integralibus, statuatur $\alpha\Phi = -\infty$, vt $e^{\alpha\Phi}$ euanescat, erit in genere

$$\int e^{\alpha\Phi} d\Phi \sin.\Phi^n = \frac{n(n-1)}{\alpha\alpha+nn}\int e^{\alpha\Phi} d\Phi \sin.\Phi^{n-2}$$

hincque integralia pro isto casu $\alpha\Phi = -\infty$ erunt:

$$\int e^{\alpha\Phi} d\Phi = -\frac{1}{\alpha}; \quad \int e^{\alpha\Phi} d\Phi \sin.\Phi = \frac{1}{\alpha\alpha+1}$$

$$\int e^{\alpha\Phi} d\Phi \sin.\Phi^2 = \frac{-1.2}{\alpha(\alpha\alpha+4)}; \quad \int e^{\alpha\Phi} d\Phi \sin.\Phi^3 = \frac{1.2.3}{(\alpha\alpha+1)(\alpha\alpha+9)}$$

$$\int e^{\alpha\Phi} d\Phi \sin\Phi^4 = \frac{-1.2.3.4}{\alpha(\alpha\alpha+4)(\alpha\alpha+16)}; \quad \int e^{\alpha\Phi} d\Phi \sin\Phi^5 = \frac{1.2.3.4.5}{(\alpha\alpha+1)(\alpha\alpha+9)(\alpha\alpha+25)}.$$

Coroll. 3.

269. Quare si proponatur haec series infinita

$$s = 1 + \frac{1.2}{\alpha\alpha+4} + \frac{1.2.3.4}{(\alpha\alpha+4)(\alpha\alpha+16)} + \frac{1.2.3.4.5.6}{(\alpha\alpha+4)(\alpha\alpha+16)(\alpha\alpha+36)} \text{ etc.}$$

erit $s = -\alpha\int e^{\alpha\Phi} d\Phi (1 + \sin.\Phi^2 + \sin.\Phi^4 + \sin.\Phi^6 + \text{etc.})$

seu $s = -\alpha\int \frac{e^{\alpha\Phi} d\Phi}{\cos.\Phi^2}$, posito post integrationem $\alpha\Phi = -\infty$.

Problema 31.

270. Formulae differentialis $e^{\alpha\Phi} d\Phi \cos.\Phi^n$ integrale inueſtigare.

Solutio

Simili modo procedendo vt ante erit

$$e^{\alpha\Phi} d\Phi \cos.\Phi^n = \tfrac{1}{\alpha} e^{\alpha\Phi} \cos.\Phi^n + \tfrac{n}{\alpha} \int e^{\alpha\Phi} d\Phi \sin.\Phi \cos.\Phi^{n-1}$$

tum vero

$$\int e^{\alpha\Phi} d\Phi \sin.\Phi \cos.\Phi^{n-1} = \tfrac{1}{\alpha} e^{\alpha\Phi} \sin.\Phi \cos.\Phi^{n-1} - \tfrac{1}{\alpha} \int e^{\alpha\Phi} d\Phi \left(\cos.\Phi^n - (n-1)\cos.\Phi^{n-2} \sin.\Phi^2\right)$$

quae poſtrema formula abit in $-(n-1)\int e^{\alpha\Phi} d\Phi \cos.\Phi^{n-2} + n\int e^{\alpha\Phi} d\Phi \cos.\Phi^n$ ita vt ſit

$$\int e^{\alpha\Phi} d\Phi \cos.\Phi^n = \tfrac{1}{\alpha} e^{\alpha\Phi} \cos.\Phi^n + \tfrac{n}{\alpha\alpha} e^{\alpha\Phi} \sin.\Phi \cos.\Phi^{n-1} + \tfrac{n(n-1)}{\alpha\alpha} \int e^{\alpha\Phi} d\Phi \cos.\Phi^{n-2} - \tfrac{nn}{\alpha\alpha} \int e^{\alpha\Phi} d\Phi \cos.\Phi^n$$

vnde colligimus:

$$\int e^{\alpha\Phi} d\Phi \cos.\Phi^n = \frac{e^{\alpha\Phi} \cos.\Phi^{n-1} (\alpha \cos.\Phi + n \sin.\Phi)}{\alpha\alpha + nn} + \frac{n(n-1)}{\alpha\alpha + nn} \int e^{\alpha\Phi} d\Phi \cos.\Phi^{n-2}$$

hinc ergo caſus ſimpliciſſimi ſunt

$$\int e^{\alpha\Phi} d\Phi = \tfrac{1}{\alpha} e^{\alpha\Phi} + C; \quad \int e^{\alpha\Phi} d\Phi \cos.\Phi = \frac{e^{\alpha\Phi}(\alpha \cos.\Phi + \sin.\Phi)}{\alpha\alpha + 1} + C$$

ad quos ſequentes omnes, vbi n eſt numerus integer poſitiuus, reducuntur.

Scholion.

Scholion.

271. Casibus simplicissimis notatis alia datur via integrale formularum propositarum, quin etiam huius magis patentis $e^{\alpha\Phi}d\Phi \sin.\Phi^m \cos.\Phi^n$ eruendi. Cum enim productum $\sin.\Phi^m \cos.\Phi^n$ resolui possit in aggregatum plurium sinuum vel cosinuum, quorum quisque est huius formae $M\sin.\lambda\Phi$ vel $M\cos.\lambda\Phi$, integratio reducitur ad alterutram harum formularum $e^{\alpha\Phi}d\Phi\sin.\lambda\Phi$ vel $e^{\alpha\Phi}d\Phi\cos.\lambda\Phi$. Ponamus ergo $\lambda\Phi = \omega$, vt habemus

$$e^{\alpha\Phi}d\Phi\cos.\lambda\Phi = \tfrac{1}{\lambda}e^{\frac{\alpha}{\lambda}\omega}d\omega\sin.\omega \text{ et}$$

$$e^{\alpha\Phi}d\Phi\cos.\lambda\Phi = \tfrac{1}{\lambda}e^{\frac{\alpha}{\lambda}\omega}d\omega\cos.\omega$$

quarum integralia per superiora ita dantur:

$$\int e^{\frac{\alpha}{\lambda}\omega}d\omega\sin.\omega = \frac{\lambda e^{\frac{\alpha}{\lambda}\omega}(\alpha\sin.\omega - \lambda\cos.\omega)}{\alpha\alpha+\lambda\lambda} = \frac{\lambda e^{\alpha\Phi}(\alpha\sin.\lambda\Phi - \lambda\cos.\lambda\Phi)}{\alpha\alpha+\lambda\lambda}$$

$$\int e^{\frac{\alpha}{\lambda}\omega}d\omega\cos.\omega = \frac{\lambda e^{\frac{\alpha}{\lambda}\omega}(\alpha\cos.\omega + \lambda\sin.\omega)}{\alpha\alpha+\lambda\lambda} = \frac{\lambda e^{\alpha\Phi}(\alpha\cos.\lambda\Phi + \lambda\sin.\lambda\Phi)}{\alpha\alpha+\lambda\lambda}.$$

Vnde tandem colligimus:

$$\int e^{\alpha\Phi}d\Phi\sin.\lambda\Phi = \frac{e^{\alpha\Phi}(\alpha\sin.\lambda\Phi - \lambda\cos.\lambda\Phi)}{\alpha\alpha+\lambda\lambda} \text{ et}$$

$$\int e^{\alpha\Phi}d\Phi\cos.\lambda\Phi = \frac{e^{\alpha\Phi}(\alpha\sin.\lambda\Phi + \lambda\sin.\lambda\Phi)}{\alpha\alpha+\lambda\lambda}$$

si ingenere statim loco $\sin.\Phi$ et $\cos.\Phi$ scripsissem $\sin.\lambda\Phi$ et $\cos.\lambda\Phi$, hac reductione non fuisset opus; sed quia hic nihil est difficultatis, breuitati consulendum existimaui.

CAPVT VI.

DE EVOLVTIONE INTEGRALIVM PER SERIES SECVNDVM SINVS COSINVSVE ANGVLORVM MVLTIPLORVM PROGREDIENTES.

Problema 32.

272.

Integrale formulae $\frac{d\Phi}{1+n\,\text{cof.}\,\Phi}$ per feriem fecundum finus angulorum multiplorum progredientem exprimere.

Solutio.

Cum fit more confueto per feriem:

$$\frac{1}{1+n\,\text{cof.}\,\Phi} = 1 - n\,\text{cof.}\,\Phi + n^2\,\text{cof.}\,\Phi^2 - n^3\,\text{cof.}\,\Phi^3 + n^4\,\text{cof.}\,\Phi^4 - \text{etc.}$$

poteftates cofinus in cofinus angulorum multiplorum conuertantur ope formularum in introductione traditarum ac primo pro poteftatibus imparibus:

$$\text{cof.}\,\Phi = \text{cof.}\,\Phi$$

$$\text{cof.}\,\Phi^3 = \tfrac{3}{4}\,\text{cof.}\,\Phi + \tfrac{1}{4}\,\text{cof.}\,3\Phi$$

$$\text{cof.}\,\Phi^5 = \tfrac{10}{16}\,\text{cof.}\,\Phi + \tfrac{5}{16}\,\text{cof.}\,3\Phi + \tfrac{1}{16}\,\text{cof.}\,5\Phi$$

$$\text{cof.}\,\Phi^7 = \tfrac{35}{64}\,\text{cof.}\,\Phi + \tfrac{21}{64}\,\text{cof.}\,3\Phi + \tfrac{7}{64}\,\text{cof.}\,5\Phi + \tfrac{1}{64}\,\text{cof.}\,7\Phi$$

$$\text{cof.}\,\Phi^9 = \tfrac{126}{256}\,\text{cof.}\,\Phi + \tfrac{84}{256}\,\text{cof.}\,3\Phi + \tfrac{36}{256}\,\text{cof.}\,5\Phi + \tfrac{9}{256}\,\text{cof.}\,7\Phi + \tfrac{1}{256}\,\text{cof.}\,9\Phi$$

vbi

vbi notandum eſt ſi ponatur in genere

$$\operatorname{cof.}\Phi^{2\lambda-1} = A\operatorname{cof.}\Phi + B\operatorname{cof.}3\Phi + C\operatorname{cof.}5\Phi + D\operatorname{cof.}7\Phi + E\operatorname{cof.}9\Phi \text{ etc.}$$

fore

$$A = 2.\frac{1.3.5\ldots\ldots(2\lambda-1)}{2.4.6\ldots\ldots 2\lambda} = \frac{2}{2^{2\lambda-1}}.\frac{6}{2}.\frac{10}{3}.\frac{14}{4}\ldots\frac{4\lambda-2}{\lambda}$$

$$B = \tfrac{\lambda-1}{\lambda+1}A;\; C = \tfrac{\lambda-2}{\lambda+2}B;\; D = \tfrac{\lambda-3}{\lambda+3}C;\; E = \tfrac{\lambda-4}{\lambda+4}D \text{ etc.}$$

Pro paribus vero poteſtatibus eſt

$$\operatorname{cof.}\Phi^0 = 1$$
$$\operatorname{cof.}\Phi^2 = \tfrac{1}{2} + \tfrac{1}{2}\operatorname{cof.}2\Phi$$
$$\operatorname{cof.}\Phi^4 = \tfrac{3}{8} + \tfrac{4}{8}\operatorname{cof.}2\Phi + \tfrac{1}{8}\operatorname{cof.}4\Phi$$
$$\operatorname{cof.}\Phi^6 = \tfrac{10}{32} + \tfrac{15}{32}\operatorname{cof.}2\Phi + \tfrac{6}{32}\operatorname{cof.}4\Phi + \tfrac{1}{32}\operatorname{cof.}6\Phi$$
$$\operatorname{cof.}\Phi^8 = \tfrac{35}{128} + \tfrac{56}{128}\operatorname{cof.}2\Phi + \tfrac{28}{128}\operatorname{cof.}4\Phi + \tfrac{8}{128}\operatorname{cof.}6\Phi + \tfrac{1}{128}\operatorname{cof.}8\Phi.$$

In genere autem ſi ponatur:

$$\operatorname{cof.}\Phi^{2\lambda} = \mathfrak{A} + \mathfrak{B}\operatorname{cof.}2\Phi + \mathfrak{C}\operatorname{cof.}4\Phi + \mathfrak{D}\operatorname{cof.}6\Phi + \mathfrak{E}\operatorname{cof.}8\Phi + \text{ etc.}$$

erit

$$\mathfrak{A} = \frac{1.3.5\ldots\ldots(2\lambda-1)}{2.4.6\ldots\ldots 2\lambda} = \frac{1}{2^{2\lambda-1}}.\frac{6}{2}.\frac{10}{3}.\frac{14}{4}\ldots\frac{4\lambda-2}{\lambda}$$

$$\mathfrak{B} = \tfrac{2\lambda}{\lambda+1}\mathfrak{A};\; \mathfrak{C} = \tfrac{\lambda-1}{\lambda+2}\mathfrak{B};\; \mathfrak{D} = \tfrac{\lambda-2}{\lambda+3}\mathfrak{C};\; \mathfrak{E} = \tfrac{\lambda-3}{\lambda+4}\mathfrak{D} \text{ etc.}$$

Quodſi nunc iſti valores ſubſtituantur, erit $\frac{1}{1+n\operatorname{cof.}\Phi} =$

$$\begin{array}{lllllll}
1 & -n\operatorname{cof.}\Phi & +\tfrac{1}{2}nn\operatorname{cof.}2\Phi & -\tfrac{1}{4}n^3\operatorname{cof.}3\Phi & +\tfrac{1}{8}n^4\operatorname{cof.}4\Phi & -\tfrac{1}{16}n^5\operatorname{cof.}5\Phi & +\tfrac{1}{32}n^6\operatorname{cof.}6\Phi \\
+\tfrac{1}{2}nn & -\tfrac{3}{4}n^3 & +\tfrac{4}{8}n^4 & -\tfrac{5}{16}n^5 & +\tfrac{6}{32}n^6 & -\tfrac{7}{64}n^7 & +\tfrac{8}{128}n^8 \\
+\tfrac{3}{8}n^4 & -\tfrac{10}{16}n^5 & +\tfrac{15}{32}n^6 & -\tfrac{21}{64}n^7 & +\tfrac{28}{128}n^8 & -\tfrac{36}{256}n^9. & \\
+\tfrac{10}{32}n^6 & -\tfrac{35}{64}n^7 & +\tfrac{56}{128}n^8 & -\tfrac{84}{256}n^9 & & & \\
+\tfrac{35}{128}n^8 & & & & & &
\end{array}$$

vnde

vnde patet, ſi ponatur

$$\frac{1}{1+n\,\text{coſ.}\,\Phi} = \text{A} - \text{B}\,\text{coſ.}\,\Phi + \text{C}\,\text{coſ.}\,2\,\Phi - \text{D}\,\text{coſ.}\,3\,\Phi + \text{E}\,\text{coſ.}\,4\,\Phi - \text{etc.}$$

eſt $\text{A} = 1 + \frac{1}{2}nn + \frac{3}{8}n^4 + \frac{10}{32}n^6 + \text{etc.}$ ſeu

$$\text{A} = 1 + \frac{1}{2}n^2 + \frac{1\cdot3}{2\cdot4}n^4 + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}n^6 + \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}n^8 + \text{etc.}$$

ſicque euidens eſt eſſe $\text{A} = \frac{1}{\surd(1-nn)}$. Simili modo eſt $\text{B} = n + \frac{3}{4}n^3 + \frac{10}{16}n^5 + \text{etc.} = \frac{2}{n}\left(\frac{1}{2}n^2 + \frac{1\cdot3}{2\cdot4}n^4 + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}n^6 + \text{etc.}\right)$ ideoque $\text{B} = \frac{2}{n}\left(\frac{1}{\surd(1-nn)} - 1\right)$. Verum et hunc valorem et ſequentes facilius hoc modo definire licet. Cum ſit

$$\frac{1}{1+n\,\text{coſ.}\,\Phi} = \text{A} - \text{B}\,\text{coſ.}\,\Phi + \text{C}\,\text{coſ.}\,2\,\Phi - \text{D}\,\text{coſ.}\,3\,\Phi \;\; \text{E}\,\text{coſ.}\,4\,\Phi - \text{etc.}$$

multiplicetur per $1 + n\,\text{coſ.}\,\Phi$, et quia $\text{coſ.}\,\Phi\,\text{coſ.}\,\lambda\Phi = \frac{1}{2}\text{coſ.}(\lambda-1)\Phi + \frac{1}{2}\text{coſ.}(\lambda+1)\Phi$, fiet

$$\begin{array}{lllll} 1 = \text{A} & - \text{B}\,\text{coſ.}\,\Phi & + \text{C}\,\text{coſ.}\,2\,\Phi & - \text{D}\,\text{coſ.}\,3\,\Phi & + \text{E}\,\text{coſ.}\,4\,\Phi - \text{etc.} \\ & + \text{A}n & - \frac{1}{2}\text{B}n & + \frac{1}{2}\text{C}n & - \frac{1}{2}\text{D}n \\ - \frac{1}{2}\text{B}n + \frac{1}{2}\text{C}n & - \frac{1}{2}\text{D}n & + \frac{1}{2}\text{E}n & - \frac{1}{2}\text{F}n & \end{array}$$

vnde quia A iam definiuimus, reliqui coefficientes ita determinantur:

$$\text{B} = \frac{2}{n}(\text{A}-1); \qquad \text{E} = \frac{2\text{D} - \text{C}n}{n}$$

$$\text{C} = \frac{2\text{B} - 2\text{A}n}{n}; \qquad \text{F} = \frac{2\text{E} - \text{D}n}{n}$$

$$\text{D} = \frac{2\text{C} - \text{B}n}{n}; \qquad \text{G} = \frac{2\text{F} - \text{E}n}{n}$$

etc.

His igitur coefficientibus inuentis, integrale facile aſſigna-

assignatur, nam cum sit $\int d\Phi \cos.\lambda\Phi = \frac{1}{\lambda}\sin.\lambda\Phi$ habebimus

$$\int \frac{d\Phi}{1+n\cos.\Phi} = A\Phi - B\sin.\Phi + \frac{1}{2}C\sin.2\Phi - \frac{1}{3}D\sin.3\Phi + \frac{1}{4}E\sin.4\Phi - \text{etc.}$$

quae series secundum sinus angulorum Φ, 2Φ, 3Φ etc. progreditur, vti desiderabatur.

Coroll. 1.

273. Primo patet hanc resolutionem locum habere non posse, nisi n sit numerus vnitate minor; si enim $n > 1$, singuli coefficientes prodeunt imaginarii. Sin autem sit $n = 1$, ob $1 + \cos.\Phi = 2\cos.\frac{1}{2}\Phi^2$, erit integrale

$$\int \frac{d\Phi}{1+\cos.\Phi} = \int \frac{\frac{1}{2}d\Phi}{\cos.\frac{1}{2}\Phi^2} = \text{tang.}\frac{1}{2}\Phi.$$

Coroll. 2.

274. Cum sit $A = \frac{1}{\sqrt{(1-nn)}}$ et $B = \frac{2}{n}\left(\frac{1}{\sqrt{(1-nn)}} - 1\right)$ reliqui coefficientes C, D, E etc. seriem recurrentem constituunt, ita vt si bini contigui sint P et Q sequens futurus sit $\frac{2}{n}Q - P$. Hinc cum aequationis $zz = \frac{2}{n}z - 1$ radices sint $\frac{1 \pm \sqrt{(1-nn)}}{n}$, quisque terminus in hac forma continetur

$$\alpha\left(\frac{1+\sqrt{(1-nn)}}{n}\right)^\lambda + \beta\left(\frac{1-\sqrt{(1-nn)}}{n}\right)^\lambda.$$

Coroll. 3.

275. Quia autem in nostra lege non A sed 2 A sumitur, posito $\lambda = 0$ prodire debet 2 A ideoque

$\alpha+\beta=\frac{2}{\surd(1-nn)}$, deinde facto $\lambda=1$ fieri debet $\frac{\alpha+\beta}{n}+\frac{(\alpha-\beta)\surd(1-nn)}{n}=\frac{2-2(1-nn)}{n\surd(1-nn)}$, vnde $\alpha-\beta=-\frac{2}{\surd(1-nn)}$. Ergo $\alpha=0$ et $\beta=\frac{2}{\surd(1-nn)}$, ſicque quilibet terminus praeter A erit $=\frac{2}{\surd(1-nn)}\left(\frac{1-\surd(1-nn)}{n}\right)^{\lambda}$.

Coroll. 4.

276. Coefficientes ergo euoluti ita ſe habebunt:

$$A=\frac{1}{\surd(1-nn)}$$

$$B=\frac{2-2\surd(1-nn)}{n\surd(1-nn)}$$

$$C=\frac{4-2nn-4\surd(1-nn)}{nn\surd(1-nn)}$$

$$D=\frac{8-6nn-2(4-nn)\surd(1-nn)}{n^3\surd(1-nn)}$$

$$E=\frac{16-16nn+2n^4-2(8-4nn)\surd(1-nn)}{n^4\surd(1-nn)}$$

$$F=\frac{32-40nn+10n^4-2(16-12nn+n^4)(\surd 1-nn)}{n^5\surd(1-nn)}$$

$$G=\frac{64-96nn+36n^4-2n^6-2(32-32nn+6n^4)\surd(1-nn)}{n^6\surd(1-nn)}.$$

Coroll. 5.

277. Quia $n<1$, hi coefficientes plerumque facilius determinantur per ſeries primum inuentas, ſcilicet:

$$A=1+\tfrac{1}{2}n^2+\tfrac{1\cdot 3}{2\cdot 4}n^4+\tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}n^6+\tfrac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}n^8+\text{etc.}$$

$$B=n(1+\tfrac{3}{4}n^2+\tfrac{3\cdot 5}{4\cdot 6}n^4+\tfrac{3\cdot 5\cdot 7}{4\cdot 6\cdot 8}n^6+\tfrac{3\cdot 5\cdot 7\cdot 9}{4\cdot 6\cdot 8\cdot 10}n^8+\text{etc.}$$

C=

$$C = \tfrac{1}{2}n^2\left(1 + \tfrac{3\cdot 4}{2\cdot 6}n^2 + \tfrac{3\cdot 4\cdot 5\cdot 6}{2\cdot 6\cdot 4\cdot 8}n^4 + \tfrac{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}{2\cdot 6\cdot 4\cdot 8\cdot 6\cdot 10}n^6 + \text{etc.}\right.$$

$$D = \tfrac{1}{4}n^3\left(1 + \tfrac{4\cdot 5}{2\cdot 8}n^2 + \tfrac{4\cdot 5\cdot 6\cdot 7}{2\cdot 8\cdot 4\cdot 10}n^4 + \tfrac{4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}{2\cdot 8\cdot 4\cdot 10\cdot 6\cdot 12}n^6 + \text{etc.}\right.$$

$$E = \tfrac{1}{8}n^4\left(1 + \tfrac{5\cdot 6}{2\cdot 10}n^2 + \tfrac{5\cdot 6\cdot 7\cdot 8}{2\cdot 10\cdot 4\cdot 12}n^4 + \tfrac{5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10}{2\cdot 10\cdot 4\cdot 12\cdot 6\cdot 14}n^6 + \text{etc.}\right.$$

$$F = \tfrac{1}{16}n^5\left(1 + \tfrac{6\cdot 7}{2\cdot 12}n^2 + \tfrac{6\cdot 7\cdot 8\cdot 9}{2\cdot 12\cdot 4\cdot 14}n^4 + \tfrac{6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11}{2\cdot 12\cdot 4\cdot 14\cdot 6\cdot 16}n^6 + \text{etc.}\right.$$

etc.

Scholion.

278. Cum ex his valoribus fit

$$\int \frac{d\Phi}{1 + n\cos.\Phi} = A\Phi - B\sin.\Phi + \tfrac{1}{2}C\sin.2\Phi - \tfrac{1}{3}D\sin.3\Phi + \tfrac{1}{4}E\sin.4\Phi - \text{etc.}$$

in hac ſerie terminus primus $A\Phi$ imprimis eſt notandus, quod creſcente angulo Φ continuo creſcat, idque in infinitum vsque, dum reliqui termini modo creſcent modo decreſcent: neque tamen certum limitem excedunt; nam $\sin.\lambda\Phi$ neque ſupra $+1$ creſcere, neque infra -1 decreſcere poteſt. Cum deinde hoc integrale ſupra inuentum ſit $\frac{1}{\sqrt{(1-nn)}}$ ang. cof. $\frac{n+\cos.\Phi}{1+n\cos.\Phi}$ ſeries illa huic angulo aequatur. Quare ſi hic angulus vocetur ω, vt ſit $d\omega = \frac{d\Phi\sqrt{(1-nn)}}{1+n\cos.\Phi}$ erit $\cos.\omega = \frac{n+\cos.\Phi}{1+n\cos.\Phi}$ hincque $n + \cos.\Phi - \cos.\omega - n\cos.\Phi\cos.\omega = 0$, ex quo eſt viciſſim $\cos.\Phi = \frac{\cos.\omega - n}{1 - n\cos.\omega}$, quae formula cum ex illa naſcatur ſumto n negatiuo, erit $d\Phi = \frac{d\omega\sqrt{(1-nn)}}{1-n\cos.\Phi}$ et

$$\frac{\Phi}{\sqrt{(1-nn)}} = A\omega + B\sin.\omega + \tfrac{1}{2}C\sin.2\omega + \tfrac{1}{3}D\sin.3\omega + \tfrac{1}{4}E\sin.4\omega \text{ etc.}$$

Quia

Quia vero est

$$\frac{\omega}{\surd(1-nn)} = A\,\Phi - B\,\text{fin.}\,\Phi + \tfrac{1}{2}C\,\text{fin.}\,2\Phi - \tfrac{1}{3}D\,\text{fin.}\,3\Phi + E\tfrac{1}{4}\,\text{fin.}\,4\Phi \text{ etc.}$$

ob $\frac{1}{\surd(1-nn)} = A$, habebimus:

$$0 = B(\text{fin.}\,\omega - \text{fin.}\,\Phi) + \tfrac{1}{2}C(\text{fin.}\,2\omega + \text{fin.}\,2\Phi) + \tfrac{1}{3}D(\text{fin.}\,3\omega - \text{fin.}\,3\Phi) + \text{etc.}$$

cuiusmodi relationes notasse iuuabit.

Problema 33.

279. Integrale formulae $d\Phi(1+n\,\text{cof.}\,\Phi)^\nu$ per feriem fecundum finus angulorum multiplorum ipfius Φ progredientem exprimere.

Solutio.

Cum fit

$$(1+n\,\text{cof.}\,\Phi)^\nu = 1 + \tfrac{\nu}{1}n\,\text{cof.}\,\Phi + \tfrac{\nu(\nu-1)}{1.\,2}n^2\,\text{cof.}\,\Phi^2 + \tfrac{\nu(\nu-1)(\nu-2)}{1.\,2.\,3}n^3\,\text{cof.}\,\Phi^3 \text{ etc.}$$

fi ponamus

$$(1+n\,\text{cof.}\,\Phi)^\nu = A + B\,\text{cof.}\,\Phi + C\,\text{cof.}\,2\Phi + D\,\text{cof.}\,3\Phi + E\,\text{cof.}\,4\Phi \text{ etc.}$$

erit per formulas fupra indicatas:

$$A = 1 + \tfrac{\nu(\nu-1)}{1.\,2}\cdot\tfrac{1}{2}n^2 + \tfrac{\nu(\nu-1)(\nu-2)(\nu-3)}{1.\,2.\,3.\,4}\cdot\tfrac{1.\,3}{2.\,4}n^4 + \tfrac{\nu(\nu-1)\ldots\ldots(\nu-5)}{1.\,2.\,\ldots\,6}\cdot\tfrac{1.\,3.\,5}{2.\,4.\,6}n^6 + \text{etc.}$$

$$B = 2n\Big(\tfrac{\nu}{2} + \tfrac{\nu(\nu-1)(\nu-2)}{1.\,2.\,3}\cdot\tfrac{1.\,3}{2.\,4}n^2 + \tfrac{\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)}{1.\,2.\,3.\,4.\,5}\cdot\tfrac{1.\,3.\,5}{2.\,4.\,6}n^4 + \text{etc.}$$

quae feries ita clarius exhibentur:

$$A = 1 + \tfrac{\nu(\nu-1)}{2.\,2}n^2 + \tfrac{\nu(\nu-1)(\nu-2)(\nu-3)}{2.\,2.\,4.\,4}n^4 + \tfrac{\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)(\nu-5)}{2.\,2.\,4.\,4.\,6.\,6}n^6$$

$$\tfrac{1}{2}B = \tfrac{\nu}{2}n + \tfrac{\nu(\nu-1)(\nu-2)}{2.\,2.\,4}n^3 + \tfrac{\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)}{2.\,2.\,4.\,4.\,6}n^5 + \text{etc.}$$

Inuen-

Inuentis autem his binis coefficientibus A et B, reliqui ex his fequenti modo commodius determinari poterunt. Cum fit

$$\nu l(1+n\,\text{cof.}\,\Phi)=l(A+B\,\text{cof.}\,\Phi+C\,\text{cof.}\,2\Phi+D\,\text{cof.}\,3\Phi+E\,\text{cof.}\,4\Phi\ \text{etc.})$$

fumantur differentialia, ac per $-d\Phi$ diuidendo prodit

$$\frac{-\nu n\,\text{fin.}\,\Phi}{1+n\,\text{cof.}\,\Phi}=\frac{B\,\text{fin.}\,\Phi+2C\,\text{fin.}\,\Phi+3D\,\text{fin.}\,\Phi+4E\,\text{fin.}\,\Phi+\text{etc.}}{A+B\,\text{cof.}\,\Phi+C\,\text{cof.}\,2\Phi+D\,\text{cof.}\,3\Phi+E\,\text{cof.}\,4\Phi\ \text{etc.}}$$

Iam per crucem multiplicando ob $\text{fin.}\,\lambda\Phi\,\text{cof.}\,\Phi=\frac{1}{2}\text{fin.}(\lambda+1)\Phi+\frac{1}{2}\text{fin.}(\lambda-1)\Phi$ et $\text{fin.}\,\Phi\,\text{cof.}\,\lambda\Phi=\frac{1}{2}\text{fin.}(\lambda+1)\Phi-\frac{1}{2}\text{fin.}(\lambda-1)\Phi$ peruenietur ad hanc aequationem:

$$\begin{array}{lllll}
o=B\,\text{fin.}\,\Phi & -2C\,\text{fin.}\,2\Phi & +3D\,\text{fin.}\,3\Phi & +4E\,\text{fin.}\,4\Phi & +5F\,\text{fin.}\,5\Phi+\text{etc.}\\
 & +\frac{1}{2}Bn & +\frac{2}{2}Cn & +\frac{3}{2}Dn & +\frac{4}{2}En\\
+\frac{2}{2}Cn & +\frac{3}{2}Dn & +\frac{4}{2}En & +\frac{5}{2}Fn & +\frac{6}{2}Gn\\
-\nu An & -\frac{\nu}{2}Bn & -\frac{\nu}{2}Cn & -\frac{\nu}{2}Dn & -\frac{\nu}{2}En\\
+\frac{\nu}{2}Cn & +\frac{\nu}{2}Dn & +\frac{\nu}{2}En & +\frac{\nu}{2}Fn & +\frac{\nu}{2}Gn
\end{array}$$

vnde hae fequuntur determinationes:

$$\begin{array}{l|l}
(\nu+2)Cn+2B-2\nu An=o & C=\frac{2\nu An-2B}{(\nu+2)n}\\
(\nu+3)Dn+4C-(\nu-1)Bn=o & D=\frac{(\nu-1)Bn-4C}{(\nu+3)n}\\
(\nu+4)En+6D-(\nu-2)Cn=o & E=\frac{(\nu-2)Cn-6D}{(\nu+4)n}\\
(\nu+5)Fn+8E-(\nu-3)Dn=o & F=\frac{(\nu-3)Dn-8E}{(\nu+5)n}\\
(\nu+6)Gn+10F-(\nu-4)En=o & G=\frac{(\nu-4)En-10F}{(\nu+6)n}
\end{array}$$

 vbi

vbi si superiores valores pro A et B substituantur, reperitur:

$$C = 4nn\left(\frac{1\nu(\nu-1)}{2\cdot 2\cdot 4} + \frac{2\nu(\nu-1)(\nu-2)(\nu-3)}{2\cdot 2\cdot 4\cdot 4\cdot 6}n^2 + \frac{3\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)(\nu-5)}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8}n^4 \text{ etc.}\right)$$

$$D = 8n^3\left(\frac{1\cdot 2\nu(\nu-1)(\nu-2)}{2\cdot 2\cdot 4\cdot 4\cdot 6} + \frac{2\cdot 3\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8}n^2 + \text{etc.}\right)$$

$$E = 16n^4\left(\frac{1\cdot 2\cdot 3\nu(\nu-1)(\nu-2)(\nu-3)}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8} + \frac{2\cdot 3\cdot 4\nu(\nu-1)\ldots(\nu-5)}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10}n^2 + \text{etc.}\right)$$

etc.

vnde forma sequentium serierum colligitur.

His autem inuentis coefficientibus erit integrale quaesitum

$$\int d\Phi(1+n\cos.\Phi)^\nu = A\Phi + B\sin.\Phi + \tfrac{1}{2}C\sin 2\Phi + \tfrac{1}{3}D\sin.3\Phi + \tfrac{1}{4}E\sin.4\Phi + \text{etc.}$$

Coroll. 1.

280. Ad similitudinem harum serierum pro C, D, E etc. datarum etiam valor ipsius B ita exprimi potest:

$$B = 2n\left(\frac{\nu}{2} + \frac{\nu(\nu-1)(\nu-2)}{2\cdot 2\cdot 4}n^2 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)}{2\cdot 2\cdot 4\cdot 4\cdot 6}n^4 + \text{etc.}\right)$$

series autem pro A inuenta formam habet singularem in hac lege non comprehensam.

Coroll. 2.

281. Si series A et B inter se comparemus, varias relationes inter eas obseruare licet, quarum haec primo se offert:

$$An + \tfrac{1}{2}B = \frac{(\nu+2)}{2}\left(1 + \frac{\nu(\nu-1)}{2\cdot 4}n^2 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2\cdot 4\cdot 4\cdot 6}n^4 + \frac{\nu(\nu-1)\ldots(\nu-5)}{2\cdot 4\cdot\ldots\cdot 8}n^6 + \text{etc.}\right)$$

quae a serie A tantum secundum denominatores differt.

Coroll. 3.

Coroll. 3.

282. Ponamus $\frac{2Ann+Bn}{\nu+2}=N$ vt fit

$$N=n^2+\frac{\nu(\nu-1)}{2\cdot 4}n^4+\frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2\cdot 4\cdot 4\cdot 6}n^6 \text{ etc.}$$

$$A=1+\frac{\nu(\nu-1)}{2\cdot 2}n^2+\frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2\cdot 2\cdot 4\cdot 4}n^4 \text{ etc.}$$

Quodfi iam n vt variabilis tractetur, differentiatio praebet:

$$\frac{dN}{n\,dn}=2+\frac{\nu(\nu-1)}{2}n^2+\frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2\cdot 4\cdot 4}n^4+\text{etc.}=2A.$$

Cum igitur fit

$$dN=\frac{4Andn+Bdn+2nndA+ndB}{\nu+2}=2Andn$$

erit $2\nu Andn=2nndA+Bdn+ndB$.

Coroll. 4.

283 Ex dato ergo coefficiente A coefficiens B ita per integrationem inueniri poteft, vt fit

$$Bn=2\int(\nu Andn-nndA)$$

vel erit etiam ex illa forma

$$B=\frac{2(\nu+2)}{n}\int Andn-2An$$

vbi notandum eft, pofito $n=0$ integrale $\int Andn$ euanefcere debere, quia hoc cafu B euanefcit.

Scholion.

284. Series pro litteris B, C, D etc. inuentas etiam fequenti modo per continuos factores exprimere licet:

$B=\nu n$

$$B = \nu n\left(1 + \frac{(\nu-1)(\nu-2)}{2.\;4}n^2 + \frac{(\nu-3)(\nu-4)}{4.\;6}Pn^2 + \frac{(\nu-5)(\nu-6)}{6.\;8}Pn^2 + \text{etc.}\right.$$

$$C = \frac{\nu(\nu-1)}{1.\;2}\cdot\frac{n^2}{2}\left(1 + \frac{(\nu-2)(\nu-3)}{2.\;6}n^2 + \frac{(\nu-4)(\nu-5)}{4.\;8}Pn^2\right.$$
$$+ \frac{(\nu-6)(\nu-7)}{6.\;10}Pn^2 + \text{etc.}$$

$$D = \frac{\nu(\nu-1)(\nu-2)}{1.\;2.\;3}\cdot\frac{n^3}{4}\left(1 + \frac{(\nu-3)(\nu-4)}{2.\;8}n^2 + \frac{(\nu-5)(\nu-6)}{4.\;10}Pn^2\right.$$
$$+ \frac{(\nu-7)(\nu-8)}{6.\;12}Pn^2 + \text{etc.}$$

$$E = \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{1.\;2.\;3.\;4}\cdot\frac{n^4}{8}\left(1 + \frac{(\nu-4)(\nu-5)}{2.\;10}n^2 + \frac{(\nu-6)(\nu-7)}{4\;12}Pn^2\right.$$
$$+ \frac{(\nu-8)(\nu-9)}{6.\;14}Pn^2 + \text{etc.}$$

$$F = \frac{\nu\ldots\ldots(\nu-4)}{1\ldots\ldots5}\cdot\frac{n^5}{16}\left(1 + \frac{(\nu-5)(\nu-6)}{2.\;12}n^2 + \frac{(\nu-7)(\nu-8)}{4.\;14}Pn^2\right.$$
$$\left.+ \frac{(\nu-9)(\nu-10)}{6.\;16}Pn^2 + \text{etc.}\right)$$

etc.

vbi in qualibet ferie littera P terminum praecedentem integrum denotat. Atque ope ferierum iftarum coefficientes plerumque facilius inueniuntur, quam ex lege ante tradita, qua quisque ex binis praecedentibus determinatur. Quin etiam haec lex defectu laborat, quod fi ν fuerit numerus integer negatiuus praeter -1, quidam coefficientes plane non definiantur, quos ergo ex his feriebus defumi oportet. Ita fi fuerit

$\nu = -2$, erit $B = \nu An = -2An$ et

$$C = \frac{2}{1.\;2}\cdot\frac{n^2}{2}\left(1 + \frac{4.\;5}{2.\;6}n^2 + \frac{4.\;5.\;6.\;7}{2.\;6.\;4.\;8}n^4 + \frac{4.\;5.\;6.\;7.\;8.\;9}{2.\;6.\;4.\;8.\;6.\;10}n^6 + \text{etc.}\right)$$

fi fit $\nu = -3$ erit $C = -Bn$ et

$$D = -\frac{4.\;5}{1.\;2}\cdot\frac{n^3}{4}\left(1 + \frac{6.\;7}{2.\;8}n^2 + \frac{6.\;7.\;8.\;9}{2.\;8.\;4.\;10}n^4 + \frac{6.\;7.\;8.\;9.\;10.\;11}{2.\;8.\;4.\;10.\;6.\;12}n^6 + \text{etc.}\right)$$

fi fit

fi fit $\nu = -4$ erit $D = -Cn$ et

$$E = \frac{5\cdot 6\cdot 7}{1\cdot 2\cdot 3}\cdot\frac{n^4}{8}\left(1 + \frac{8\cdot 9}{2\cdot 10}n^2 + \frac{8\cdot 9\cdot 10\cdot 11}{2\cdot 10\cdot 4\cdot 12}n^4 + \frac{8\cdot 9\cdot 10\cdot 11\cdot 12\cdot 13}{2\cdot 10\cdot 4\cdot 12\cdot 6\cdot 14}n^6 + \text{etc}\right)$$

fi fit $\nu = -5$ erit $E = -Dn$ et

$$F = -\frac{6\cdot 7\cdot 8\cdot 9}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{n^4}{16}\left(1 + \frac{10\cdot 11}{2\cdot 12}n^2 + \frac{10\cdot 11\cdot 12\cdot 13}{2\cdot 12\cdot 4\cdot 14}n^4 + \frac{10\cdot 11\cdot 12\cdot 13\cdot 14\cdot 15}{2\cdot 12\cdot 4\cdot 14\cdot 6\cdot 16}n^6 + \text{etc.}\right)$$

et ita de reliquis.

Exemplum 1.

285. *Formulae* $d\Phi(1 + n\cos\Phi)^\nu$ *integrale euoluere, fi* ν *fit numerus integer pofitiuus.*

Pofito $(1 + n\cos\Phi)^\nu = A + B\cos\Phi + C\cos 2\Phi + D\cos 3\Phi + E\cos 4\Phi + \text{etc.}$

pro fingulis valoribus exponentis ν habebimus:

1) fi $\nu = 1$; $A = 1$; $B = 2$; $C = 0$; etc.

2) fi $\nu = 2$; $A = 1 + \frac{1}{2}n^2$; $B = 2n$; $C = \frac{1}{2}nn$; $D = 0$; etc.

3) fi $\nu = 3$; $A = 1 + \frac{3}{2}n^2$; $B = 3n(1 + \frac{1}{4}n^2)$; $C = \frac{3}{2}n^2$; $D = \frac{1}{4}n^3$; $E = 0$; etc.

4) fi $\nu = 4$; $A = 1 + \frac{6}{2}n^2 + \frac{3}{8}n^4$; $B = 4n(1 + \frac{3}{4}n^2)$; $C = 3n^2(1 + \frac{1}{6}n^2)$; $D = n^3$; $E = \frac{1}{8}n^4$; $F = 0$.

Hi autem cafus nihil habent difficultatis. Ad vfum fequentem tantum iuuabit primum terminum abfolutum A notaffe:

fi $\nu = 1$; $A = 1$

fi $\nu = 2$; $A = 1 + \frac{2\cdot 1}{2\cdot 2}n^2$

A 2 fi $\nu = 3$

si $\nu = 3$; $A = 1 + \frac{3\cdot 2}{2\cdot 2} n^2$

si $\nu = 4$; $A = 1 + \frac{4\cdot 3}{2\cdot 2} n^2 + \frac{4\cdot 3\cdot 2\cdot 1}{2\cdot 2\cdot 4\cdot 4} n^4$

si $\nu = 5$; $A = 1 + \frac{5\cdot 4}{2\cdot 2} n^2 + \frac{5\cdot 4\cdot 3\cdot 2}{2\cdot 2\cdot 4\cdot 4} n^4$

si $\nu = 6$; $A = 1 + \frac{6\cdot 5}{2\cdot 2} n^2 + \frac{6\cdot 5\cdot 4\cdot 3}{2\cdot 2\cdot 4\cdot 4} n^4 + \frac{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} n^6$

si $\nu = 7$; $A = 1 + \frac{7\cdot 6}{2\cdot 2} n^2 + \frac{7\cdot 6\cdot 5\cdot 4}{2\cdot 2\cdot 4\cdot 4} n^4 + \frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} n^6$

etc.

Exemplum 2.

286. *Formulae* $\frac{d\Phi}{(1+n\cos.\Phi)^\mu}$ *integrale per seriem euoluere*

Posito
$$\frac{1}{(1+n\cos.\Phi)^\mu} = A + B\cos.\Phi + C\cos.2\Phi + D\cos.3\Phi + E\cos.4\Phi + \text{etc.}$$

ex praecedentibus formulis ponendo $\nu = -\mu$ erit

$$A = 1 + \frac{\mu(\mu+1)}{2\cdot 2} n^2 + \frac{\mu(\mu+1)(\mu+2)(\mu+3)}{2\cdot 2\cdot 4\cdot 4} n^4 + \frac{\mu(\mu+1)(\mu+2)(\mu+3)(\mu+4)(\mu+5)}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} n^6$$

$$B = -\mu n\left(1 + \frac{(\mu+1)(\mu+2)}{2\cdot 4} n^2 + \frac{(\mu+3)(\mu+4)}{4\cdot 6} P n^2 + \frac{(\mu+5)(\mu+6)}{6\cdot 8} P n^2 + \text{etc.}\right)$$

$$C = \frac{\mu(\mu+1)}{1\cdot 2}\cdot\frac{n^2}{2}\left(1 + \frac{(\mu+2)(\mu+3)}{2\cdot 6} n^2 + \frac{(\mu+4)(\mu+5)}{4\cdot 8} P n^2 + \frac{(\mu+6)(\mu+7)}{6\cdot 10} P n^2 + \text{etc.}\right)$$

$$D = -\frac{\mu(\mu+1)(\mu+2)}{1\cdot 2\cdot 3}\cdot\frac{n^3}{4}\left(1 + \frac{(\mu+3)(\mu+4)}{2\cdot 8} n^2 + \frac{(\mu+5)(\mu+6)}{4\cdot 10} P n^2 + \frac{(\mu+7)(\mu+8)}{6\cdot 12} P n^2 + \text{etc.}\right)$$

etc.

vbi

vbi vt ante in quaque ſerie P terminum praecedentem denotat. Hi autem coefficientes ita a ſe inuicem pendent, vt ſit

$$B = \frac{-2(\mu - 2)}{n} \int A n dn - 2 A n \text{ et}$$

$$C = \frac{2B + 2\mu A n}{(\mu - 2) n}; \quad D = \frac{4C + (\mu + 1) B n}{(\mu - 3) n}$$

$$E = \frac{6D + (\mu + 2) C n}{(\mu - 4) n}; \quad F = \frac{8E + (\mu + 3) D n}{(\mu - 5) n}$$

$$G = \frac{10F + (\mu + 4) E n}{(\mu - 6) n}; \quad H = \frac{12G + (\mu + 5) F n}{(\mu - 7) n}$$

etc.

Vbi incommodo, quando μ eſt numerus integer, ſupra iam remedium eſt alltaum. Hic igitur praecipue inueſtigamus quomodo coefficientes cuiusque caſus ex caſu praecedente determinari queant, quod ita fieri poterit. Cum ſit

$$\frac{1}{(1 + n \operatorname{cof.} \Phi)^{\mu}} = A + B \operatorname{cof.} \Phi + C \operatorname{cof.} 2\Phi + D \operatorname{cof.} 3\Phi + \text{etc.}$$

ponatur

$$\frac{1}{(1 + n \operatorname{cof.} \Phi)^{\mu+1}} = A' + B' \operatorname{cof.} \Phi + C' \operatorname{cof.} 2\Phi + D' \operatorname{cof.} 3\Phi + \text{etc.}$$

haec igitur ſeries per $1 + n \operatorname{cof.} \Phi$ multiplicata in illam abire debet, eſt autem productum

$$\begin{array}{llll} A' & + B' \operatorname{cof.} \Phi & + C' \operatorname{cof.} 2\Phi & + D' \operatorname{cof.} 3\Phi + \text{etc.} \\ & + A' n & + \tfrac{1}{2} B' n & + \tfrac{1}{2} C' n \\ + \tfrac{1}{2} B' n & + \tfrac{1}{2} C' n & + \tfrac{1}{2} D' n & + \tfrac{1}{2} E' n \end{array}$$

 vnde

vnde colligimus

$$B' = \frac{2(A - A')}{n}; \qquad C' = \frac{2(B - B') - 2A'n}{n}$$

$$D' = \frac{2(C - C') - B'n}{n}; \quad E' = \frac{2(D - D') - C'n}{n} \text{ etc.}$$

dummodo ergo coefficiens A' constaret, sequentes B', C', D etc. haberemus. Videamus igitur quomodo A' ex A determinari possit: quia est

$$A = 1 + \frac{\mu(\mu+1)}{2. \quad 2.} n^2 + \frac{\mu(\mu+1)(\mu+2)(\mu+3)}{2. \quad 2. \quad 4. \quad 4} n^4 \text{ etc.}$$

$$A' = 1 + \frac{(\mu+1)(\mu+2)}{2. \quad 2.} n^2 + \frac{(\mu+1)(\mu+2)(\mu+3)(\mu+4)}{2. \quad 2. \quad 4. \quad 4} n^4 \text{ etc.}$$

tractetur n vt variabilis, ac prior series per n^μ multiplicata differentietur, vt prodeat

$$\frac{d.An^\mu}{dn} = \mu n^{\mu-1} + \frac{\mu(\mu+1)(\mu+2)n^{\mu+1}}{2. \quad 2}$$

$$+ \frac{\mu(\mu+1)(\mu+2)(\mu+3)(\mu+4)}{2. \quad 2. \quad 4. \quad 4} n^{\mu+3} \text{ etc.}$$

quae series manifesto est $= \mu n^{\mu-1} A'$; quocirca A' ita per A determinatur, vt sit

$$A' = \frac{d.An^\mu}{d.n^\mu} = A + \frac{n\,dA}{\mu\,dn}.$$

Cum igitur pro casu $\mu = 1$ inuenerimus

$$A = \frac{1}{\sqrt{(1-nn)}}; \text{ ob } \frac{dA}{dn} = \frac{n}{(1-nn)^{\frac{3}{2}}}$$

$$\text{erit } A' = \frac{1}{\sqrt{(1-nn)}} + \frac{nn}{(1-nn)^{\frac{3}{2}}} = \frac{1}{(1-nn)^{\frac{3}{2}}}.$$

Hic

Hic iam eſt valor ipſius A pro $\mu = 2$, vnde ob $\frac{dA}{dn} = \frac{3n}{(1-nn)^{\frac{5}{2}}}$ fiet pro $\mu = 3$;

$$A = \frac{1}{(1-nn)^{\frac{3}{2}}} + \frac{3nn}{2(1-nn)^{\frac{5}{2}}} = \frac{1+\frac{1}{2}nn}{(1-nn)^{\frac{5}{2}}}.$$

Hoc modo ſi vlterius progrediamur, reperiemus:

ſi $\mu = 1$; $A = \frac{1}{\sqrt{(1-nn)}}$

ſi $\mu = 2$; $A = \frac{1}{(1-nn)\sqrt{(1-nn)}}$

ſi $\mu = 3$; $A = \frac{1+\frac{1}{2}nn}{(1-nn)^2\sqrt{(1-nn)}}$

ſi $\mu = 4$; $A = \frac{1+\frac{3}{2}nn}{(1-nn)^3\sqrt{(1-nn)}}$

ſi $\mu = 5$; $A = \frac{1+3nn+\frac{3}{8}n^4}{(1-nn)^4\sqrt{(1-nn)}}$.

Coroll. 1.

287. Eodem modo etiam reliqui coefficientes B′, C′ etc. ex analogis B, C etc. definientur, eruntque omnes iſtae relationes inter ſe ſimiles, ſcilicet

vti eſt $A' = \frac{d.An^\mu}{d.n^\mu} = A + \frac{n\,dA}{\mu\,dn}$

ita erit $B' = \frac{d\,Bn^\mu}{d.n^\mu} = B + \frac{n\,dB}{\mu\,dn}$

$C' = \frac{d.Cn^\mu}{d.n^\mu} = C + \frac{n\,dC}{\mu\,dn}$

etc.

 Coroll. 2.

Coroll. 2.

288. At ante inuenimus $B' = \frac{2(A - A')}{n}$, vnde fiet

$$B' = -\frac{2\,dA}{\mu\,dn} = B + \frac{n\,dB}{\mu\,dn}, \text{ hincque}$$

$$\mu\,B\,dn + n\,dB + 2\,dA = 0$$

mult. per $n^{\mu-1}$ vt fit

$$d\,Bn^{\mu} + 2\,n^{\mu-1}\,dA = 0,$$

vnde integrando

$$Bn^{\mu} = -2\int n^{\mu-1}\,dA = -2\,n^{\mu-1}\,A + 2(\mu-1)\int A\,n^{\mu-2}\,dn$$

ideoque

$$B = -\frac{2\,A}{n} + \frac{2(\mu-1)}{n^{\mu}}\int A\,n^{\mu-2}\,dn.$$

At ante habueramus

$$B = -2\,An - \frac{2(\mu-2)}{n}\int A\,n\,dn.$$

Coroll. 3.

289. His valoribus aequatis obtinetur aequatio inter A et n, qua quantitas A per n determinatur, erit enim

$$n^{-\mu}\int n^{\mu-1}\,dA = An + \frac{(\mu-2)}{n}\int A\,n\,dn$$

vnde per duplicem differentiationem prodit

$$(1-nn)\,ddA + \frac{dn\,dA}{n} - 2(\mu+1)\,n\,dn\,dA - \mu(\mu+1)\,A\,dn^2 = 0.$$

Scholion 1.

290. Si hos valores ipsius A cum superioribus, vbi μ erat numerus integer negatiuus, inter

inter se comparemus, eximiam conuenientiam deprehendemus.

Pro superioribus.	Pro his formulis.
si $\nu=0$; $A=1$	si $\mu=1$; $A=\frac{1}{\sqrt{(1-nn)}}$
$\nu=1$; $A=1$	$\mu=2$; $A=\frac{1}{(1-nn)\sqrt{(1-nn)}}$
$\nu=2$; $A=1+\frac{1}{2}nn$	$\mu=3$; $A=\frac{1+\frac{1}{2}nn}{(1-nn)^2\sqrt{(1-nn)}}$
$\nu=3$; $A=1+\frac{3}{2}n^2$	$\mu=4$; $A=\frac{1+\frac{3}{2}nn}{(1-nn)^3\sqrt{(1-nn)}}$
$\nu=4$; $A=1+3n^2+\frac{3}{8}n^4$	$\mu=5$; $A=\frac{1+3nn+\frac{3}{8}n^4}{(1-nn)^4\sqrt{(1-nn)}}$

etc.

vnde concludimus si fuerit

$$(1+n\cos\Phi)^{\nu}=A+B\cos\Phi+C\cos 2\Phi+\text{etc.}$$

$$(1+n\cos\Phi)^{-\nu-1}=\mathfrak{A}+\mathfrak{B}\cos\Phi+\mathfrak{C}\cos 2\Phi+\text{etc.}$$

fore $\mathfrak{A}=\frac{A}{(1-nn)^{\nu}\sqrt{(1-nn)}}$.

Quare cum pro casibus, quibus ν est numerus integer positiuus, valor ipsius A facile definiatur, etiam pro casibus, quibus est negatiuus, inde expedite assignabitur.

Scholion 2.

291. Cum pro casu $\mu=1$, supra valores singularum litterarum A, B, C, D etc. sint inventi, scilicet posito breuitatis gratia $\frac{1-\sqrt{(1-nn)}}{n}=m$,

$A=\frac{1}{\sqrt{(1-nn)}}$; $B=\frac{2m}{\sqrt{(1-nn)}}$; $C=\frac{2mm}{\sqrt{(1-nn)}}$; $D=\frac{2m^3}{\sqrt{(1-nn)}}$ et

et in genere pro termino quocunque $N = \frac{2m^\lambda}{\sqrt{(1-nn)}}$, si pro simili termino casu $\mu = 2$, scribamus N' erit $N' = \frac{d.Nn}{dn}$. Nunc autem est $\frac{d.Nn}{dn} = \frac{2m^\lambda}{(1-nn)^{\frac{3}{2}}}$ $+ \frac{2\lambda n m^{\lambda-1} dm}{dn\sqrt{(1-nn)}}$: tum vero $\frac{dm}{dn} = \frac{m}{n\sqrt{(1-nn)}}$, vnde colligimus

$$N' = \frac{2m^\lambda}{(1-nn)^{\frac{3}{2}}} + \frac{2\lambda m^\lambda}{1-nn} = \frac{2m^\lambda(1+\lambda\sqrt{(1-nn)})}{(1-nn)\sqrt{(1-nn)}}.$$

Quare si statuamus:

$$\frac{1}{(1+n\cos.\phi)^2} = A + B\cos.\phi + C\cos.2\phi + D\cos.3\phi + E\cos.4\phi \text{ etc.}$$

erit

$$A = \frac{1}{(1-nn)^{\frac{3}{2}}};\ B = \frac{2m(1+\sqrt{(1-nn)})}{(1-nn)^{\frac{3}{2}}};\ C = \frac{2m^2(1+2\sqrt{(1-nn)})}{(1-nn)^{\frac{3}{2}}};$$
$$D = \frac{2m^3(1+3\sqrt{(1-nn)})}{(1-nn)^{\frac{3}{2}}} \text{ etc.}$$

Verum si exponens μ fuerit numerus fractus, coefficientes A, B, C, D, E etc. haud aliter, ac per series supra datas definiri posse videntur. Primus autem A modo peculiari vero proxime assignari potest, quemadmodum in problemate sequente docemus.

Problema 34.

Problema 34.

292. Pro euolutione formulae $(1+n\cos.\Phi)^{\nu}$ in huiusmodi feriem $A+B\cos.\Phi+C\cos.2\Phi+D\cos.3\Phi+E\cos.4\Phi$ etc. terminum abfolutum A vero proxime definire.

Solutio.

Cum neceffario fit $n<1$, feries quidem fupra inuenta pro A conuergit, verum fi n parum ab vnitate deficiat, permultos terminos actu euolui oportet, antequam valor ipfius A fatis exacte prodeat, praecipue fi ν fuerit numerus mediocriter magnus tam pofitiuus quam negatiuus. Quoniam tamen pofita euolutione huius formulae $(1+n\cos.\Phi)^{-\nu-1}$ $=\mathfrak{A}+\mathfrak{B}\cos.\Phi+\mathfrak{C}\cos.2\Phi+$etc. a termino $\mathfrak{A}$ ille A ita pendet vt fit $A=(1-nn)^{\nu+\frac{1}{2}}\mathfrak{A}$ pro hoc termino A inueniendo duplicem habemus feriem

$$A=1+\frac{\nu(\nu-1)}{2\cdot 2}n^2+\frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2\cdot 2\cdot 4\cdot 4}n^4+\frac{\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)(\nu-5)}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}n^6 \text{ etc.}$$

$$A=(1-nn)^{\nu+\frac{1}{2}}\Big(1+\frac{(\nu+1)(\nu+2)}{2\cdot 2}n^2+\frac{(\nu+1)(\nu+2)(\nu+3)(\nu+4)}{2\cdot 2\cdot 4\cdot 4}n^4$$
$$+\frac{(\nu+1)(\nu+2)(\nu+3)(\nu+4)(\nu+5)(\nu+6)}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}n^6 \text{ etc.}\Big)$$

quouis cafu ea vfurpari poteft, quae magis conuergit. Verum tamen quia reliqui coefficientes B, C, D, E etc. tandem conuergere debent, hinc alia via ad valorem ipfius A appropinquandi patet. Quoniam enim hi coefficientes alternatim per pares et impa-

res poteſtates ipſius n definiuntur, ſumto angulo quocunque α erit

$$(1+n\cos.\alpha)^\nu = A + B\cos.\alpha + C\cos.2\alpha + D\cos.3\alpha + E\cos 4\alpha + \text{etc.}$$

et $$(1+n\cos.\alpha)^\nu = A - B\cos.\alpha + C\cos.2\alpha - D\cos.3\alpha + E\cos.4\alpha - \text{etc.}$$

His igitur additis prodit

$$\tfrac{1}{2}(1+n\cos.\alpha)^\nu + \tfrac{1}{2}(1-n\cos.\alpha)^\nu = A + C\cos.2\alpha + E\cos.4\alpha + G\cos.6\alpha + \text{etc.}$$

vbi ſi pro α ſcribamus $90^\circ - \alpha$, erit

$$\tfrac{1}{2}(1+n\sin.\alpha)^\nu + \tfrac{1}{2}(1-\sin.\alpha)^\nu = A - C\cos 2\alpha + E\cos.4\alpha - G\cos.6\alpha + \text{etc.}$$

vnde his additis ſemiſſis terminorum denuo tollitur. Formemus plures huiusmodi expreſſiones, ac ponamus breuitatis gratia:

$$\tfrac{1}{4}(1+n\cos.\alpha)^\nu + \tfrac{1}{4}(1-n\cos.\alpha)^\nu + \tfrac{1}{4}(1+n\sin.\alpha)^\nu + \tfrac{1}{4}(1-n\sin.\alpha)^\nu = \mathfrak{A}$$
$$\tfrac{1}{4}(1+n\cos.\beta)^\nu + \tfrac{1}{4}(1-n\cos.\beta)^\nu + \tfrac{1}{4}(1+n\sin.\beta)^\nu + \tfrac{1}{4}(1-n\sin\beta)^\nu = \mathfrak{B}$$
$$\tfrac{1}{4}(1+n\cos\gamma)^\nu + \tfrac{1}{4}(1-n\cos\gamma)^\nu + \tfrac{1}{4}(1+n\sin\gamma)^\nu + \tfrac{1}{4}(1-n\sin.\gamma)^\nu = \mathfrak{C}$$
etc.

et pro coefficientibus B, C, D, E etc. ſcribamus reſpectiue (1), (2), (3), (4) etc. quo facilius terminos ab initio quantumuis remotos repraeſentare poſſimus. Habebimus ergo

$$\mathfrak{A} = A + (4)\cos.4\alpha + (8)\cos.8\alpha + (12)\cos.12\alpha \text{ etc.}$$
$$\mathfrak{B} = A + (4)\cos.4\beta + (8)\cos.8\beta + (12)\cos.12\beta \text{ etc.}$$
$$\mathfrak{C} = A + (4)\cos.4\gamma + (8)\cos.8\gamma + (12)\cos.12\gamma \text{ etc.}$$
etc.

Atque

Atque hinc ſequentes approximationes adipiſcimur.

I. Si capiamus $4\alpha = \frac{\pi}{2}$ ſeu $\alpha = \frac{\pi}{8}$, prodit

$$\mathfrak{A} = A - (8) + (16) - (24) + \text{etc.}$$

Ergo $A = \mathfrak{A} + (8) - (16) + (24) - \text{etc.}$

Quare ſi termini (8) et ſequentes ob paruitatem contemni queant, erit ſatis exacte $A = \mathfrak{A}$.

II. Sumamus duas ſeries ac ſtatuamus $4\alpha = \frac{\pi}{4}$ et $4\beta = \frac{3\pi}{4}$ vt ſit $\alpha = \frac{\pi}{16}$ et $\beta = \frac{3\pi}{16}$ erit $\cos. 4\alpha + \cos. 4\beta = 0$, $\cos. 8\alpha + \cos. 8\beta = 0$, $\cos. 12\alpha + \cos. 12\beta = 0$ et $\cos. 16\alpha + \cos. 16\beta = -2$, vnde ſequitur:

$$\mathfrak{A} + \mathfrak{B} = 2A - 2(16) + 2(32) - 2(48) + \text{etc.}$$

ideoque

$$A = \tfrac{1}{2}(\mathfrak{A} + \mathfrak{B}) + (16) - (32) \text{ etc.}$$

vbi numeri (16), (32) plerumque tam erunt parvi, vt negligi queant.

III. Addamus tres ſeries ac ſtatuamus $4\alpha = \frac{\pi}{6}$; $4\beta = \frac{3\pi}{6}$; $4\gamma = \frac{5\pi}{6}$ vt ſit $\alpha = \frac{\pi}{24}$; $\beta = \frac{\pi}{8}$; $\gamma = \frac{5\pi}{24}$ eritque

$\cos. 4\alpha + \cos. 4\beta + \cos. 4\gamma = 0$	$\cos. 16\alpha + \cos. 16\beta + \cos. 16\gamma = 0$
$\cos. 8\alpha + \cos. 8\beta + \cos. 8\gamma = 0$	$\cos. 20\alpha + \cos. 20\beta + \cos. 20\gamma = 0$
$\cos. 12\alpha + \cos. 12\beta + \cos. 12\gamma = 0$	$\cos. 24\alpha + \cos. 24\beta + \cos. 24\gamma = -3$

vnde colligitur

$$A = \tfrac{1}{3}(\mathfrak{A} + \mathfrak{B} + \mathfrak{C}) + (24) - (48) \text{ etc.}$$

IV. Si haec determinatio non satis exacta videatur, addantur quatuor eiusmodi expressiones $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$, sitque

$$4\alpha = \frac{\pi}{8};\ 4\beta = \frac{3\pi}{8};\ 4\gamma = \frac{5\pi}{8},\ 4\delta = \frac{7\pi}{8}$$

ac reperietur

$$\mathfrak{A} + \mathfrak{B} + \mathfrak{C} + \mathfrak{D} = 4\mathrm{A} - 4(32) + 4(64) \text{ etc.}$$

ergo multo propius

$$\mathrm{A} = \tfrac{1}{4}(\mathfrak{A} + \mathfrak{B} + \mathfrak{C} + \mathfrak{D}).$$

Coroll. 1.

293. Ex inuento autem valore A sequens B satis expedite reperitur, cum sit

$$\mathrm{B} = \frac{2(\nu+2)}{n}\int \mathrm{A}\,n\,dn - 2\,\mathrm{A}\,n.$$

Quatenus ergo in A ingreditur membrum $(1 \pm n\cos.\alpha)^\nu$ vel $(1 + nf)^\nu$ dum f omnes illos sinus et cosinus complectitur, inde pro B oritur

$$\frac{2(\nu+2)}{n}\int n\,dn(1+nf)^\nu - 2n(1+nf)^\nu = \frac{2-2(1-\nu nf)(1+nf)^\nu}{(\nu+1)nff}.$$

Coroll. 2.

294. Cognitis autem coefficientibus A et B, quemadmodum sequentes omnes ex illis deriuari possint, supra ostendimus. Iis vero inuentis integratio formulae $d\Phi(1 + n\cos.\Phi)^\nu$ per se est manifesta.

Problema 35.

295. Integrale formulae $d\Phi\, l(1 + n\cos.\Phi)$ per seriem secundum sinus angulorum $\Phi, 2\Phi, 3\Phi$ etc. progredientem euoluere.

Solutio.

Solutio.

Cum sit

$$l(1+n\cos.\Phi)=n\cos.\Phi-\tfrac{1}{2}n^2\cos.\Phi^2+\tfrac{1}{3}n^3\cos.\Phi^3-\tfrac{1}{4}n^4\cos.\Phi^4 \text{ etc.}$$

erit his poteſtatibus ad ſimplices coſinus reductis.

$$\begin{array}{llll} l(1+n\cos.\Phi)= & +n\cos.\Phi-\tfrac{1}{2}\cdot\tfrac{1}{2}nn\cos.2\Phi+\tfrac{1}{3}\cdot\tfrac{1}{4}n^3\cos.3\Phi-\tfrac{1}{4}\cdot\tfrac{1}{8}n^4\cos.4\Phi \\ & -\tfrac{1}{2}\cdot\tfrac{1}{2}n^2+\tfrac{1}{3}\cdot\tfrac{3}{4}n^3-\tfrac{1}{4}\cdot\tfrac{4}{8}n^4 \qquad +\tfrac{1}{5}\cdot\tfrac{5}{16}n^5 \\ & -\tfrac{1}{4}\cdot\tfrac{3}{8}n^4+\tfrac{1}{5}\cdot\tfrac{10}{16}n^5-\tfrac{1}{6}\cdot\tfrac{15}{32}n^6 \\ & -\tfrac{1}{6}\cdot\tfrac{10}{32}n^6+\tfrac{1}{7}\cdot\tfrac{35}{64}n^7 \\ & -\tfrac{1}{8}\cdot\tfrac{35}{128}n^8. \end{array}$$

Quare ſi ponamus

$$l(1+n\cos.\Phi)=-A+B\cos.\Phi-C\cos.2\Phi+D\cos.3\Phi-\text{etc.}$$

erit

$$A=+\tfrac{1}{2}\cdot\tfrac{n^2}{2}+\tfrac{1\cdot3}{2\cdot4}\cdot\tfrac{n^4}{4}+\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\tfrac{n^6}{6}+\tfrac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\cdot\tfrac{n^8}{8}+\text{etc.}$$

conſiderato ergo numero n vt variabili, erit

$$\frac{n\,dA}{dn}=\tfrac{1}{2}nn+\tfrac{1\cdot3}{2\cdot4}n^4+\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}n^6+\text{etc.}=\frac{1}{\sqrt{(1-nn)}}-1.$$

Hinc $dA=\frac{dn}{n\sqrt{(1-nn)}}-\frac{dn}{n}$, vnde integratio praebet;

$$A=l\frac{1-\sqrt{(1-nn)}}{n}-ln+C=l\frac{2-2\sqrt{(1-nn)}}{nn}$$

hoc enim modo euaneſcente n fit $A=l1=0$.

Tum vero erit

$$\tfrac{1}{2}B=\tfrac{1}{2}\cdot n+\tfrac{1\cdot3}{2\cdot4}\cdot\tfrac{n^3}{3}+\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\tfrac{n^5}{5}+\text{etc.}$$

vnde differentiatio praebet

$$\frac{nn\,dB}{2\,dn}=\tfrac{1}{2}nn+\tfrac{1\cdot3}{2\cdot4}n^4+\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}n^6+\text{etc.}=\frac{1}{\sqrt{(1-nn)}}-1$$

 ergo

ergo $\frac{1}{2}dB = \frac{dn}{nn\surd(1-nn)} - \frac{dn}{nn}$ et integrando

$$\tfrac{1}{2}B = \frac{-\surd(1-nn)}{n} + \frac{1}{n} + C = \frac{1-\surd(1-nn)}{n}$$

integrali ita determinato vt euanefcat pofito $n=0$.
Quocirca pro binis primis terminis habemus:

$$A = l\,\frac{2-2\surd(1-nn)}{nn} \text{ et } B = \frac{2-2\surd(1-nn)}{n}$$

vt fit $A = l\frac{B}{n}$. At pro reliquis differentiemus aequationem affumtam:

$$\frac{-n\,d\Phi\,\text{fin.}\Phi}{1+n\,\text{cof.}\Phi} = -Bd\Phi\,\text{fin.}\Phi + 2Cd\Phi\,\text{fin.}2\Phi - 3Dd\Phi\,\text{fin.}3\Phi$$
$$+ 4Ed\Phi\,\text{fin.}4\Phi$$

feu

$$0 = \frac{n\,\text{fin.}\Phi}{1+n\,\text{cof.}\Phi} - B\,\text{fin.}\Phi + 2C\,\text{fin.}2\Phi - 3D\,\text{fin.}3\Phi$$
$$+ 4E\,\text{fin.}4\Phi - \text{etc.}$$

Quare per $2+2n\,\text{cof.}\Phi$ multiplicando prodit:

$$\begin{array}{rcccc} 0 = 2n\,\text{fin.}\Phi - 2B\,\text{fin.}\Phi & +4C\,\text{fin.}2\Phi & -6D\,\text{fin}\,3\Phi & +8E\,\text{fin.}4\Phi & -\text{etc.} \\ & -Bn & +2Cn & -3Dn & \\ +2Cn & -3Dn & +4En & -5Fn & \end{array}$$

vnde colligimus:

$$C = \frac{B-n}{n};\; D = \frac{4C-Bn}{3n};\; E = \frac{6D-2Cn}{4n};\; F = \frac{8E-3Dn}{5n}.$$

Cum igitur fit $B = \frac{2-2\surd(1-nn)}{n}$ erit $C = \frac{2-nn-2\surd(1-nn)}{nn}$, feu $C = \left(\frac{1-\surd(1-nn)}{n}\right)^2$, tum vero $D = \frac{2}{3}\left(\frac{1-\surd(1-nn)}{n}\right)^3$; $E = \frac{2}{4}\left(\frac{1-\surd(1-nn)}{n}\right)^4$; $F = \frac{2}{5}\left(\frac{1-\surd(1-nn)}{n}\right)^5$ etc.
Hinc fi breuitatis gratia ponamus $\frac{1-\surd(1-nn)}{n} = m$ erit

$$l(1+n\,\text{cof.}\Phi) = -l\frac{2m}{n} + \tfrac{2}{1}m\,\text{cof.}\Phi - \tfrac{2}{2}m^2\,\text{cof.}2\Phi + \tfrac{2}{3}m^3\,\text{cof.}3\Phi$$
$$- \tfrac{2}{4}m^4\,\text{cof.}4\Phi$$

ideoque

ideoque integrale quaesitum:

$$\int d\Phi(1+n\,\text{cos}.\Phi) = \text{Const.} - \Phi\, l\tfrac{2m}{n} + \tfrac{2}{1}m\,\text{sin}.\Phi - \tfrac{2}{4}m^2\,\text{sin}.2\Phi + \tfrac{2}{9}m^3\,\text{sin}.3\Phi - \tfrac{2}{16}m^4\,\text{sin}.4\Phi + \tfrac{2}{25}m^5\,\text{sin}.5\Phi - \text{etc.}$$

Corollarium.

296. Quodsi ergo ponamus $n=1$ erit $m=1$ et

$$l(1+\text{cos}.\Phi) = -l2 + \tfrac{2}{1}\text{cos}.\Phi - \tfrac{2}{2}\text{cos}.2\Phi + \tfrac{2}{3}\text{cos}.3\Phi - \tfrac{2}{4}\text{cos}.4\Phi + \text{etc.}$$

et

$$l(1-\text{cos}.\Phi) = -l2 - \tfrac{2}{1}\text{cos}.\Phi - \tfrac{2}{2}\text{cos}.2\Phi - \tfrac{2}{3}\text{cos}.3\Phi - \tfrac{2}{4}\text{cos}.4\Phi - \text{etc.}$$

Cum iam sit $1+\text{cos}.\Phi = 2\,\text{cos}.\tfrac{1}{2}\Phi^2$ et $1-\text{cos}.\Phi = 2\,\text{sin}.\tfrac{1}{2}\Phi^2$ erit

$$l\,\text{cos}.\tfrac{1}{2}\Phi = -l2 + \text{cos}.\Phi - \tfrac{1}{2}\text{cos}.2\Phi + \tfrac{1}{3}\text{cos}.3\Phi - \tfrac{1}{4}\text{cos}.4\Phi + \text{etc.}$$ et

$$l\,\text{sin}.\tfrac{1}{2}\Phi = -l2 - \text{cos}.\Phi - \tfrac{1}{2}\text{cos}.2\Phi - \tfrac{1}{3}\text{cos}.3\Phi - \tfrac{1}{4}\text{cos}.4\Phi - \text{etc.}$$ hinc

$$l\,\text{tang}.\tfrac{1}{2}\Phi = -2\,\text{cos}.\Phi - \tfrac{2}{3}\text{cos}.3\Phi - \tfrac{2}{5}\text{cos}.5\Phi - \tfrac{2}{7}\text{cos}.7\Phi - \text{etc.}$$

CAPVT VII.

CAPVT VII.

METHODVS GENERALIS INTEGRALIA QVAECVNQVE PROXIME INVENIENDI.

Problema 36.

297.

Formulae integralis cuiuscunque $y = \int X dx$ valorem vero proxime indagare.

Solutio.

Cum omnis formula integralis per se sit indeterminata, ea semper ita determinari solet, vt si variabili x certus quidam valor puta a tribuatur, ipsum integrale $y = \int X dx$ datum valorem puta b obtineat. Integratione igitur hoc modo determinata, quaestio huc redit, si variabili x alius quicunque valor ab a diuersus tribuatur, valor, quem tum integrale y sit habiturum, definiatur. Tribuamus ergo ipsi x primo valorem parum ab a discrepantem, puta $x = a + \alpha$, vt α sit quantitas valde parva: et quia functio X parum variatur, siue pro x scribatur a siue $a + \alpha$ eam tanquam constantem spectare licebit. Hinc ergo formulae differentialis $X dx$ integrale

integrale erit $Xx + \text{Conſt.} = y$; ſed quia poſito $x = a$ fieri debet $y = b$, et valor ipſius X quaſi manet immutatus, erit $Xa + \text{Conſt.} = b$, ideoque $\text{Conſt.} = b - Xa$, vnde conſequimur $y = b + X(x-a)$. Quare ſi ipſi x valorem $a + \alpha$ tribuamus, habebimus valorem conuenientem ipſius y, qui ſit $= b + \beta$; ac iam ſimili modo ex hoc caſu definire poterimus y, ſi ipſi x tribuatur alius valor parum ſuperans $a + \alpha$, poſito igitur $a + \alpha$ loco x, valor ipſius X inde ortus denuo pro conſtante haberi poterit, indeque fiet $y = b + \beta + X(x - a - \alpha)$. Hanc igitur operationem continuare licet quousque lubuerit, cuius ratio quo melius perſpiciatur, rem ita repraeſentemus:

ſi $x = a$ fiat $X = A$ et $y = b$

ſi $x = a'$ . . $X = A'$ . . $y = b' = b + A(a' - a)$

ſi $x = a''$ . . $X = A''$ . . $y = b'' = b' + A'(a'' - a')$

ſi $x = a'''$ . . $X = A'''$ . . $y = b''' = b'' + A''(a''' - a'')$

etc.

vbi valores a, a', a'', a''' etc. ſecundum differentias valde paruas procedere ponuntur. Erit ergo $b' = b + A(a' - a)$ quippe in quam abit formula inuenta $y = b + X(x - a)$ fit enim $X = A$, quia ponitur $x = a$, tum vero tribuitur ipſi x valor $= a'$; cui reſpondet $y = b'$, ſimili modo erit $b'' = A'(a'' - a')$; tum $b''' = b'' + A''(a''' - a'')$ etc. vti ſupra poſuimus.

mns. Reſtituendo ergo valores praecedentes habebimus:

$$b' = b + A(a' - a)$$
$$b'' = b + A(a' - a) + A'(a'' - a')$$
$$b''' = b + A(a' - a) + A'(a'' - a') + A''(a''' - a'')$$
$$b'''' = b + A(a' - a) + A'(a'' - a') + A''(a''' - a'') + A(a'''' - a''')$$
etc.

vnde ſi x quantumuis excedet a, ſeries a', a'', a''' etc. creſcendo continuetur ad x, et vltimum aggregatum dabit valorem ipſius y.

Coroll. 1.

298. Si incrementa, quibus x augetur, aequalia ſtatuantur ſcilicet α, vt ſit $a' = a + \alpha$, $a'' = a + 2\alpha$, $a''' = a + 3\alpha$, etc. quibus valoribus pro x ſubſtitutis functio X abeat in A', A'', A''' etc. atque vltimus illorum valorum puta $a + n\alpha$ ſit $= x$ horum vero X, erit

$$y = b + \alpha(A + A' + A'' + A''' \ldots\ldots + X).$$

Coroll. 2.

299. Valor ergo integralis y per ſummationem ſeriei A, A', $A'' \ldots\ldots X$, cuius termini ex formula X formantur ponendo loco x ſucceſſiue a, $a + \alpha$, $a + 2\alpha \ldots\ldots a + n\alpha$, eruitur. Summa enim illius ſeriei per differentiam α multiplicata et ad b adiecta dabit valorem ipſius y, qui ipſi $x = a + n\alpha$ reſpondet.

Coroll. 3.

Coroll. 3.

300. Quo minores ſtatuuntur differentiae, ſecundum quas valor ipſius x increſcat, eo accuratius hoc modo valor ipſius y definitur. Siquidem termini ſeriei A, A′, A″, etc. inde etiam ſecundum paruas differentias progrediantur, niſi enim hoc eueniat, illa determinatio nimis erit incerta.

Coroll. 4.

301. Haec ergo approximatio ex doctrina ſerierum ita explicatur:

Ex indicibus $a, a', a'', a''' \ldots\ldots x$ formetur
ſeries $\mathrm{A}, \mathrm{A}', \mathrm{A}'', \mathrm{A}''' \ldots\ldots \mathrm{X}$

cuius ergo terminus generalis X ex formula differentiali $dy = \mathrm{X}dx$ datur. Tum in hac ſerie ſit terminus vltimum praecedens $'\mathrm{X}$, reſpondens indici $'x$; hincque noua formetur ſeries

$$\mathrm{A}(a'-a);\ \mathrm{A}'(a''-a');\ \mathrm{A}''(a'''-a'')\ \ldots\ldots\ '\mathrm{X}(x-'x)$$

cuius ſumma ſi ponatur $=\mathrm{S}$ erit integrale $y = \int \mathrm{X}dx = b + \mathrm{S}$, proxime

Scholion 1.

302. Hoc modo integratio vulgo explicari ſolet, vt dicatur eſſe ſummatio omnium valorum formulae differentialis $\mathrm{X}dx$ ſi variabili x ſucceſſiue omnes valores a dato quodam a vsque ad x tribuantur, qui ſecundum differentiam dx procedunt,

 hanc

hanc differentiam autem infinite paruam accipi oportere. Similis igitur haec ratio integrationem repraesentandi est illi, qua in Geometria lineae vt aggregata infinitorum punctorum concipi solent, quae idea, quemadmodum si rite explicetur, admitti potest, ita etiam illa integrationis explicatio tolerari potest, dummodo ad vera principia, vti hic fecimus, reuocetur, vt omni cauillationi occurratur. Ex methodo igitur exposita vtique patet integrationem per summationem vero proxime obtineri posse, neque vero exacte expediri, nisi differentiae infinite paruae, hoc est nullae, statuantur. Atque ex hoc fonte tam nomen integrationis, quae etiam summatio vocari solet, quam signum integralis $\int$ est natum, quae re bene explicata, omnino retineri possunt.

Scholion 2.

303. Si pro singulis interuallis, in quae saltum ab a ad x distinximus, quantitates A, A', A'', A''' etc. reuera essent constantes, integrale $\int X dx$ accurate impetraremus. Eatenus ergo error inest quatenus pro singulis illa interuallis istae quantitates non sunt constantes. Ac pro primo quidem interuallo, quo variabilis x a termino a ad a' procedit, A est valor ipsius X termino a conueniens, alteri autem termino a' respondet A'; vnde quatenus non est $A' = A$, eatenus error irrepit: cum igitur in istius interualli initio sit $X = A$, in fine autem $X = A'$, conueniret potius medium quoddam inter

inter A et A′ assumi, id quod in correctione huius methodi mox tradenda obseruabitur. Interim hic notasse iuuabit, pari iure pro quouis interuallo valorem tam finalem quam initialem capi posse, vbi simul hoc perspicitur, si altero modo in excessu peccetur, altero plerumque in defectu errari. Ex quo hinc binas expressiones erruere licet, quarum altera valorem ipsius y nimis magnum, altera nimis paruum sit praebitura, ita vt illae quasi limites veri valoris ipsius y constituant. Quemadmodum ergo rem repraesentauimus §. 301. valor ipsius $y = \int X dx$ intra hos duos limites continebitur

$$b + A(a'-a) + A'(a''-a') + A''(a'''-a'') \ldots + {}'X(x-{}'x) \text{ et}$$

$$b + A'(a'-a) + A''(a''-a') + A'''(a'''-a'') \ldots + X(x-{}'x)$$

quibus cognitis ad veritatem propius accedere licet.

Scholion. 3.

304. Iam notauimus interualla illa, per quae x successiue increscere assumimus, ideo valde parua statui debere, vt valores respondentes A, A′, A″ etc. parum a se inuicem discrepent: atque hinc potissimum iudicari oportet, vtrum illa interualla $a'-a$, $a''-a'$, $a'''-a''$ etc. inter se aequalia an inaequalia capi conueniat. Vbi enim valor ipsius X mutando x parum mutatur, ibi interualla, per quae x procedit, tuto maiora constitui possunt, vbi autem euenit, vt ipsi x leui mutatione inducta, functio X

vehementer varietur, ibi interualla minima accipi debent. Veluti si sit $X = \frac{1}{\sqrt{(1-xx)}}$ perspicuum est, vbi x proxime ad vnitatem accedit, quantumuis paruum interuallum, per quod x augeatur, accipiatur, functionem X maximam mutationem pati posse, quia tandem sumto $x = 1$, ea adeo in infinitum excrescit. His igitur casibus ista approximatione pro eo saltem interuallo, in cuius altero termino X fit infinita, vti non licet; sed huic incommodo facile remedium affertur, dum formula ope idoneae substitutionis in aliam transformatur, vel dum pro hoc saltem interuallo peculiaris integratio instituitur. Veluti si proposita sit formula $\frac{x\,dx}{\sqrt{(1-x^2)}}$, pro interuallo ab $x = 1 - \omega$ ad $x = 1$ illa methodo integrale non reperitur, at posito $x = 1 - z$, quia termini ipsius z sunt o et ω, erit z quantitas minima, vnde formula erit $\frac{dz(1-z)}{\sqrt{(2z-3z^2+z^3)}} = \frac{dz}{\sqrt{2z}}$, cuius integrale $\frac{2\sqrt{z}}{\sqrt{2}}$ pro interuallo illo praebet partem integralis $\frac{2\sqrt{\omega}}{\sqrt{2}}$. Quod artificium in omnibus huiusmodi casibus adhiberi potest; ipsam autem methodum descriptam aliquot exemplis illustrari opus est.

Exemplum 1.

305. *Integrale* $y = \int x^n dx$ *ita sumtum, vt euanescat posito* $x = 0$, *proxime exhibere.*

Hic est $a = 0$ et $b = 0$, tum $X = x^n$, iam valores ipsius x a o crescant per communem differentiam

tiam α, vt sint

indices $0, \alpha, 2\alpha, 3\alpha, 4\alpha \ldots\ldots x$

series $0, \alpha^n, 2^n\alpha^n, 3^n\alpha^n, 4^n\alpha^n \ldots\ldots x^n$

et terminus vltimum praecedens est $(x-\alpha)^n$, quare integralis $y = \int x^n dx = \frac{1}{n+1} x^{n+1}$ limites sunt

$\alpha(0+\alpha^n+2^n\alpha^n+3^n\alpha^n+\ldots\ldots+(x-\alpha)^n)$ et

$\alpha(\alpha^n+2^n\alpha^n+3^n\alpha^n \ldots\ldots\ldots+x^n)$

qui eo erunt arctiores, quo minus interuallum α accipiatur. Ita si $\alpha = 1$, erunt limites:

$0+1+2^n+3^n+4^n \ldots\ldots+(x-1)^n$

et $1+2^n+3^n+4^n+ \ldots\ldots\ldots+x^n$

si sumatur $\alpha = \frac{1}{2}$ erunt limites:

$$\frac{1}{2^{n+1}}(0+1+2^n+3^n+4^n+ \ldots\ldots (2x-1)^n)$$

et $$\frac{1}{2^{n+1}}(1+2^n+3^n+4^n+ \ldots\ldots\ldots+(2x)^n)$$

ac si in genere sit $\alpha = \frac{1}{m}$ erunt limites:

$$\frac{1}{m^{n+1}}(0+1+2^n+3^n+4^n \ldots\ldots+(mx-1)^n)$$

et $$\frac{1}{m^{n+1}}(1+2^n+3^n+4^n \ldots\ldots\ldots+(mx)^n)$$

quorum hic illum superat excessu $\frac{x^n}{m}$, vnde patet si numerus m sumatur infinitus, vtrumque limitem verum integralis $y = \frac{1}{n+1} x^{n+1}$ esse praebiturum valorem.

Coroll 1.

Coroll. 1.

306. Seriei ergo $1+2^n+3^n+4^n+\dots(mx)^n$ summa eo propius ad $\frac{1}{n+1}(mx)^{n+1}$ accedit, quo maior capiatur numerus m, quare posito $mx=z$, huius progressionis

$$1+2^n+3^n+4^n+\dots\dots z^n$$

summa eo propius ad $\frac{1}{n+1}z^{n+1}$ accedit, quo maior fuerit numerus z.

Coroll. 2.

307. Ex priore autem limite posito $mx=z$, eadem quantitas $\frac{1}{n+1}z^{n+1}$ proxime exhibet summam huius seriei

$$0+1+2^n+3^n+4^n+\dots(z-1)^n$$

vnde medium sumendo erit accuratius:

$$1+2^n+3^n+4^n\dots\dots+(z-1)^n+\tfrac{1}{2}z^n=\frac{1}{n+1}z^{n+1}$$

seu addendo vtrinque $\frac{1}{2}z^n$ habebimus

$$1+2^n+3^n+4^n\dots\dots+z^n=\frac{1}{n+1}z^{n+1}+\tfrac{1}{2}z^n \text{ proxime}$$

quod congruit cum iis, quae de vera huius progressionis summa sunt cognita.

Exemplum 2.

308. *Integrale* $y=\int\frac{dx}{x^n}$ *ita sumtum, vt euanescat posito* $x=1$, *proxime exhibere.*

Erit

Erit ergo $a=1$ et $b=0$, vnde si ab a ad x intervallum progressionis statuatur $=\alpha$, erunt indices a, $a+\alpha$, $a+2\alpha$, $a+3\alpha \ldots x$ et termini seriei $\frac{1}{a^n}, \frac{1}{(a+\alpha)^n}, \frac{1}{(a+2\alpha)^n}, \frac{1}{(a+3\alpha)^n} \ldots \frac{1}{x^n} = X$

vbi terminus vltimum praecedens est $\frac{1}{(x-\alpha)^n} = {}'X$.

Cum nunc nostrum integrale sit $y = \frac{1}{n-1} - \frac{1}{(n-1)x^{n-1}}$ eius valor intra hos limites continebitur:

$$\alpha\left(1+\frac{1}{(1+\alpha)^n}+\frac{1}{(1+2\alpha)^n}+\frac{1}{(1+3\alpha)^n}+\ldots+\frac{1}{(x-\alpha)^n}\right) \text{ et}$$

$$\alpha\left(\frac{1}{(1+\alpha)^n}+\frac{1}{(1+2\alpha)^n}+\frac{1}{(1+3\alpha)^n}\ldots\ldots+\frac{1}{x^n}\right).$$

Quare posito $\alpha = \frac{1}{m}$, erunt hi limites:

$$m^{n-1}\left(\frac{1}{m^n}+\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\frac{1}{(m+3)^n}+\ldots+\frac{1}{(mx-1)^n}\right) \text{ et}$$

$$m^{n-1}\left(\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\frac{1}{(m+3)^n}+\frac{1}{(m+4)^n}+\ldots\frac{1}{(mx)^n}\right)$$

qui, quo maior accipiatur numerus m, eo propius ad valorem integralis $\frac{1}{n-1} - \frac{1}{(n-1)x^{n-1}}$ accedunt. Notandum autem est casu $n=1$ integrale fore $=lx$.

Coroll. 1.

309. Quodsi ponamus $mx = m + z$, vt sit $x = \frac{m+z}{m}$ prodibunt hae progressiones:

$$m^{n-1}\left(\frac{1}{m^n} + \frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \dots\dots + \frac{1}{(m+z-1)^n}\right) \text{ et}$$

$$m^{n-1}\left(\frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \frac{1}{(m+3)^n} + \dots\dots + \frac{1}{(m+z)^n}\right)$$

quarum summa alterius maior est, alterius minor quam $\frac{1}{n-1} - \frac{m^{n-1}}{(n-1)(m+z)^{n-1}} = \frac{(m+z)^{n-1} - m^{n-1}}{(n-1)(m+z)^{n-1}}$; casu autem $n = 1$ haec expressio abit in $l(1 + \frac{z}{m})$.

Coroll. 2.

310. Cum prior progressio maior sit quam posterior, erit

$$\frac{1}{m^n} + \frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \dots + \frac{1}{(m+z-1)^n} > \frac{(m+z)^{n-1} - m^{n-1}}{(n-1)m^{n-1}(m+z)^{n-1}}$$

$$\frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \frac{1}{(m+3)^n} + \dots + \frac{1}{(m+z)^n} < \frac{(m+z)^{n-1} - m^{n-1}}{(n-1)m^{n-1}(m+z)^{n-1}}$$

addatur hic vtrinque $\frac{1}{m^n}$ ibi vero $\frac{1}{(m+z)^n}$ et sumatur medium arithmeticum erit exactius:

$$\frac{1}{m^n} + \frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \frac{1}{(m+3)^n} + \dots + \frac{1}{(m+z)^n}$$
$$= \frac{(2m+n-1)(m+z)^n - (2z+2m-n+1)m^n}{2(n-1)m^n(m+z)^n}$$

quae expressio casu $n = 1$ abit in $l(1 + \frac{z}{m}) + \frac{1}{2m} + \frac{1}{2(m+z)}$.

Coroll. 3.

Coroll. 3.

311. Ponatur $z = mv$, et habebimus sequentis seriei summam proxime expressam:

$$\frac{1}{m^n} + \frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \ldots + \frac{1}{m^n(1+v)^n}$$
$$= \frac{(2m+n-1)(1+v)^n - 2m(1+v) + n - 1}{2(n-1)m^n(1+v)^n}$$

et casu $n = 1$

$$\frac{1}{m} + \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{m+mv} = l(1+v) + \frac{2+v}{2m(1+v)}$$

vnde si $v = 1$ erit proxime

$$\frac{1}{m^n} + \frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \ldots + \frac{1}{2^n m^n}$$
$$= \frac{2^n(2m+n-1) - 4m + n - 1}{2^{n+1}(n-1)m^n}$$ et

$$\frac{1}{m} + \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{2m} = l2 + \frac{3}{4m}.$$

Coroll. 4.

312. Hinc nascitur regula logarithmos quantumuis magnorum numerorum proxime assignandi, dum series vulgares tantum pro numeris parum ab vnitate differentibus, valent. Scribamus enim u pro $1+v$, et habebimus:

$$lu = \frac{1}{m} + \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{mu} - \frac{1-u}{2mu}$$

vnde lu eo accuratius definitur, quo maior sumatur numerus m.

Exemplum 3.

313 *Integrale* $y = \int \frac{c\,dx}{cc+xx}$ *ita sumtum, vt euanescat posito* $x=0$, *proxime exprimere.*

Hoc integrale vt nouimus est $y = \text{Ang. tang.} \frac{x}{c}$, ad quem valorem proxime exhibendum est $a=0$, et $b=0$; si ergo valor ipsius x ab 0 per differentiam constantem α crescere statuatur, ob $X = \frac{c}{cc+xx}$ erunt eius valores

pro indicibus	0;	α	2α	....	x
series	$\frac{1}{c}$;	$\frac{c}{cc+\alpha\alpha}$;	$\frac{c}{cc+4\alpha\alpha}$;	,	$\frac{c}{cc+xx}$

cuius terminus vltimum praecedens est $'X = \frac{c}{cc+(x-\alpha)^2}$.

Quare integralis nostri $y = \text{Ang. tang.} \frac{x}{c}$ valor proxime est

$$\alpha\left(\frac{1}{c} + \frac{c}{cc+\alpha\alpha} + \frac{c}{cc+4\alpha\alpha} + \ldots\ldots + \frac{c}{cc+(x-\alpha)^2}\right)$$

alter vero proxime minor, quia hic est nimis magnus, est

$$\alpha\left(\frac{c}{cc+\alpha\alpha} + \frac{c}{cc+4\alpha\alpha} + \frac{c}{cc+9\alpha\alpha} + \ldots\ldots \frac{c}{cc+xx}\right).$$

Inter quos si medium capiatur, ibi $\alpha.\frac{1}{2c}$ hic vero $\alpha.\frac{c}{2(cc+xx)}$ adiiciendo propius erit:

$$\alpha\left(\frac{c}{cc} + \frac{c}{cc+\alpha\alpha} + \frac{c}{cc+4\alpha\alpha} + \frac{c}{cc+9\alpha\alpha} + \ldots\ldots + \frac{c}{cc+xx}\right)$$

$$= \text{Ang. tang.} \frac{x}{c} + \frac{\alpha}{2}\left(\frac{1}{c} + \frac{c}{cc+xx}\right) = \text{Ang. tang.} \frac{x}{c} + \frac{\alpha(2cc+xx)}{2c(cc+xx)}.$$

Pro

Pro hoc ergo angulo valorem proxime verum habemus:

$$\text{Ang. tang.} \tfrac{x}{c} = \alpha c \left(\tfrac{1}{cc} + \tfrac{1}{cc+\alpha\alpha} + \tfrac{1}{cc+4\alpha\alpha} + \ldots + \tfrac{1}{cc+xx} \right) - \tfrac{\alpha(2cc+xx)}{2c(cc+xx)}$$

qui eo minus a veritate discrepabit, quo minor fuerit α numerus ratione ipsius c. Quodsi ergo pro c numerum valde magnum sumamus, pro α vnitatem accipere licet, vnde posito $x = cv$ erit

$$\text{Ang. tang.}\, v = c \left(\tfrac{1}{cc} + \tfrac{1}{cc+1} + \tfrac{1}{cc+4} + \tfrac{1}{cc+9} \ldots + \tfrac{1}{cc+ccvv} \right) - \tfrac{(2+vv)}{2c(1+vv)}$$

idque eo exactius, quo maior capiatur numerus c.

Coroll. 1.

314. Si ponamus $c = 1$, quo casu error insignis esse debet, fiet

$$\text{Ang. tang.}\, v = 1 + \tfrac{1}{1+1} + \tfrac{1}{1+4} + \tfrac{1}{1+9} + \ldots + \tfrac{1}{1+vv} - \tfrac{(2+vv)}{2(1+vv)}.$$

Sit $v = 1$, erit Ang. tang. $1 = \frac{\pi}{4} = 1 + \frac{1}{2} - \frac{3}{4} = \frac{3}{4}$, hincque $\pi = 3$, quod non multum abhorret a vero; si ponamus $c = 2$; prodit

$$\text{Ang. tang.}\, v = 2 \left(\tfrac{1}{4} + \tfrac{1}{4+1} + \tfrac{1}{4+4} + \tfrac{1}{4+9} + \ldots + \tfrac{1}{4+4vv} \right) - \tfrac{(2+vv)}{4(1+vv)}$$

vnde si $v = 1$ colligitur

$$\text{Ang. taug. } 1 = \tfrac{\pi}{4} = 2 \left(\tfrac{1}{4} + \tfrac{1}{4+1} + \tfrac{1}{4+4} \right) - \tfrac{3}{8} = \tfrac{23}{20} - \tfrac{3}{8} = \tfrac{31}{40},$$

sicque $\pi = \frac{31}{10} = 3, 1$, propius accedens.

 Coroll. 2.

Coroll. 2.

315. Sit $c=6$, eritque

$$\text{Ang. tang. } v = 6\left(\frac{1}{36}+\frac{1}{36+1}+\frac{1}{36+4}+\dots+\frac{1}{36+36vv}\right) - \frac{(2+vv)}{1.(1+vv)}$$

vnde si $v=\frac{1}{2}$ et $v=\frac{1}{3}$ oritur:

$$\text{Ang. tang. } \tfrac{1}{2} = 6\left(\frac{1}{36}+\frac{1}{36+1}+\frac{1}{36+4}+\frac{1}{36+9}\right)-\frac{3}{20}$$

$$\text{Ang. tang. } \tfrac{1}{3} = 6\left(\frac{1}{36}+\frac{1}{36+1}+\frac{1}{36+4}\right)-\frac{19}{120}.$$

At est Ang. tang. $\frac{1}{2}$ + Ang. tang. $\frac{1}{3}$ = Ang. tang. $1=\frac{\pi}{4}$. Ergo

$$\frac{\pi}{4} = 12\left(\frac{1}{36}+\frac{1}{37}+\frac{1}{40}\right)+\frac{2}{15}-\frac{37}{120} = \frac{1063}{1110}-\frac{7}{40} = \frac{695}{888}$$

seu $\pi = \frac{695}{222} = 3,1306.$

Coroll. 3.

316. Sin autem ibi statim ponamus $v=1$, erit

$$\frac{\pi}{4} = 6\left(\frac{1}{36}+\frac{1}{37}+\frac{1}{40}+\frac{1}{45}+\frac{1}{52}+\frac{1}{61}+\frac{1}{72}\right)-\frac{1}{8}$$

vnde fit $\pi = 3,13696$ multo propius veritati, plurium scilicet terminorum additio propius ad veritatem perducit.

Problema 37.

317. Methodum ad integralium valores appropinquandi ante expositam, perfectiorem reddere, vt minus a veritate aberretur.

Solutio.

Sit $y=\int X\,dx$ formula integralis proposita, cuius valorem iam constet esse $y=b$, si ponatur $x=a$, siue

siue is sit datus per ipsam integrationis conditionem, siue iam per aliquot operationes inde deriuatus; ac tribuamus iam ipsi x valorem parum superantem illum a, cui respondet $y=b$, tum vero fiat $X=A$ si ponatur $x=a$. In superiori autem methodo assumsimus, dum x parum supra a excrescit, manere X constantem $=A$, ideoque fore $\int X dx = A(x-a)$. At quatenus X non est constans, eatenus non est $\int X dx = X(x-a)$, sed reuera habetur $\int X dx = X(x-a) - \int (x-a) dX$. Ponamus igitur $dX = P dx$ eritque $\int (x-a) dX = \int P(x-a) dx$ et si iam $P = \frac{dX}{dx}$, quamdiu x non multum a excedit, vt constantem spectemus, habebimus $\int P(x-a) dx = \frac{1}{2} P(x-a)^2$, sicque fiet $y = \int X dx = b + X(x-a) - \frac{1}{2} P(x-a)^2$, qui valor iam propius ad veritatem accedit, etsi pro X et P ii valores capiantur, quos induunt vel posito $x=a$, vel posito $x=a+\alpha$, maiore scilicet valore, ad quem hac operatione x crescere statuimus: ex quo hinc prout vel $x=a$ vel $x=a+\alpha$ ponimus, geminos limites obtinebimus, inter quos veritas subsistit. Simili autem modo vlterius progredi poterimus: cum enim P non sit constans, erit $\int P(x-a) dx = \frac{1}{2} P(x-a)^2 - \frac{1}{2} \int (x-a)^2 dP$, vnde si statuamus $dP = Q dx$, erit $\int (x-a)^2 dP = \int Q (x-a)^2 dx = \frac{1}{3} Q(x-a)^3$, si quidem Q, vt quantitatem constantem spectemus, ita vt sit

$$y = \int X dx = b + X(x-a) - \frac{1}{2} P(x-a)^2 + \frac{1}{2 \cdot 3} Q(x-a)^3.$$

Eadem

Eadem ergo methodo si vlterius procedamus, **ponendo**

$$X=\frac{dy}{dx};\ P=\frac{dX}{dx};\ Q=\frac{dP}{dx};\ R=\frac{dQ}{dx};\ S=\frac{dR}{dx}\ \text{etc.}$$

inueniemus:

$$y=b+X(x-a)-\tfrac{1}{2}P(x-a)^2+\tfrac{1}{2.3}Q(x-a)^3-\tfrac{1}{2.3.4}R(x-a)^4$$
$$+\tfrac{1}{2.3.4.5}S(x-a)^5-\text{etc.}$$

quae series vehementer conuergit, si modo x non multum superet a, atque adeo si in infinitum continuetur, verum valorem ipsius y exhibebit, siquidem in functionibus X, P, Q, R etc. valor extremus $x=a+\alpha$ substituatur. Nisi autem eam seriem in infinitum extendere velimus, praestabit per interualla procedere tribuendo ipsi x successiue valores a, a', a'', a''', a'''' etc. ac tum pro singulis valores litteris X, P, Q, R, S etc. conuenientes quaeri oportet, qui fiunt, vt sequuntur:

si fuerit $x=a,\ a',\ a'',\ a''',\ a^{IV},\ a^{V}$ etc.

fiat $X=A,\ A',\ A'',\ A''',\ A^{IV},\ A^{V}$ etc.

$\frac{dX}{dx}=P=B,\ B',\ B'',\ B''',\ B^{IV},\ B^{V}$ etc.

$\frac{dP}{dx}=Q=C,\ C',\ C'',\ C''',\ C^{IV},\ C^{V}$ etc.

$\frac{dQ}{dx}=R=D,\ D',\ D'',\ D''',\ D^{IV},\ D^{V}$ etc.

etc.

tum vero fit

$y=b,\ b',\ b'',\ b''',\ b^{IV},\ b^{V}$ etc.

quibus

quibus constitutis erit vt ex antecedentibus colligere licet:

$$b' = b + A'(a' - a) - \tfrac{1}{2}B'(a' - a)^2 + \tfrac{1}{6}C'(a' - a)^3 - \tfrac{1}{24}D'(a' - a)^4 + \text{etc.}$$

$$b'' = b' + A''(a'' - a') - \tfrac{1}{2}B''(a'' - a')^2 + \tfrac{1}{6}C''(a'' - a')^3 - \tfrac{1}{24}D''(a'' - a')^4 + \text{etc.}$$

$$b''' = b'' + A'''(a''' - a'') - \tfrac{1}{2}B'''(a''' - a'')^2 + \tfrac{1}{6}C'''(a''' - a'')^3 - \tfrac{1}{24}D'''(a''' - a'')^4 + \text{etc.}$$

$$b^{IV} = b''' + A^{IV}(a^{IV} - a''') - \tfrac{1}{2}B^{IV}(a^{IV} - a''')^2 + \tfrac{1}{6}C^{IV}(a^{IV} - a''')^3 - \tfrac{1}{24}D^{IV}(a^{IV} - a''')^4 + \text{etc.}$$

etc.

quae expressiones eousque continuentur, donec pro valore ipsius x quantumuis ab initiali a discrepante valor ipsius y obtineatur.

Coroll. 1.

318. Haec igitur approximandi methodus eo vtitur Theoremate, cuius veritas iam in calculo differentiali est demonstrata, quod si y eiusmodi fuerit functio ipsius x, quae posito $x = a$, fiat $= b$, ac statuatur $\frac{dy}{dx} = X$, $\frac{dX}{dx} = P$, $\frac{dP}{dx} = Q$, $\frac{dQ}{dx} = R$ etc. fore generaliter:

$$y = b + X(x-a) - \tfrac{1}{2}P(x-a)^2 + \tfrac{1}{6}Q(x-a)^3 - \tfrac{1}{24}R(x-a)^4 + \tfrac{1}{120}S(x-a)^5 \text{ etc.}$$

Coroll. 2.

319. Si hanc feriem in infinitum continuare vellemus, non opus effet, valorem ipfius x parum tantum ab a diuerfum affumere. Verum quo ifta feries magis conuergens reddatur, expedit faltum ab a ad x in interualla difpefci, et pro fingulis operationem hic defcriptam inftitui.

Coroll. 3.

320. Si valores ipfius x ab a per differentias conftantes $=\alpha$ crefcere faciamus, fitque vltimus $a+n\alpha=x$, ita vt

fi fuerit $x=a,\ a+\alpha,\ a+2\alpha,\ a+3\alpha,\ \ldots\ldots x$

$$\begin{array}{llllll}
\text{fiat } X=A, & A', & A'', & A''', & \ldots\ldots & X \\
\frac{dX}{dx}=P=B, & B', & B'', & B''', & \ldots\ldots & P \\
\frac{dP}{dx}=Q=C, & C', & C'', & C''', & \ldots\ldots & Q \\
\frac{dQ}{dx}=R=D, & D', & D'', & D''', & \ldots\ldots & R
\end{array}$$

etc.

indeque $y=b,\ \ b',\ \ b'',\ \ b''',\ \ldots\ldots y$

erit pro valore $x=x$ omnes feries colligendo:

$$\begin{aligned}
y=b+\alpha\ &(A'+A''+A'''+\ldots\ldots+X) \\
-\tfrac{1}{2}\alpha^2\ &(B'+B''+B'''+\ldots\ldots+P) \\
+\tfrac{1}{6}\alpha^3\ &(C'+C''+C'''+\ldots\ldots+Q) \\
-\tfrac{1}{24}\alpha^4\ &(D'+D''+D'''+\ldots\ldots+R)
\end{aligned}$$

etc.

Scho-

Scholion 1.

321. Demonſtratio theorematis Coroll. 1. memorati, cui haec methodus approximandi innititur, ex natura differentialium ita inſtruitur. Sit y functio ipſius x, quae poſito $x = a$, fiat $y = b$; et quaeramus valorem ipſius y, ſi x vtcunque excedat a: incipiamus a valore ipſius maximo, qui eſt x, et per differentialia deſcendamus, atque ex differentialibus patet:

ſi fuerit x	fore y
$x - dx$	$y - dy + ddy - d^3y + d^4y -$ etc.
$x - 2dx$	$y - 2dy + 3ddy - 4d^3y + 5d^4y -$ etc.
$x - 3dx$	$y - 3dy + 6ddy - 10d^3y + 15d^4y -$ etc.
.	.
.	.
.	.
.	.
$x - ndx$	$y - ndy + \frac{n(n+1)}{1 \cdot 2} ddy - \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3} d^3y + \frac{n(n+1)(n+2)(n+3)}{1 \cdot 2 \cdot 3 \cdot 4} d^4y -$ etc.

Ponamus nunc $x - ndx = a$, erit $n = \frac{x - a}{dx}$ ideoque numerus infinitus; tum vero valor pro y reſultans per hypotheſin eſſe debet $= b$, quamobrem habebimus:

$$b = y - \frac{(x-a)dy}{dx} + \frac{(x-a)^2 ddy}{1 \cdot 2\, dx^2} - \frac{(x-a)^3 d^3y}{1 \cdot 2 \cdot 3\, dx^3} + \frac{(x-a)^4 d^4y}{1 \cdot 2 \cdot 3 \cdot 4\, dx^4} - \text{etc.}$$

Quod ſi iam ſtatuamus $\frac{dy}{dx} = X$, $\frac{dX}{dx} = P$, $\frac{dP}{dx} = Q$, $\frac{dQ}{dx} = R$ etc. reperimus vt ante:

$$y = b + X(x-a) - \tfrac{1}{2}P(x-a)^2 + \tfrac{1}{6}Q(x-a)^3 - \tfrac{1}{24}R(x-a)^4 + \text{etc.}$$

 Vnde

Vnde patet si x quam minime superet a, sufficere statui $y = b + X(x-a)$ quod est fundamentum approximationis primum propositae, illius scilicet limitis, quo X ex valore maiore ipsius x definitur.

Scholion 2.

322. Quemadmodum hoc ratiocinium nobis alterum tantum limitem supra assignatum patefecit, ita ad alterum limitem hoc ratiocinium nos manuducet. Scilicet vti ante ab x ad a descendimus, ita nunc ab a ad x ascendamus

si abeat a	tum b abibit in
in $a+da$	$b+db$
$a+2da$	$b+2db+ddb$
$a+3da$	$b+3db+3ddb+d^3b$
.	.
.	.
.	.
$a+nda$	$b+ndb+\frac{n(n-1)}{1.2}ddb+\frac{n(n-1)(n-2)}{1.2.3}d^3b$ etc.

Sit iam $a+nda=x$, seu $n=\frac{x-a}{da}$, et valor ipsius b fiet $=y$. Sint autem A, B, C, D etc. valores superiorum functionum X, P, Q, R etc. si loco x scribatur a, eritque pro praesenti casu $A=\frac{db}{da}$; $B=\frac{ddb}{da^2}$; $C=\frac{d^3b}{da^3}$ etc. Quocirca habebimus

$$y=b+A(x-a)+\tfrac{1}{2}B(x-a)^2+\tfrac{1}{6}C(x-a)^3+\tfrac{1}{24}D(x-a)^4+\text{etc.}$$

quae series superiori praeter signa omnino est simi-

lis;

lis; ac si x parum excedat a vt $b + A(x-a)$ satis exacte valorem ipsius y indicet, hinc alter limes supra assignatus nascitur. Quodsi autem progressum ab a ad x vt supra §. 320. in interualla aequalia secundum differentiam α dispescamus, et termini in singulis seriebus vltimos praecedentes notentur per $'X$, $'P$, $'Q$, $'R$ etc. habebimus pro y quasi alterum limitem:

$$
\begin{aligned}
y = b &+ \alpha\,(A + A' + A'' + \ldots\ldots + 'X) \\
&+ \tfrac{1}{2}\alpha^2(B + B' + B'' + \ldots\ldots + 'P) \\
&+ \tfrac{1}{6}\alpha^3(C + C' + C'' + \ldots\ldots + 'Q) \\
&+ \tfrac{1}{24}\alpha^4(D + D' + D'' + \ldots\ldots + 'R) \\
&\quad \text{etc.}
\end{aligned}
$$

ita vt etiam in hac methodo emendata binos habebimus limites, inter quos verus valor ipsius y contineatur. Propius ergo valorem assequemur, si inter hos limites medium arithmeticum capiamus vnde prodibit:

$$
\begin{aligned}
y = b &+ \alpha\,(A + A' + A'' + \ldots\ldots + X) - \tfrac{1}{2}\alpha(A+X) \\
&\qquad + \tfrac{1}{4}\alpha^2(B-P) \\
&+ \tfrac{1}{6}\alpha^3(C + C' + C'' + \ldots\ldots + Q) - \tfrac{1}{12}\alpha^3(C+Q) \\
&\qquad + \tfrac{1}{48}\alpha^4(D-R) \\
&+ \tfrac{1}{120}\alpha^5(E + E' + E'' + \ldots\ldots + S) - \tfrac{1}{240}\alpha^6(E+S) \\
&\qquad + \tfrac{1}{1440}\alpha^7(F-T) \\
&\quad \text{etc.}
\end{aligned}
$$

 Atque

Atque hinc ſuperiores approximationes tantum addendo membrum $\frac{1}{4}\alpha^2(B-P)$ non mediocriter corrigentur.

Exemplum 1.

323. *Logarithmum cuiusuis numeri* x *proxime exprimere.*

Hic igitur eſt $y=\int\frac{dx}{x}$, quod integrale ita capitur vt euaneſcat poſito $x=1$, erit ergo $a=1$ et $b=0$ et $X=\frac{1}{x}$. Sumamus iam ab vnitate ad x per interualla $=\alpha$ aſcendi; et cum ſit $P=\frac{dX}{dx}=-\frac{1}{xx}$; $Q=\frac{dP}{dx}=\frac{2}{x^3}$; $R=\frac{dQ}{dx}=-\frac{6}{x^4}$ pro indicibus:

$$x= 1;\ 1+\alpha;\ 1+2\alpha;\ 1+3\alpha;\ \ldots\ x$$

$$\text{erit } X= 1;\ \frac{1}{1+\alpha};\ \frac{1}{1+2\alpha};\ \frac{1}{1+3\alpha};\ \ldots\ \frac{1}{x}$$

$$P=-1;\ \frac{1}{(1+\alpha)^2};\ \frac{1}{(1+2\alpha)^2};\ \frac{1}{(1+3\alpha)^2};\ \ldots -\frac{1}{xx}$$

$$Q= 2;\ \frac{2}{(1+\alpha)^3};\ \frac{2}{(1+2\alpha)^3};\ \frac{2}{(1+3\alpha)^3};\ \ldots +\frac{2}{x^3}$$

$$R=-6;\ \frac{6}{(1+\alpha)^4};\ \frac{6}{(1+2\alpha)^4};\ \frac{6}{(1+3\alpha)^4};\ \ldots -\frac{6}{x^4}$$

etc.

vnde adipiſcimur:

$$lx= \alpha\left(1+\frac{1}{1+\alpha}+\frac{1}{1+2\alpha}+\frac{1}{1+3\alpha}+\ldots+\frac{1}{x}\right)-\frac{1}{2}\alpha\left(1+\frac{1}{x}\right)$$

$$-\frac{1}{4}\alpha\alpha\left(1-\frac{1}{xx}\right)$$

$$+\frac{1}{3}\alpha^3\left(1+\frac{1}{(1+\alpha^3)}+\frac{1}{(1+2\alpha)^3}+\frac{1}{(1+3\alpha)^3}+\ldots+\frac{1}{x^3}\right)-\frac{1}{6}\alpha^3\left(1+\frac{1}{x^3}\right)$$

$$-\frac{1}{8}\alpha^4\left(1-\frac{1}{x^4}\right)$$

$$+\frac{1}{5}\alpha^5\left(1+\frac{1}{(1+\alpha)^5}+\frac{1}{(1+2\alpha)^5}+\frac{1}{(1+3\alpha)^5}+\ldots+\frac{1}{x^5}\right)-\frac{1}{10}\alpha^5\left(1+\frac{1}{x^5}\right)$$

$$-\frac{1}{12}\alpha^6\left(1-\frac{1}{x^6}\right)$$

etc.

Quare

Quare si sumamus $\alpha = \frac{1}{m}$ erit

$$lx = \frac{1}{m} + \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{mx} - \frac{(x+1)}{2mx}$$
$$- \frac{(xx-1)}{4mmxx}$$
$$+ \tfrac{1}{3}\left(\frac{1}{m^3} + \frac{1}{(m+1)^3} + \frac{1}{(m+2)^3} + \ldots + \frac{1}{(mx)^3}\right) - \frac{(x^3+1)}{6m^3x^3}$$
$$- \frac{(x^4-1)}{4m^4x^4}$$
$$+ \tfrac{1}{5}\left(\frac{1}{m^5} + \frac{1}{(m+1)^5} + \frac{1}{(m+2)^5} + \ldots + \frac{1}{(mx)^5}\right) - \frac{(x^5+1)}{10m^5x^5}$$

etc.

Corollarium.

324. Si hae progressiones in infinitum continuentur, erit postremarum partium summa $= -\frac{1}{2} l \frac{m}{m-1} - \frac{1}{2} l \frac{mx+1}{mx} = -\frac{1}{2} l \frac{mx+1}{(m-1)x}$ primarum vero $\frac{1}{2} l \frac{m+1}{m-1}$, vnde cum sit $lx + \frac{1}{2} l \frac{mx+1}{(m-1)x} + \frac{1}{2} l \frac{m-1}{m+1} = \frac{1}{2} l \frac{x(mx+1)}{m+1}$, erit

$$l \frac{x(mx+1)}{m+1} = 2\left(\frac{1}{m+1} + \frac{1}{m+2} + \frac{1}{m+3} + \ldots + \frac{1}{mx}\right)$$
$$+ \tfrac{2}{3}\left(\frac{1}{(m+1)^3} + \frac{1}{(m+2)^3} + \frac{1}{(m+3)^3} + \ldots + \frac{1}{m^3x^3}\right)$$
$$+ \tfrac{2}{5}\left(\frac{1}{(m+1)^5} + \frac{1}{(m+2)^5} + \frac{1}{(m+3)^5} + \ldots + \frac{1}{m^5x^5}\right)$$

etc.

quae expressio adeo, si in infinitum continuetur, verum valorem log. $\frac{x(mx+1)}{m+1}$ praebet.

Exemplum 2.

325. *Arcum circuli cuius tangens est $= \frac{x}{a}$ hac methodo proxime exprimere.*

Quaestio

Coroll. 1.

309. Quodsi ponamus $mx = m + z$, vt sit $x = \frac{m+z}{m}$ prodibunt hae progressiones:

$$m^{n-1}\left(\frac{1}{m^n} + \frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \ldots\ldots + \frac{1}{(m+z-1)^n}\right) \text{ et}$$

$$m^{n-1}\left(\frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \frac{1}{(m+3)^n} + \ldots\ldots + \frac{1}{(m+z)^n}\right)$$

quarum summa alterius maior est, alterius minor quam $\frac{1}{n-1} - \frac{m^{n-1}}{(n-1)(m+z)^{n-1}} = \frac{(m+z)^{n-1} - m^{n-1}}{(n-1)(m+z)^{n-1}}$; casu autem $n = 1$ haec expressio abit in $l(1 + \frac{z}{m})$.

Coroll. 2.

310. Cum prior progressio maior sit quam posterior, erit

$$\frac{1}{m^n} + \frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \ldots + \frac{1}{(m+z-1)^n} > \frac{(m+z)^{n-1} - m^{n-1}}{(n-1)m^{n-1}(m+z)^{n-1}}$$

$$\frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \frac{1}{(m+3)^n} + \ldots + \frac{1}{(m+z)^n} < \frac{(m+z)^{n-1} - m^{n-1}}{(n-1)m^{n-1}(m+z)^{n-1}}$$

addatur hic vtrinque $\frac{1}{m^n}$ ibi vero $\frac{1}{(m+z)^n}$ et sumatur medium arithmeticum erit exactius:

$$\frac{1}{m^n} + \frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \frac{1}{(m+3)^n} + \ldots + \frac{1}{(m+z)^n}$$
$$= \frac{(2m+n-1)(m+z)^n - (2z+2m-n+1)m^n}{2(n-1)m^n(m+z)^n}$$

quae expressio casu $n = 1$ abit in $l(1 + \frac{z}{m}) + \frac{1}{2m} + \frac{1}{2(m+z)}$.

Coroll. 3.

Coroll. 3.

311. Ponatur $x = mv$, et habebimus sequentis seriei summam proxime expressam:

$$\frac{1}{m^n} + \frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \ldots + \frac{1}{m^n(1+v)^n}$$
$$= \frac{(2m+n-1)(1+v)^n - 2m(1+v) + n - 1}{2(n-1)m^n(1+v)^n}$$

et casu $n = 1$

$$\frac{1}{m} + \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{m+mv} = l(1+v) + \frac{2+v}{2m(1+v)}$$

vnde si $v = 1$ erit proxime

$$\frac{1}{m^n} + \frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \ldots + \frac{1}{2^n m^n}$$
$$= \frac{2^n(2m+n-1) - 4m + n - 1}{2^{n+1}(n-1)m^n} \text{ et}$$

$$\frac{1}{m} + \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{2m} = l2 + \frac{3}{4m}.$$

Coroll. 4.

312. Hinc nascitur regula logarithmos quantumuis magnorum numerorum proxime assignandi, dum series vulgares tantum pro numeris parum ab vnitate differentibus, valent. Scribamus enim u pro $1+v$, et habebimus:

$$lu = \frac{1}{m} + \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{mu} - \frac{1-u}{2mu}$$

vnde lu eo accuratius definitur, quo maior sumatur numerus m.

Exemplum 3.

313 *Integrale $y = \int \frac{c\,dx}{cc+xx}$ ita sumtum, vt euanescat posito $x = 0$, proxime exprimere.*

Hoc integrale vt nouimus est $y = \text{Ang.tang.}\frac{x}{c}$, ad quem valorem proxime exhibendum est $a = 0$, et $b = 0$; si ergo valor ipsius x ab o per differentiam constantem α crescere statuatur, ob $X = \frac{c}{cc+xx}$ erunt eius valores

pro indicibus 0; α 2α x

series $\frac{1}{c}$; $\frac{c}{cc+\alpha\alpha}$; $\frac{c}{cc+4\alpha\alpha}$; , $\frac{c}{cc+xx}$

cuius terminus vltimum praecedens est $'X = \frac{c}{cc+(x-\alpha)^2}$.

Quare integralis nostri $y = \text{Ang.tang.}\frac{x}{c}$ valor proxime est

$$\alpha\left(\frac{1}{c} + \frac{c}{cc+\alpha\alpha} + \frac{c}{cc+4\alpha\alpha} + \ldots + \frac{c}{cc+(x-\alpha)^2}\right)$$

alter vero proxime minor, quia hic est nimis magnus, est

$$\alpha\left(\frac{c}{cc+\alpha\alpha} + \frac{c}{cc+4\alpha\alpha} + \frac{c}{cc+9\alpha\alpha} + \ldots + \frac{c}{cc+xx}\right).$$

Inter quos si medium capiatur, ibi $\alpha.\frac{1}{c}$ hic vero $\alpha.\frac{c}{cc+xx}$ adiiciendo propius erit:

$$\alpha\left(\frac{c}{cc} + \frac{c}{cc+\alpha\alpha} + \frac{c}{cc+4\alpha\alpha} + \frac{c}{cc+9\alpha\alpha} + \ldots + \frac{c}{cc+xx}\right)$$

$$= \text{Ang.tang.}\frac{x}{c} + \frac{\alpha}{2}\left(\frac{1}{c} + \frac{c}{cc+xx}\right) = \text{Ang.tang.}\frac{x}{c} + \frac{\alpha(2cc+xx)}{2c(cc+xx)}.$$

Pro-

Pro hoc ergo angulo valorem proxime verum habemus:

$$\text{Ang. tang.}\,\frac{x}{c} = \alpha c\left(\frac{1}{cc} + \frac{1}{cc+\alpha\alpha} + \frac{1}{cc+4\alpha\alpha} + \ldots + \frac{1}{cc+xx}\right) - \frac{\alpha(2cc+xx)}{2c(cc+xx)}$$

qui eo minus a veritate discrepabit, quo minor fuerit α numerus ratione ipsius c. Quodsi ergo pro c numerum valde magnum sumamus, pro α vnitatem accipere licet, vnde posito $x = cv$ erit

$$\text{Ang. tang.}\,v = c\left(\frac{1}{cc} + \frac{1}{cc+1} + \frac{1}{cc+4} + \frac{1}{cc+9} \ldots + \frac{1}{cc+ccvv}\right) - \frac{(2+vv)}{2c(1+vv)}$$

idque eo exactius, quo maior capiatur numerus c.

Coroll. 1.

314. Si ponamus $c = 1$, quo casu error insignis esse debet, fiet

$$\text{Ang. tang.}\,v = 1 + \frac{1}{1+1} + \frac{1}{1+4} + \frac{1}{1+9} + \ldots + \frac{1}{1+vv} - \frac{(2+vv)}{2(1+vv)}.$$

Sit $v = 1$, erit Ang. tang. $1 = \frac{\pi}{4} = 1 + \frac{1}{2} - \frac{3}{4} = \frac{3}{4}$, hincque $\pi = 3$, quod non multum abhorret a vero; si ponamus $c = 2$; prodit

$$\text{Ang. tang.}\,v = 2\left(\frac{1}{4} + \frac{1}{4+1} + \frac{1}{4+4} + \frac{1}{4+9} + \ldots + \frac{1}{4+4vv}\right) - \frac{(2+vv)}{4(1+vv)}$$

vnde si $v = 1$ colligitur

$$\text{Ang. tang.}\,1 = \frac{\pi}{4} = 2\left(\frac{1}{4} + \frac{1}{4+1} + \frac{1}{4+4}\right) - \frac{3}{8} = \frac{23}{20} - \frac{3}{8} = \frac{31}{40},$$

sicque $\pi = \frac{31}{10} = 3,1$, propius accedens.

Coroll. 2.

315. Sit $c=6$, eritque

$$\text{Ang. tang. } v = 6\left(\tfrac{1}{36}+\tfrac{1}{36+1}+\tfrac{1}{36+4}+\ldots+\tfrac{1}{36+16vv}\right) - \tfrac{(1+vv)}{2.(1+vv)}$$

vnde si $v=\frac{1}{2}$ et $v=\frac{1}{3}$ oritur:

$$\text{Ang. tang. } \tfrac{1}{2} = 6\left(\tfrac{1}{36}+\tfrac{1}{36+1}+\tfrac{1}{36+4}+\tfrac{1}{36+9}\right)-\tfrac{3}{20}$$

$$\text{Ang. tang. } \tfrac{1}{3} = 6\left(\tfrac{1}{36}+\tfrac{1}{36+1}+\tfrac{1}{36+4}\right)-\tfrac{19}{120}.$$

At est Ang. tang. $\frac{1}{2}$ + Ang. tang. $\frac{1}{3}$ = Ang. tang. $1=\frac{\pi}{4}$. Ergo

$$\tfrac{\pi}{4} = 12\left(\tfrac{1}{36}+\tfrac{1}{37}+\tfrac{1}{40}\right)+\tfrac{2}{15}-\tfrac{37}{120}=\tfrac{1063}{1110}-\tfrac{7}{40}=\tfrac{695}{888}$$

seu $\pi=\frac{695}{222}=3,1306.$

Coroll. 3.

316. Sin autem ibi statim ponamus $v=1$, erit

$$\tfrac{\pi}{4} = 6\left(\tfrac{1}{36}+\tfrac{1}{37}+\tfrac{1}{40}+\tfrac{1}{45}+\tfrac{1}{52}+\tfrac{1}{61}+\tfrac{1}{72}\right)-\tfrac{1}{8}$$

vnde fit $\pi=3,13696$ multo propius veritati, plurium scilicet terminorum additio propius ad veritatem perducit.

Problema 37.

317. Methodum ad integralium valores appropinquandi ante expositam, perfectiorem reddere, vt minus a veritate aberretur.

Solutio.

Sit $y=\int X\,dx$ formula integralis proposita, cuius valorem iam constet esse $y=b$, si ponatur $x=a$, siue

ſiue is ſit datus per ipſam integrationis conditionem, ſiue iam per aliquot operationes inde deriuatus; ac tribuamus iam ipſi x valorem parum ſuperantem illum a, cui reſpondet $y=b$, tum vero fiat $X=A$ ſi ponatur $x=a$. In ſuperiori autem methodo aſſumſimus, dum x parum ſupra a excreſcit, manere X conſtantem $=A$, ideoque fore $\int X dx = A(x-a)$. At quatenus X non eſt conſtans, eatenus non eſt $\int X dx = X(x-a)$, ſed reuera habetur $\int X dx = X(x-a) - \int (x-a) dX$. Ponamus igitur $dX = P dx$ eritque $\int (x-a) dX = \int P(x-a) dx$ et ſi iam $P = \frac{dX}{dx}$, quamdiu x non multum a excedit, vt conſtantem ſpectemus, habebimus $\int P(x-a) dx = \frac{1}{2} P(x-a)^2$, ſicque fiet $y = \int X dx = b + X(x-a) - \frac{1}{2} P(x-a)^2$, qui valor iam propius ad veritatem accedit, etſi pro X et P ii valores capiantur, quos induunt vel poſito $x=a$, vel poſito $x=a+\alpha$, maiore ſcilicet valore, ad quem hac operatione x creſcere ſtatuimus: ex quo hinc prout vel $x=a$ vel $x=a+\alpha$ ponimus, geminos limites obtinebimus, inter quos veritas ſubſiſtit. Simili autem modo vlterius progredi poterimus: cum enim P non ſit conſtans, erit $\int P(x-a) dx = \frac{1}{2} P(x-a)^2 - \frac{1}{2} \int (x-a)^2 dP$, vnde ſi ſtatuamus $dP = Q dx$, erit $\int (x-a)^2 dP = \int Q (x-a)^2 dx = \frac{1}{3} Q(x-a)^3$, ſi quidem Q, vt quantitatem conſtantem ſpectemus, ita vt ſit

$$y = \int X dx = b + X(x-a) - \frac{1}{2} P(x-a)^2 + \frac{1}{2 \cdot 3} Q(x-a)^3.$$

Eadem

Eadem ergo methodo si vlterius procedamus, ponendo

$$X = \frac{dy}{dx};\ P = \frac{dX}{dx};\ Q = \frac{dP}{dx};\ R = \frac{dQ}{dx};\ S = \frac{dR}{dx} \text{ etc.}$$

inueniemus:

$$y = b + X(x-a) - \tfrac{1}{2}P(x-a)^2 + \tfrac{1}{2\cdot 3}Q(x-a)^3 - \tfrac{1}{2\cdot 3\cdot 4}R(x-a)^4 + \tfrac{1}{2\cdot 3\cdot 4\cdot 5}S(x-a)^5 - \text{etc.}$$

quae series vehementer conuergit, si modo x non multum superet a, atque adeo si in infinitum continuetur, verum valorem ipsius y exhibebit, siquidem in functionibus X, P, Q, R etc. valor extremus $x = a + \alpha$ substituatur. Nisi autem eam seriem in infinitum extendere velimus, praestabit per interualla procedere tribuendo ipsi x successiue valores a, a', a'', a''', a'''' etc. ac tum pro singulis valores litteris X, P, Q, R, S etc. conuenientes quaeri oportet, qui sint, vt sequuntur:

si fuerit $x = a,\ a',\ a'',\ a''',\ a^{IV},\ a^{V}$ etc.

fiat $X = A,\ A',\ A'',\ A''',\ A^{IV},\ A^{V}$ etc.

$\frac{dX}{dx} = P = B,\ B',\ B'',\ B''',\ B^{IV},\ B^{V}$ etc.

$\frac{dP}{dx} = Q = C,\ C',\ C'',\ C''',\ C^{IV},\ C^{V}$ etc.

$\frac{dQ}{dx} = R = D,\ D',\ D'',\ D''',\ D^{IV},\ D^{V}$ etc.

etc.

tum vero fit

$$y = b,\ b',\ b'',\ b''',\ b^{IV},\ b^{V} \text{ etc.}$$

quibus

quibus constitutis erit vt ex antecedentibus colligere licet:

$$b' = b + A'(a' - a) - \tfrac{1}{2}B'(a' - a)^2 + \tfrac{1}{6}C'(a' - a)^3 - \tfrac{1}{24}D'(a' - a)^4 + \text{etc.}$$

$$b'' = b' + A''(a'' - a') - \tfrac{1}{2}B''(a'' - a')^2 + \tfrac{1}{6}C''(a'' - a')^3 - \tfrac{1}{24}D''(a'' - a')^4 + \text{etc.}$$

$$b''' = b'' + A'''(a''' - a'') - \tfrac{1}{2}B'''(a''' - a'')^2 + \tfrac{1}{6}C'''(a''' - a'')^3 - \tfrac{1}{24}D'''(a''' - a'')^4 + \text{etc.}$$

$$b^{IV} = b''' + A^{IV}(a^{IV} - a''') - \tfrac{1}{2}B^{IV}(a^{IV} - a''')^2 + \tfrac{1}{6}C^{IV}(a^{IV} - a''')^3 - \tfrac{1}{24}D^{IV}(a^{IV} - a''')^4 + \text{etc.}$$

etc.

quae expressiones eousque continuentur, donec pro valore ipsius x quantumuis ab initiali a discrepante valor ipsius y obtineatur.

Coroll. 1.

318. Haec igitur approximandi methodus eo vtitur Theoremate, cuius veritas iam in calculo differentiali est demonstrata, quod si y eiusmodi fuerit functio ipsius x, quae posito $x = a$, fiat $= b$, ac statuatur $\frac{dy}{dx} = X$, $\frac{dX}{dx} = P$, $\frac{dP}{dx} = Q$, $\frac{dQ}{dx} = R$ etc. fore generaliter:

$$y = b + X(x-a) - \tfrac{1}{2}P(x-a)^2 + \tfrac{1}{6}Q(x-a)^3 - \tfrac{1}{24}R(x-a)^4 + \tfrac{1}{120}S(x-a)^5 \text{ etc.}$$

Coroll. 2.

319. Si hanc feriem in infinitum continuare vellemus, non opus effet, valorem ipfius x parum tantum ab a diuerfum affumere. Verum quo ifta feries magis conuergens reddatur, expedit faltum ab a ad x in interualla difpefci, et pro fingulis operationem hic defcriptam inftitui.

Coroll. 3.

320. Si valores ipfius x ab a per differentias conftantes $= \alpha$ crefcere faciamus, fitque vltimus $a + n\alpha = x$, ita vt

fi fuerit $x = a,\ a+\alpha,\ a+2\alpha,\ a+3\alpha,\ \ldots\ x$

$$\begin{array}{llllll} \text{fiat} & X = A, & A', & A'', & A''', & \ldots\ X \\ \frac{dX}{dx} = & P = B, & B', & B'', & B''', & \ldots\ P \\ \frac{dP}{dx} = & Q = C, & C', & C'', & C''', & \ldots\ Q \\ \frac{dQ}{dx} = & R = D, & D', & D'', & D''', & \ldots\ R \\ & & & \text{etc.} & & \end{array}$$

indeque $y = b,\ \ b',\ \ b'',\ \ b''',\ \ldots\ y$

erit pro valore $x = x$ omnes feries colligendo:

$$\begin{aligned} y = b + \alpha\ & (A' + A'' + A''' + \ldots + X) \\ - \tfrac{1}{2}\alpha^2\ & (B' + B'' + B''' + \ldots + P) \\ + \tfrac{1}{6}\alpha^3\ & (C' + C'' + C''' + \ldots + Q) \\ - \tfrac{1}{24}\alpha^4 & (D' + D'' + D''' + \ldots + R) \\ & \text{etc.} \end{aligned}$$

Scho-

Scholion 1.

321. Demonstratio theorematis Coroll. 1. memorati, cui haec methodus approximandi innititur, ex natura differentialium ita instruitur. Sit y functio ipsius x, quae posito $x=a$, fiat $y=b$; et quaeramus valorem ipsius y, si x vtcunque excedat a: incipiamus a valore ipsius maximo, qui est x, et per differentialia descendamus, atque ex differentialibus patet:

si fuerit x	fore y
$x-dx$	$y-dy+ddy-d^3y+d^4y-$ etc.
$x-2dx$	$y-2dy+3ddy-4d^3y+5d^4y-$ etc.
$x-3dx$	$y-3dy+6ddy-10d^3y+15d^4y-$ etc.
.	.
.	.
.	.
.	.
$x-ndx$	$y-ndy+\frac{n(n+1)}{1\cdot 2}ddy-\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}d^3y+\frac{n(n+1)(n+2)(n+3)}{1\cdot 2\cdot 3\cdot 4}d^4y-$ etc.

Ponamus nunc $x-ndx=a$, erit $n=\frac{x-a}{dx}$ ideoque numerus infinitus; tum vero valor pro y resultans per hypothesin esse debet $=b$, quamobrem habebimus:

$$b=y-\frac{(x-a)dy}{dx}+\frac{(x-a)^2ddy}{1\cdot 2\,dx^2}-\frac{(x-a)^3d^3y}{1\cdot 2\cdot 3\,dx^3}+\frac{(x-a)^4d^4y}{1\cdot 2\cdot 3\cdot 4\,dx^4}-\text{etc.}$$

Quod si iam statuamus $\frac{dy}{dx}=X$, $\frac{dX}{dx}=P$, $\frac{dP}{dx}=Q$, $\frac{dQ}{dx}=R$ etc. reperimus vt ante:

$$y=b+X(x-a)-\tfrac{1}{2}P(x-a)^2+\tfrac{1}{6}Q(x-a)^3-\tfrac{1}{24}R(x-a)^4+\text{etc.}$$

Vnde patet ſi x quam minime ſuperet a, ſufficere ſtatui $y = b + X(x-a)$ quod eſt fundamentum approximationis primum propoſitae, illius ſcilicet limitis, quo X ex valore maiore ipſius x definitur.

Scholion 2.

322. Quemadmodum hoc ratiocinium nobis alterum tantum limitem ſupra aſſignatum patefecit, ita ad alterum limitem hoc ratiocinium nos manuducet. Scilicet vti ante ab x ad a deſcendimus, ita nunc ab a ad x aſcendamus

ſi abeat a	tum b abibit in
in $a+da$	$b+db$
$a+2da$	$b+2db+ddb$
$a+3da$	$b+3db+3ddb+d^3b$
.	.
.	.
.	.
$a+nda$	$b+ndb+\frac{n(n-1)}{1.2}ddb+\frac{n(n-1)(n-2)}{1.2.3}d^3b$ etc.

Sit iam $a+nda=x$, ſeu $n=\frac{x-a}{da}$, et valor ipſius b fiet $=y$. Sint autem A, B, C, D etc. valores ſuperiorum functionum X, P, Q, R etc. ſi loco x ſcribatur a, eritque pro praeſenti caſu $A=\frac{db}{da}$; $B=\frac{ddb}{da^2}$; $C=\frac{d^3b}{da^3}$ etc. Quocirca habebimus

$$y=b+A(x-a)+\tfrac{1}{2}B(x-a)^2+\tfrac{1}{6}C(x-a)^3+\tfrac{1}{24}D(x-a)^4+\text{etc.}$$

quae ſeries ſuperiori praeter ſigna omnino eſt ſimilis;

lis; ac si x parum excedat a vt $b+A(x-a)$ satis exacte valorem ipsius y indicet, hinc alter limes supra assignatus nascitur. Quodsi autem progressum ab a ad x vt supra §. 320. in interualla aequalia secundum differentiam α dispescamus, et termini in singulis seriebus vltimos praecedentes notentur per $'X$, $'P$, $'Q$, $'R$ etc. habebimus pro y quasi alterum limitem:

$$\begin{aligned} y = b &+ \alpha\,(A+A'+A''+\ldots\ldots+{}'X) \\ &+ \tfrac{1}{2}\alpha^2(B+B'+B''+\ldots\ldots+{}'P) \\ &+ \tfrac{1}{6}\alpha^3(C+C'+C''+\ldots\ldots+{}'Q) \\ &+ \tfrac{1}{24}\alpha^4(D+D'+D''+\ldots\ldots+{}'R) \\ &\quad \text{etc.} \end{aligned}$$

ita vt etiam in hac methodo emendata binos habebimus limites, inter quos verus valor ipsius y contineatur. Propius ergo valorem assequemur, si inter hos limites medium arithmeticum capiamus vnde prodibit:

$$\begin{aligned} y = b &+ \alpha\,(A+A'+A''+\ldots\ldots+X) - \tfrac{1}{2}\alpha(A+X) \\ &\qquad + \tfrac{1}{4}\alpha^2(B-P) \\ &+ \tfrac{1}{6}\alpha^3(C+C'+C''+\ldots\ldots+Q) - \tfrac{1}{12}\alpha^3(C+Q) \\ &\qquad + \tfrac{1}{48}\alpha^4(D-R) \\ &+ \tfrac{1}{120}\alpha^5(E+E'+E''+\ldots\ldots+S) - \tfrac{1}{240}\alpha^6(E+S) \\ &\qquad + \tfrac{1}{1440}\alpha^7(F-T) \\ &\quad \text{etc.} \end{aligned}$$

 Atque

Atque hinc superiores approximationes tantum addendo membrum $\frac{1}{4}\alpha^2(B-P)$ non mediocriter corrigentur.

Exemplum 1.

323. *Logarithmum cuiusuis numeri* x *proxime exprimere.*

Hic igitur est $y=\int\frac{dx}{x}$, quod integrale ita capitur vt euanescat posito $x=1$, erit ergo $a=1$ et $b=0$ et $X=\frac{1}{x}$. Sumamus iam ab vnitate ad x per interualla $=\alpha$ ascendi; et cum sit $P=\frac{dX}{dx}=-\frac{1}{xx}$; $Q=\frac{dP}{dx}=\frac{2}{x^3}$; $R=\frac{dQ}{dx}=-\frac{6}{x^4}$ pro indicibus:

$$x = 1;\ 1+\alpha;\ 1+2\alpha;\ 1+3\alpha;\ \ldots\ x$$

$$\text{erit } X = 1;\ \frac{1}{1+\alpha};\ \frac{1}{1+2\alpha};\ \frac{1}{1+3\alpha};\ \ldots\ \frac{1}{x}$$

$$P = -1;\ \frac{1}{(1+\alpha)^2};\ \frac{1}{(1+2\alpha)^2};\ \frac{1}{(1+3\alpha)^2};\ \ldots -\frac{1}{xx}$$

$$Q = 2;\ \frac{2}{(1+\alpha)^3};\ \frac{2}{(1+2\alpha)^3};\ \frac{2}{(1+3\alpha)^3};\ \ldots +\frac{2}{x^3}$$

$$R = -6;\ \frac{6}{(1+\alpha)^4};\ \frac{6}{(1+2\alpha)^4};\ \frac{6}{(1+3\alpha)^4};\ \ldots -\frac{6}{x^4}$$

etc.

vnde adipiscimur:

$$\begin{aligned} lx = {} & \alpha\left(1+\frac{1}{1+\alpha}+\frac{1}{1+2\alpha}+\frac{1}{1+3\alpha}+\ldots+\frac{1}{x}\right)-\frac{1}{2}\alpha\left(1+\frac{1}{x}\right) \\ & -\frac{1}{4}\alpha\alpha\left(1-\frac{1}{xx}\right) \\ & +\frac{1}{3}\alpha^3\left(1+\frac{1}{(1+\alpha)^3}+\frac{1}{(1+2\alpha)^3}+\frac{1}{(1+3\alpha)^3}+\ldots+\frac{1}{x^3}\right)-\frac{1}{6}\alpha^3\left(1+\frac{1}{x^3}\right) \\ & -\frac{1}{8}\alpha^4\left(1-\frac{1}{x^4}\right) \\ & +\frac{1}{5}\alpha^5\left(1+\frac{1}{(1+\alpha)^5}+\frac{1}{(1+2\alpha)^5}+\frac{1}{(1+3\alpha)^5}+\ldots+\frac{1}{x^5}\right)-\frac{1}{10}\alpha^5\left(1+\frac{1}{x^5}\right) \\ & -\frac{1}{12}\alpha^6\left(1-\frac{1}{x^6}\right) \end{aligned}$$

etc. Quare

Quare si sumamus $\alpha = \frac{1}{m}$ erit

$$lx = \frac{1}{m} + \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{mx} - \frac{(x+1)}{2mx}$$
$$- \frac{(xx-1)}{4mmxx}$$
$$+ \frac{1}{3}\left(\frac{1}{m^3} + \frac{1}{(m+1)^3} + \frac{1}{(m+2)^3} + \ldots + \frac{1}{(mx)^3}\right) - \frac{(x^3+1)}{6m^3x^3}$$
$$- \frac{(x^4-1)}{8m^4x^4}$$
$$+ \frac{1}{5}\left(\frac{1}{m^5} + \frac{1}{(m+1)^5} + \frac{1}{(m+2)^5} + \ldots + \frac{1}{(mx)^5}\right) - \frac{(x^5+1)}{10m^5x^5}$$

etc.

Corollarium.

324. Si hae progressiones in infinitum continuentur, erit postremarum partium summa $= -\frac{1}{2}l\frac{m}{m-1} - \frac{1}{2}l\frac{mx+1}{mx} = -\frac{1}{2}l\frac{mx+1}{(m-1)x}$ primarum vero $\frac{1}{2}l\frac{m+1}{m-1}$, vnde cum sit $lx + \frac{1}{2}l\frac{mx+1}{(m-1)x} + \frac{1}{2}l\frac{m-1}{m+1} = \frac{1}{2}l\frac{x(mx+1)}{m+1}$, erit

$$l\frac{x(mx+1)}{m+1} = 2\left(\frac{1}{m+1} + \frac{1}{m+2} + \frac{1}{m+3} + \ldots + \frac{1}{mx}\right)$$
$$+ \frac{2}{3}\left(\frac{1}{(m+1)^3} + \frac{1}{(m+2)^3} + \frac{1}{(m+3)^3} + \ldots + \frac{1}{m^3x^3}\right)$$
$$+ \frac{2}{5}\left(\frac{1}{(m+1)^5} + \frac{1}{(m+2)^5} + \frac{1}{(m+3)^5} + \ldots + \frac{1}{m^5x^5}\right)$$

etc.

quae expressio adeo, si in infinitum continuetur, verum valorem log. $\frac{x(mx+1)}{m+1}$ praebet.

Exemplum 2.

325. *Arcum circuli cuius tangens est* $= \frac{x}{a}$ *hac methodo proxime exprimere.*

Quaestio

Quaeſtio igitur eſt de integrali $y = \int \frac{c\,dx}{cc+xx}$, quod poſito $x = 0$ euaneſcit; eritque $a = 0$, et $b = 0$, tum vero $X = \frac{c}{cc+xx}$; $P = \frac{dX}{dx} = \frac{-2cx}{(cc+xx)^2}$; $Q = \frac{dP}{dx} = \frac{-2c(cc-3xx)}{(cc+xx)^3}$; $R = \frac{dQ}{dx} = \frac{6cx(3cc-4xx)}{(cc+xx)^4}$; $S = \frac{dR}{dx} = \frac{6c(3c^4-33ccxx+20x^4)}{(cc+xx)^5}$ etc. quae formae in infinitum continuatae dant:

$$y = \frac{cx}{cc+xx} + \frac{cx^2}{(cc+xx)^2} - \frac{cx^3(cc-3xx)}{3(cc+xx)^3} - \frac{cx^4(3cc-4xx)}{4(cc+xx)^4} -$$
$$+ \frac{cx^5(3c^4-33ccxx+20x^4)}{20(cc+xx)^5} + \text{etc.}$$

Verum ſi x per interualla $= 1$, vt ſit $\alpha = 1$, creſcere ponamus erit

$A = \frac{c}{cc}$; $B = 0$; $C = \frac{-2c^3}{c^6}$; $D = 0$

$A' = \frac{c}{cc+1}$; $B' = \frac{-2c}{(cc+1)^2}$; $C' = \frac{-2c(cc-3)}{(cc+1)^3}$; $D' = \frac{6c(3cc-4)}{(cc+1)^4}$

$A'' = \frac{c}{cc+4}$; $B'' = \frac{-4c}{(cc+4)^2}$; $C'' = \frac{-2c(cc-12)}{(cc+4)^3}$; $D'' = \frac{12c(3cc-16)}{(cc+4)^4}$

$A''' = \frac{c}{cc+9}$; $B''' = \frac{-6c}{(cc+9)^2}$; $C''' = \frac{-2c(cc-27)}{(cc+9)^3}$; $D''' = \frac{18c(3cc-36)}{(cc+9)^4}$

.

.

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$X = \frac{c}{cc+xx}$; $P = \frac{-2cx}{(cc+xx)^2}$; $Q = \frac{-2c(cc-3xx)}{(cc+xx)^3}$; $R = \frac{6cx(3cc-4xx)}{(cc+xx)^4}$

hincque

$$y = c\left(\frac{1}{cc} + \frac{1}{cc+1} + \frac{1}{cc+4} + \frac{1}{cc+9} \cdots\cdots + \frac{1}{cc+xx}\right) - \frac{1}{2c} - \frac{c}{2(cc+xx)}$$
$$+ \frac{cx}{2(c+xx)^2}$$
$$-\frac{c}{3}\left(\frac{1}{c^4} + \frac{cc-3}{(cc+1)^3} + \frac{cc-12}{(cc+4)^3} + \frac{cc-27}{(cc+9)^3} + \ldots + \frac{cc-3xx}{(cc+xx)^3}\right) + \frac{1}{6c^3} + \frac{c(cc-3xx)}{6(cc+xx)^3}$$
$$- \frac{cx(3cc-4xx)}{8(cc+xx)^4}$$

etc.

Corol-

Corollarium.

326. Poſito ergo $c=x=4$ vt fiat $y=$ Ang. tang. $1=\frac{\pi}{4}$ erit

$$\frac{\pi}{4}=\frac{1}{4}+\frac{4}{17}+\frac{4}{20}+\frac{4}{25}+\frac{1}{8}-\frac{1}{8}-\frac{1}{16}+\frac{1}{128}$$

$$-\frac{4}{3}\left(\frac{1}{256}+\frac{13}{17^3}+\frac{4}{20^3}-\frac{11}{25^3}-\frac{32}{32^3}\right)+\frac{1}{384}-\frac{1}{1536}+\frac{1}{128,256}$$

cuius valor non multum a veritate diſcedit, ſed haec exempla tantum illuſtrationis cauſa afferro, non vt approximatio facilior, quam aliae methodi ſuppeditant, inde expectetur.

Exemplum 3.

327. *Integrale* $y=\int\frac{e^{-\frac{1}{x}}dx}{x}$ *ita ſumtum, vt euaneſcet poſito* $x=0$, *vero proxime aſſignare.*

Per reductiones ſupra expoſitas eſt $\int\frac{e^{-\frac{1}{x}}dx}{x}=e^{-\frac{1}{x}}x-\int e^{-\frac{1}{x}}dx$ et pars $e^{-\frac{1}{x}}x$ euaneſcit poſito $x=0$. Quaeramus ergo integrale $z=\int e^{-\frac{1}{x}}dx$, quia eo invento habetur $y=e^{-\frac{1}{x}}x-z$; ac ſupra iam obſeruauimus alias methodos approximandi in hoc exemplo fruſtra tentari. Cum igitur poſito $x=0$ euaneſcat z, erit $a=0$ et $b=0$, tum vero $X=e^{-\frac{1}{x}}$, hincque $P=\frac{dX}{dx}=e^{-\frac{1}{x}}\cdot\frac{1}{xx}$; $Q=\frac{dP}{dx}=e^{-\frac{1}{x}}\left(\frac{1}{x^4}-\frac{2}{x^3}\right)$; $R=\frac{dQ}{dx}=e^{-\frac{1}{x}}\left(\frac{1}{x^6}-\frac{6}{x^5}+\frac{6}{x^4}\right)$;

$S=$

$S = \frac{dR}{dx} = e^{-\frac{1}{x}}\left(\frac{1}{x^8} - \frac{12}{x^7} + \frac{36}{x^6} - \frac{24}{x^5}\right)$ etc. quibus valoribus in infinitum continuatis erit

$$z = e^{-\frac{1}{x}}\left(x - \frac{1}{2} + \frac{1}{6}x^3\left(\frac{1}{x^4} - \frac{2}{x^3}\right) - \frac{1}{24}x^4\left(\frac{1}{x^6} - \frac{6}{x^5} + \frac{6}{x^4}\right) + \frac{1}{120}x^5\left(\frac{1}{x^8} - \frac{12}{x^7} + \frac{36}{x^6} - \frac{24}{x^5}\right)\right) \text{ seu}$$

$$z = e^{-\frac{1}{x}}\left(x - \frac{1}{2} + \frac{1}{6}\left(\frac{1}{x} - 2\right) - \frac{1}{24}\left(\frac{1}{xx} - \frac{6}{x} + 6\right) + \frac{1}{120}\left(\frac{1}{x^3} - \frac{12}{x^2} + \frac{36}{x} - 24\right) - \frac{1}{720}\left(\frac{1}{x^4} - \frac{20}{x^3} + \frac{120}{x^2} - \frac{240}{x} + 120\right) \text{ etc.}\right)$$

quae series parum conuergit, quicunque valor ipsi x tribuatur. Per interualla igitur a o vsque ad x ascendamus, ponendo pro x successiue 0, α, 2α, 3α etc. vbi notandum fore $A = 0$, $B = 0$, $C = 0$, $D = 0$ etc. ac regula nostra praebet:

$$z = \alpha\left(e^{-\frac{1}{\alpha}} + e^{-\frac{1}{2\alpha}} + e^{-\frac{1}{3\alpha}} + \ldots + e^{-\frac{1}{x}}\right) - \frac{1}{2}\alpha e^{-\frac{1}{x}} - \frac{1}{4}\alpha^2 e^{-\frac{1}{x}}\frac{1}{xx}$$
$$+ \frac{1}{6}\alpha^3\left(e^{-\frac{1}{\alpha}}\left(\frac{1}{\alpha^4} - \frac{2}{\alpha^3}\right) + e^{-\frac{1}{2\alpha}}\left(\frac{1}{16\alpha^4} - \frac{2}{8\alpha^3}\right) + e^{-\frac{1}{3\alpha}}\left(\frac{1}{81\alpha^4} - \frac{2}{27\alpha^3}\right) \ldots + e^{-\frac{1}{x}}\left(\frac{1}{x^4} - \frac{2}{x^3}\right)\right)$$
$$- \frac{1}{12}\alpha^3 e^{-\frac{1}{x}}\left(\frac{1}{x^4} - \frac{2}{x^3}\right) - \frac{1}{48}\alpha^4 e^{-\frac{1}{x}}\left(\frac{1}{x^6} - \frac{6}{x^5} + \frac{6}{x^4}\right).$$

Si hinc valorem ipsius z pro casu $x = 1$ determinare velimus, et pro α fractionem paruam $\frac{1}{n}$ assumamus, habebimus:

$$z = \frac{1}{n}\left(e^{-\frac{n}{1}} + e^{-\frac{n}{2}} + e^{-\frac{n}{3}} + e^{-\frac{n}{4}} + \ldots + e^{-\frac{n}{n}}\right) - \frac{1}{2ne} - \frac{1}{4nne}$$
$$+ \frac{1}{6}\left(e^{-\frac{n}{1}}\left(\frac{n-2}{1}\right) + e^{-\frac{n}{2}}\left(\frac{n-4}{16}\right) + e^{-\frac{n}{3}}\left(\frac{n-6}{81}\right) + \ldots + e^{-\frac{n}{n}}\left(\frac{n-2n}{n^4}\right)\right)$$
$$+ \frac{1}{18n^3e} - \frac{1}{48n^4e}.$$

Si

Si hic pro n ſumatur numerus mediocriter magnus vel vti 10, valor ipſius x ad partem millioneſimam vnitatis exactus reperitur, ac vicies exactior prodiret, ſi pro n ſumeremus 20.

Scholion. 1.

328. Hoc exemplum ſufficiat eximium vſum huius methodi approximandi oſtendiſſe. Interim tamen occurrunt caſus, quibus ne hac quidem methodo vti licet, etiamſi totum ſpatium, per quod variabilis x creſcit, in minima interualla diuidamus. Euenit hoc, quando functio X pro quopiam interuallo dum variabili x, certus quidam valor tribuitur in infinitum excreſcit, cum tamen ipſa quantitas integralis $y = \int X dx$ hoc caſu non fiat infinita: veluti ſi fuerit $y = \int \frac{dx}{\sqrt{(a-x)}}$, vbi $X = \frac{1}{\sqrt{(a-x)}}$ quae poſito $x = a$ fit infinita, integrale vero $y = C - 2\sqrt{(a-x)}$ hoc caſu eſt finitum. Hoc autem ſemper vſu venit, quoties huiusmodi factor $a-x$ in denominatore habet exponentem vnitate minorem, tum enim idem factor in integrali in numeratorem tranſit; ſin autem eiusdem factoris exponens in denominatore eſt vnitas, vel adeo vnitate maior, tum etiam ipſum integrale caſu $x = a$ fit infinitum, quo caſu quia approximatio ceſſat, hic tantum de iis ſermo eſt, vbi exponens vnitate eſt minor; quoniam tum approximatio reuera turbatur. Verum huic incommodo facile medela afferri poteſt, cum enim differentiale eius-

modi formam fit habiturum $\frac{X\,dx}{(a-x)^{\lambda:\mu}}$ exiftente $\lambda < \mu$, ponatur $a-x=z^{\mu}$, vt fit $x=a-z^{\mu}$ et $dx=-\mu z^{\mu-1}dz$ et differentiale noftrum erit $=-\mu X z^{\mu-\lambda-1}dz$, quod cafu $x=a$ feu $z=0$ non amplius fit infinitum. Vel quod eodem redit, pro iis interuallis, quibus functio X fit infinita, integratio feorfim reuera inftituatur, ponendo $x=a\pm\omega$, tum enim formula $X\,dx$ fatis fiet fimplex ob ω valde paruum, vt integratio nihil habeat difficultatis. Veluti fi valorem ipfius $y=\int\frac{xx\,dx}{\sqrt{(a^4-x^4)}}$ per interualla ab $x=0$, vsque ad $x=a-\alpha$ iam fimus confecuti, pro hoc vltimo interuallo ponamus $x=a-\omega$, et integrari oportebit $\frac{(a-\omega)^2 d\omega}{\sqrt{(4a^3\omega-6aa\omega\omega+4a\omega^3-\omega^4)}}$ quod ob ω valde paruum abit in $\frac{d\omega\sqrt{a}}{2\sqrt{\omega}}\left(1-\frac{\omega}{2a}+\frac{7\omega\omega}{8aa}\right)$ cuius integrale fumto $\omega=\alpha$ eft $\sqrt{a\alpha}-\frac{\alpha\sqrt{\alpha}}{6\sqrt{a}}+\frac{7\alpha\alpha\sqrt{\alpha}}{4aa\sqrt{a}}$, quod fi ad plures terminos continuetur, non folum pro vltimo interuallo fed pro duobus pluribusue poftremis ponendo $\omega=2\alpha$ vel $\omega=3\alpha$ adhiberi poteft. Pro quibus enim interuallis denominator iam fit fatis paruus, praeftat hac methodo vti, quam ea quae ante eft expofita.

Scholion 2.

329. Interdum etiam aliud incommodum occurrit, vt denominator duobus cafibus euanefcat, veluti fi fuerit $y=\int\frac{X\,dx}{\sqrt{(a-x)(x-b)}}$, vbi variabilis x femper inter limites b et a contineri debet, ita vt cum a b ad a creuerit, deinceps iterum ab a ad b decre-

decrescat; interea autem integrale y continuo crescere pergat; cuius igitur valor per interualla commode determinari non potest. Hoc ergo casu in subsidium vocetur haec substitutio $x=\frac{1}{2}(a+b)-\frac{1}{2}(a-b)\operatorname{cos}\Phi$, qua fit $dx=+\frac{1}{2}(a-b)d\Phi\operatorname{fin}.\Phi$ et $(a-x)(x-b)$ $=(\frac{1}{2}(a-b)+\frac{1}{2}(a-b)\operatorname{cof}.\Phi)(\frac{1}{2}(a-b)-\frac{1}{2}(a-b)\operatorname{cof}.\Phi)$ seu $(a-x)(x-b)=\frac{1}{4}(a-b)^2\operatorname{fin}.\Phi^2$, vnde oritur $y=\int X\,d\Phi$ quae nullo amplius incommodo laborat, cum angulum Φ continuo vlterius aequabiliter augere licet. Hoc etiam ad casus patet, vbi bini factores in denominatore non eundem habent exponentem, veluti si fuerit $y=\int\frac{X\,dx}{\sqrt[2\lambda]{(a-x)^\mu(x-b)^\nu}}$, ita vt μ et ν sint minores quam 2λ, quem exponentem parem suppono. Si iam μ et ν non sint aequales sed $\nu<\mu$ ad aequalitatem reducantur, hoc modo $y=\int\frac{X\,dx\sqrt[2\lambda]{(x-b)^{\mu-\nu}}}{\sqrt[2\lambda]{(a-x)^\mu(x-b)^\mu}}$. Quodsi iam vt ante ponatur $x=\frac{1}{2}(a+b)-\frac{1}{2}(a-b)\operatorname{cof}.\Phi$ obtinebitur $y=(\frac{a-b}{2})^{\frac{2\lambda-\mu-\nu}{2\lambda}}\int X\,d\Phi\operatorname{fin}.\Phi^{\frac{\lambda-\mu}{\lambda}}(1-\operatorname{cof}.\Phi)^{\frac{\mu-\nu}{2\lambda}}$, vbi angulum Φ quousque libuerit continuare et methodo per interualla procedente vti licet. Quibus obseruatis vix quicquam amplius hanc methodum approximandi remorabitur.

 CAPVT VIII.

CAPVT VIII.

DE VALORIBVS INTEGRALIVM QVOS CERTIS TANTVM CASIBVS RECIPIVNT.

Problema 38.

330.

Integralis $\int \frac{x^m dx}{\sqrt{(1-xx)}}$ valorem, quem posito $x=1$ recipit, assignare, integrali scilicet ita determinato, vt euanescat posito $x=0$.

Solutio.

Pro casibus simplicissimis, quibus $m=1$ vel $m=2$, habemus posito $x=1$, post integrationem

$$\int \frac{dx}{\sqrt{(1-xx)}} = \frac{\pi}{2} \text{ et } \int \frac{xdx}{\sqrt{(1-xx)}} = 1.$$

Deinde supra §. 119. vidimus esse in genere

$$\int \frac{x^{m+1}dx}{\sqrt{(1-xx)}} = \frac{m}{m+1}\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}} - \frac{1}{m+1}x^m\sqrt{(1-xx)}$$

casu ergo $x=1$ erit

$$\int \frac{x^{m+1}dx}{\sqrt{(1-xx)}} = \frac{m}{m+1}\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}},$$

vnde

vnde a simplicissimis ad maiores exponentis m valores progrediendo obtinebimus:

$$\int\frac{dx}{\sqrt{(1-xx)}}=\frac{\pi}{2}$$
$$\int\frac{xxdx}{\sqrt{(1-xx)}}=\frac{1}{2}\cdot\frac{\pi}{2}$$
$$\int\frac{x^4dx}{\sqrt{(1-xx)}}=\frac{1.3}{2.4}\cdot\frac{\pi}{2}$$
$$\int\frac{x^6dx}{\sqrt{(1-xx)}}=\frac{1.3.5}{2.4.6}\cdot\frac{\pi}{2}$$
$$\int\frac{x^8dx}{\sqrt{(1-xx)}}=\frac{1.3.5.7}{2.4.6.8}\cdot\frac{\pi}{2}$$
$$\vdots$$
$$\int\frac{x^{2n}dx}{\sqrt{(1-xx)}}=\frac{1.3.5\ldots\ldots(2n-1)}{2.4.6\ldots\ldots 2n}\cdot\frac{\pi}{2}$$

$$\int\frac{xdx}{\sqrt{(1-xx)}}=1$$
$$\int\frac{x^3dx}{\sqrt{(1-xx)}}=\frac{2}{3}$$
$$\int\frac{x^5dx}{\sqrt{(1-xx)}}=\frac{2.4}{3.5}$$
$$\int\frac{x^7dx}{\sqrt{(1-xx)}}=\frac{2.4.6}{3.5.7}$$
$$\int\frac{x^9dx}{\sqrt{(1-xx)}}=\frac{2.4.6.8}{3.5.7.9}$$
$$\vdots$$
$$\int\frac{x^{2n+1}dx}{\sqrt{(1-xx)}}=\frac{2.4.6\ldots\ldots 2n}{3.5.7\ldots\ldots(2n+1)}.$$

Coroll. 1.

331. Integrale ergo $\int\frac{x^m dx}{\sqrt{(1-xx)}}$ posito $x=1$ algebraice exprimitur casibus quibus exponens m est numerus integer impar; casibus autem quibus est par quadraturam circuli inuoluit; semper enim π designat peripheriam circuli, cuius diameter $=1$.

Coroll. 2.

Coroll. 2.

332. Si binas postremas formulas in se multiplicemus prodit:

$$\int \frac{x^{2n}dx}{\sqrt{(1-xx)}} \cdot \int \frac{x^{2n+1}dx}{\sqrt{(1-xx)}} = \frac{1}{2n+1} \cdot \frac{\pi}{2} \text{ posito scilicet } x=1,$$

quam veram esse patet etiamsi n non sit numerus integer.

Coroll. 3.

333. Haec ergo aequalitas subsistet si ponamus $x=z^{\nu}$ iisdem conditionibus, quia sumto $x=0$ vel $x=1$ fit $z=0$ vel $z=1$. Erit ergo:

$$\nu\nu \int \frac{z^{2n\nu+\nu-1}dz}{\sqrt{(1-z^{2\nu})}} \cdot \int \frac{z^{2n\nu+2\nu-1}dz}{\sqrt{(1-z^{2\nu})}} = \frac{1}{2n+1} \cdot \frac{\pi}{2}$$

et posito $2n\nu+\nu-1=\mu$ fiet posito $z=1$

$$\int \frac{z^{\mu}dx}{\sqrt{(1-z^{2\nu})}} \cdot \int \frac{z^{\mu+\nu}dz}{\sqrt{(1-z^{2\nu})}} = \frac{1}{\nu(\mu+1)} \cdot \frac{\pi}{2}.$$

Scholion 1.

334. Quod tale productum binorum integralium exhiberi queat, eo magis est notatu dignum, quod aequalitas haec subsistit, etiamsi neutra formula neque algebraice neque per π exhiberi queat. Veluti si $\nu=2$ et $\mu=0$ fit

$$\int \frac{dz}{\sqrt{(1-z^4)}} \cdot \int \frac{zzdz}{\sqrt{(1-z^4)}} = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4},$$

simili-

ſimilique modo:

$\nu=3$, $\mu=0$ fit $\int\frac{dz}{\sqrt{(1-z^6)}}\cdot\int\frac{z^3\,dz}{\sqrt{(1-z^6)}}=\frac{1}{3}\cdot\frac{\pi}{2}=\frac{\pi}{6}$

$\nu=3$, $\mu=1$ fit $\int\frac{z\,dz}{\sqrt{(1-z^6)}}\cdot\int\frac{z^4\,dz}{\sqrt{(1-z^6)}}=\frac{1}{6}\cdot\frac{\pi}{2}=\frac{\pi}{12}$

$\nu=4$, $\mu=0$ fit $\int\frac{dz}{\sqrt{(1-z^8)}}\cdot\int\frac{z^4\,dz}{\sqrt{(1-z^8)}}=\frac{1}{4}\cdot\frac{\pi}{2}=\frac{\pi}{8}$

$\nu=4$, $\mu=2$ fit $\int\frac{zz\,dz}{\sqrt{(1-z^8)}}\cdot\int\frac{z^6\,dz}{\sqrt{(1-z^8)}}=\frac{1}{12}\cdot\frac{\pi}{2}=\frac{\pi}{24}$

$\nu=5$, $\mu=0$ fit $\int\frac{dz}{\sqrt{(1-z^{10})}}\cdot\int\frac{z^5\,dz}{\sqrt{(1-z^{10})}}=\frac{1}{5}\cdot\frac{\pi}{2}=\frac{\pi}{10}$

$\nu=5$, $\mu=1$ fit $\int\frac{z\,dz}{\sqrt{(1-z^{10})}}\cdot\int\frac{z^6\,dz}{\sqrt{(1-z^{10})}}=\frac{1}{10}\cdot\frac{\pi}{2}=\frac{\pi}{20}$

$\nu=5$, $\mu=2$ fit $\int\frac{zz\,dz}{\sqrt{(1-z^{10})}}\cdot\int\frac{z^7\,dz}{\sqrt{(1-z^{10})}}=\frac{1}{15}\cdot\frac{\pi}{2}=\frac{\pi}{30}$

$\nu=5$, $\mu=3$ fit $\int\frac{z^3\,dz}{\sqrt{(1-z^{10})}}\cdot\int\frac{z^8\,dz}{\sqrt{(1-z^{10})}}=\frac{1}{20}\cdot\frac{\pi}{2}=\frac{\pi}{40}$

quae Theoremata ſine dubio omni attentione ſunt digna.

Scholion 2.

335. Facile hinc etiam colligitur valor integralis $\int\frac{x^m\,dx}{\sqrt{(x-xx)}}$ poſito $x=1$, ſi enim ſcribamus $x=zz$, fiet hoc integrale $2\int\frac{z^{2m}\,dz}{\sqrt{(1-zz)}}$; quocirca pro caſu $x=1$ nanciſcimur ſequentes valores:

$$\int \frac{dx}{\sqrt{(x-xx)}} = \pi$$

$$\int \frac{x\,dx}{\sqrt{(x-xx)}} = \tfrac{1}{2}\pi$$

$$\int \frac{xx\,dx}{\sqrt{(x-xx)}} = \frac{1.3}{2.4}.\pi$$

$$\int \frac{x^3\,dx}{\sqrt{(x-xx)}} = \frac{1.3.5}{2.4.6}.\pi$$

$$\int \frac{x^4\,dx}{\sqrt{(x-xx)}} = \frac{1.3.5.7}{2.4.6.8}\pi$$

$$\int \frac{x^5\,dx}{\sqrt{(x-xx)}} = \frac{1.3.5.7.9}{2.4.6.8.10}\pi$$

$$\vdots$$

$$\int \frac{x^m\,dx}{\sqrt{(x-xx)}} = \frac{1.3.5\ldots\ldots(2m-1)}{2.4.6\ldots\ldots\ 2m}\pi.$$

Hinc ergo integralium huiusmodi formulas inuolventium, quae magis sunt complicata, valores, quos posito $x=1$ recipiunt per series succincte exprimi possunt, quem vsum aliquot exemplis declaremus.

Exemplum 1.

336. *Valorem integralis* $\int \frac{dx}{\sqrt{(1-x^4)}}$ *posito* $x=1$, *per seriem exhibere.*

Integrali detur haec forma $\int \frac{dx}{\sqrt{(1-xx)}}.(1+xx)^{-\frac{1}{2}}$, vt habeamus:

$$\int \frac{dx}{\sqrt{(1-x^4)}} = \int \frac{dx}{\sqrt{(1-xx)}}\left(1-\tfrac{1}{2}xx+\tfrac{1.3}{2.4}x^4-\tfrac{1.3.5}{2.4.6}x^6+\tfrac{1.3.5.7}{2.4.6.8}x^8-\text{etc.}\right)$$

singulis ergo terminis pro casu $x=1$ integratis orietur:

$$\int \frac{dx}{\sqrt{(1-x^4)}} = \frac{\pi}{2}\left(1-\tfrac{1}{4}+\tfrac{1.9}{4.16}-\tfrac{1.9.25}{4.16.36}+\tfrac{1.9.25.49}{4.16.36.64}-\text{etc.}\right)$$

Corol-

Corollarium.

337. Simili modo pro eodem casu $x = 1$, reperitur:

$$\int \frac{x\,dx}{\sqrt{(1-x^4)}} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \text{etc.} = \tfrac{\pi}{4}$$

$$\int \frac{xx\,dx}{\sqrt{(1-x^4)}} = \tfrac{\pi}{2}\left(\tfrac{1}{2} - \tfrac{1^2 \cdot 3}{2^2 \cdot 4} + \tfrac{1^2 \cdot 3^2 \cdot 5}{2^2 \cdot 4^2 \cdot 6} - \tfrac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} + \text{etc.}\right)$$

$$\int \frac{x^3\,dx}{\sqrt{(1-x^4)}} = \tfrac{2}{3} - \tfrac{4}{3 \cdot 5} + \tfrac{6}{5 \cdot 7} - \tfrac{8}{7 \cdot 9} + \tfrac{10}{9 \cdot 11} - \text{etc.}$$

est autem $\int \frac{x^3\,dx}{\sqrt{(1-x^4)}} = \frac{1}{2} - \frac{1}{2}\sqrt{(1-x^4)}$ ideoque $= \frac{1}{2}$ posito $x = 1$, vnde haec postrema series est $= \frac{1}{2}$.

Exemplum 2.

338. *Valorem integralis* $\int dx \sqrt{\frac{1+axx}{1-xx}}$ *casu* $x = 1$, *per seriem exhibere.*

Cum sit $\sqrt{(1+axx)} = 1 + \frac{1}{2}axx - \frac{1 \cdot 1}{2 \cdot 4}a^2x^4 + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}a^3x^6 - \text{etc.}$ erit per $\int \frac{dx}{\sqrt{(1-xx)}}$ multiplicando et integrando

$$\int dx \sqrt{\frac{1+axx}{1-xx}} = \tfrac{\pi}{2}\left(1 + \tfrac{1 \cdot 1}{2 \cdot 2}a - \tfrac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4}a^2 + \tfrac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6}a^3 - \text{etc.}\right)$$

vnde peripheriam ellipsis cognoscere licet.

Exemplum 3.

339. *Valorem integralis* $\int \frac{dx}{\sqrt{x(1-xx)}}$ *casu* $x = 1$ *per seriem exhibere.*

Repraesentetur haec formula ita $\int \frac{dx(1+x)^{-\frac{1}{2}}}{\sqrt{(x-xx)}}$ vt sit:

$$\int \frac{dx}{\sqrt{(x-xx)}}\left(1 - \tfrac{1}{2}x + \tfrac{1 \cdot 3}{2 \cdot 4}x^2 - \tfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \text{etc.}\right)$$

vnde

vnde feries haec obtinetur :

$$\int \frac{dx}{\sqrt{x(1-xx)}} = \pi \left(1 - \frac{1}{4} + \frac{1 \cdot 9}{4 \cdot 16} - \frac{1 \cdot 9 \cdot 25}{4 \cdot 16 \cdot 36} + \text{etc.} \right)$$

quae ab exemplo primo haud differt: quod non mirum, cum pofito $x = zz$ haec formula ad illam reducatur.

Problema 39.

340. Valorem integralis $\int x^{m-1} dx (1-xx)^{n-\frac{1}{2}}$, quod pofito $x = 0$ euanefcat, definire.

Solutio.

Reductiones fupra §. 128. datae praebent pro hoc cafu

$$\int x^{m-1} dx (1-xx)^{\frac{\mu}{2}+1} = \frac{x^m (1-xx)^{\frac{\mu}{2}+1}}{m+\mu+2} + \frac{\mu+1}{m+\mu+2} \int x^{m-1} dx (1-xx)^{\frac{\mu}{2}}$$

fumto ergo $\mu = 2n - 1$, erit

$$\int x^{m-1} dx (1-xx)^{n+\frac{1}{2}} = \frac{2n+1}{m+2n+1} \int x^{m-1} dx (1-xx)^{n-\frac{1}{2}}$$ pofito $x = 1$.

Cum igitur in praecedente problemate valor $\int \frac{x^{m-1} dx}{\sqrt{(1-xx)}}$ fit affignatus, quem breuitatis gratia ponamus $= M$, hinc ad fequentes progrediamur:

$$\int \frac{x^{m-1} dx}{\sqrt{(1-xx)}} = M$$

$$\int x^{m-1} dx (1-xx)^{\frac{1}{2}} = \frac{1}{m+1} M$$

$\int x^{m-1}$

$$\int x^{m-1}dx(1-xx)^{\frac{3}{2}} = \frac{1\cdot 3}{(m+1)(m+3)}M$$

$$\int x^{m-1}dx(1-xx)^{\frac{5}{2}} = \frac{1\cdot 3\cdot 5}{(m+1)(m+3)(m+5)}M$$

et in genere

$$\int x^{m-1}dx(1-xx)^{n-\frac{1}{2}} = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{(m+1)(m+3)(m+5)\ldots(m+2n-1)}M.$$

Iam duo caſus ſunt perpendendi, prout $m-1$ eſt vel numerus par vel impar: ſi enim

$m-1$ ſit par, erit $M = \frac{1\cdot 3\cdot 5\ldots(m-2)}{2\cdot 4\cdot 6\ldots(m-1)}\cdot\frac{\pi}{2}$

$n-1$ ſit impar, erit $M = \frac{2\cdot 4\cdot 6\ldots(m-2)}{3\cdot 5\cdot 7\ldots(m-1)}$.

Hinc ſequentes deducuntur valores:

$\int dx\sqrt{(1-xx)} = \frac{\pi}{4}$	$\int x\,dx\sqrt{(1-xx)} = \frac{1}{3}$
$\int x^2dx\sqrt{(1-xx)} = \frac{1}{4}\cdot\frac{\pi}{4}$	$\int x^3dx\sqrt{(1-xx)} = \frac{1}{3}\cdot\frac{2}{5}$
$\int x^4dx\sqrt{(1-xx)} = \frac{1\cdot 3}{4\cdot 6}\cdot\frac{\pi}{4}$	$\int x^5dx\sqrt{(1-xx)} = \frac{1}{3}\cdot\frac{2\cdot 4}{5\cdot 7}$
$\int x^6dx\sqrt{(1-xx)} = \frac{1\cdot 3\cdot 5}{4\cdot 6\cdot 8}\cdot\frac{\pi}{4}$	$\int x^7dx\sqrt{(1-xx)} = \frac{1}{3}\cdot\frac{2\cdot 4\cdot 6}{5\cdot 7\cdot 9}$
$\int dx(1-xx)^{\frac{3}{2}} = \frac{3\pi}{16}$	$\int x\,dx(1-xx)^{\frac{3}{2}} = \frac{1}{5}$
$\int xxdx(1-xx)^{\frac{3}{2}} = \frac{1}{6}\cdot\frac{3\pi}{16}$	$\int x^3dx(1-xx)^{\frac{3}{2}} = \frac{1}{5}\cdot\frac{2}{7}$
$\int x^4dx(1-xx)^{\frac{3}{2}} = \frac{1\cdot 3}{6\cdot 8}\cdot\frac{3\pi}{16}$	$\int x^5dx(1-xx)^{\frac{3}{2}} = \frac{1}{5}\cdot\frac{2\cdot 4}{7\cdot 9}$
$\int x^6dx(1-xx)^{\frac{3}{2}} = \frac{1\cdot 3\cdot 5}{6\cdot 8\cdot 10}\cdot\frac{3\pi}{16}$	$\int x^7dx(1-xx)^{\frac{3}{2}} = \frac{1}{5}\cdot\frac{2\cdot 4\cdot 6}{7\cdot 9\cdot 11}$

$\int dx(1-xx)^{\frac{5}{2}} = \frac{5\pi}{32}$	$\int x dx(1-xx)^{\frac{5}{2}} = \frac{1}{7}$
$\int x^2 dx(1-xx)^{\frac{5}{2}} = \frac{1}{8}\cdot\frac{5\pi}{32}$	$\int x^3 dx(1-xx)^{\frac{5}{2}} = \frac{1}{7}\cdot\frac{2}{9}$
$\int x^4 dx(1-xx)^{\frac{5}{2}} = \frac{1\cdot 3}{8\cdot 10}\cdot\frac{5\pi}{32}$	$\int x^5 dx(1-xx)^{\frac{5}{2}} = \frac{1}{7}\cdot\frac{2\cdot 4}{9\cdot 11}$
$\int x^6 dx(1-xx)^{\frac{5}{2}} = \frac{1\cdot 3\cdot 5}{8\cdot 10\cdot 12}\cdot\frac{5\pi}{32}$	$\int x^7 dx(1-xx)^{\frac{5}{2}} = \frac{1}{7}\cdot\frac{2\cdot 4\cdot 6}{9\cdot 11\cdot 13}$

etc.

Problema 40.

341. Valores integralium $\int \frac{x^m dx}{\sqrt[3]{(1-x^3)}}$ et $\int \frac{x^m dx}{\sqrt[3]{(1-x^3)^2}}$ posito $x=1$ assignare.

Solutio.

Ponamus pro casibus simplicissimis:

$$\int \frac{dx}{\sqrt[3]{(1-x^3)}} = A; \quad \int \frac{xdx}{\sqrt[3]{(1-x^3)}} = B; \quad \int \frac{xxdx}{\sqrt[3]{(1-x^3)}} = C$$

$$\int \frac{dx}{\sqrt[3]{(1-x^3)^2}} = A'; \quad \int \frac{xdx}{\sqrt[3]{(1-x^3)^2}} = B'; \quad \int \frac{xxdx}{\sqrt[3]{(1-x^3)^2}} = C'$$

et ex reductione prima §. 122. posito $a=1$ et $b=-1$, pro casu $x=1$ habemus

$$\int x^{m+n-1}dx(1-x^n)^{\frac{\mu}{\nu}} = \frac{m\nu}{m\nu+n\mu+n\nu}\int x^{m-1}dx(1-x^n)^{\frac{\mu}{\nu}}$$

ergo pro priori vbi $n=3$, $\nu=3$ et $\mu=-1$

$$\int x^{m+2}dx(1-x^3)^{-\frac{1}{3}} = \frac{m}{m+2}\int x^{m-1}dx(1-x^3)^{-\frac{1}{3}}$$

et pro posteriori vbi $n=3$, $\nu=3$ et $\mu=-2$

$$\int x^{m+2}dx(1-x^3)^{-\frac{2}{3}} = \frac{m}{m+1}\int x^{m-1}dx(1-x^3)^{-\frac{2}{3}}$$

hinc

hinc obtinemus pro forma priori:

$\int \frac{dx}{\sqrt[3]{(1-x^3)}} = A$	$\int \frac{x\,dx}{\sqrt[3]{(1-x^3)}} = B$	$\int \frac{xx\,dx}{\sqrt[3]{(1-x^3)}} = C$
$\int \frac{x^3dx}{\sqrt[3]{(1-x^3)}} = \frac{1}{3} A$	$\int \frac{x^4dx}{\sqrt[3]{(1-x^3)}} = \frac{2}{4} B$	$\int \frac{x^5dx}{\sqrt[3]{(1-x^3)}} = \frac{3}{5} C$
$\int \frac{x^6dx}{\sqrt[3]{(1-x^3)}} = \frac{1.4}{3.6} A$	$\int \frac{x^7dx}{\sqrt[3]{(1-x^3)}} = \frac{2.5}{4.7} B$	$\int \frac{x^8dx}{\sqrt[3]{(1-x^3)}} = \frac{3.6}{5.8} C$
$\int \frac{x^9dx}{\sqrt[3]{(1-x^3)}} = \frac{1.4.7}{3.6.9} A$	$\int \frac{x^{10}dx}{\sqrt[3]{(1-x^3)}} = \frac{2.5.8}{4.7.10} B$	$\int \frac{x^{11}dx}{\sqrt[3]{(1-x^3)}} = \frac{3.6.9}{5.8.11} C$
$\int \frac{x^{12}dx}{\sqrt[3]{(1-x^3)}} = \frac{1.4.7.10}{3.6.9.12} A$	$\int \frac{x^{13}dx}{\sqrt[3]{(1-x^3)}} = \frac{2.5.8.11}{4.7.10.13} B$	$\int \frac{x^{14}dx}{\sqrt[3]{(1-x^3)}} = \frac{3.6.9.12}{5.8.11.14} C$

etc.

at pro forma posteriori

$\int \frac{dx}{\sqrt[3]{(1-x^3)^2}} = A'$	$\int \frac{x\,dx}{\sqrt[3]{(1-x^3)^2}} = B'$	$\int \frac{xx\,dx}{\sqrt[3]{(1-x^3)^2}} = C'$
$\int \frac{x^3dx}{\sqrt[3]{(1-x^3)^2}} = \frac{1}{2} A'$	$\int \frac{x^4dx}{\sqrt[3]{(1-x^3)^2}} = B'$	$\int \frac{x^5dx}{\sqrt[3]{(1-x^3)^2}} = \frac{3}{4} C'$
$\int \frac{x^6dx}{\sqrt[3]{(1-x^3)^2}} = \frac{1.4}{2.5} A'$	$\int \frac{x^7dx}{\sqrt[3]{(1-x^3)^2}} = \frac{2.5}{3.6} B'$	$\int \frac{x^8dx}{\sqrt[3]{(1-x^3)^2}} = \frac{3.6}{4.7} C'$
$\int \frac{x^9dx}{\sqrt[3]{(1-x^3)^2}} = \frac{1.4.7}{2.5.8} A'$	$\int \frac{x^{10}dx}{\sqrt[3]{(1-x^3)^2}} = \frac{2.5.8}{3.6.9} B'$	$\int \frac{x^{11}dx}{\sqrt[3]{(1-x^3)^2}} = \frac{3.6.9}{4.7.10} C'$
$\int \frac{x^{12}dx}{\sqrt[3]{(1-x^3)^2}} = \frac{1.4.7.10}{2.5.8.11} A'$	$\int \frac{x^{13}dx}{\sqrt[3]{(1-x^3)^2}} = \frac{2.5.8.11}{3.6.9.12} B'$	$\int \frac{x^{14}dx}{\sqrt[3]{(1-x^3)^2}} = \frac{3.6.9.12}{4.7.10.13} C'$

vnde

vnde concludimus fore generaliter:

$$\int \frac{x^{3n}dx}{\sqrt[3]{(1-x^3)}} = \frac{1.4.7.\ldots.(3n-2)}{3.6.9.\ldots.\ 3n}A \qquad \int \frac{x^{3n}dx}{\sqrt[3]{(1-x^3)^2}} = \frac{1.4.7.\ldots.(3n-2)}{2.5.8.\ldots.(3n-1)}A'$$

$$\int \frac{x^{3n+1}dx}{\sqrt[3]{(1-x^3)}} = \frac{2.5.8.\ldots.(3n-1)}{4.7.10.\ldots.(3n+1)}B \qquad \int \frac{x^{3n+1}dx}{\sqrt[3]{(1-x^3)^2}} = \frac{2.5.8.\ldots.(3n-1)}{3.6.9.\ldots.(3n)}B'$$

$$\int \frac{x^{3n+2}dx}{\sqrt[3]{(1-x^3)}} = \frac{3.6.9.\ldots.3n}{5.8.11.\ldots.(3n+2)}C \qquad \int \frac{x^{3n+2}dx}{\sqrt[3]{(1-x^3)^2}} = \frac{3.6.9.\ldots.3n}{4.7.10.\ldots.(3n+1)}C'$$

notandum autem est esse $C = \frac{1}{2}$ et $C' = 1$.

Coroll. 1.

342. Hae formulae variis modis combinari possunt, vt egregia Theoremata inde oriantur, erit scilicet:

$$\int \frac{x^{3n}dx}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{x^{3n+2}dx}{\sqrt[3]{(1-x^3)^2}} = \frac{AC'}{3n+1} = \frac{1}{3n+1}\int \frac{dx}{\sqrt[3]{(1-x^3)}}$$

$$\int \frac{x^{3n+1}dx}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{x^{3n}dx}{\sqrt[3]{(1-x^3)^2}} = \frac{A'B}{3n+1} = \frac{1}{3n+1}\int \frac{dx}{\sqrt[3]{(1-x^3)^2}} \cdot \int \frac{xdx}{\sqrt[3]{(1-x^3)}}$$

$$\int \frac{x^{3n+2}dx}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{x^{3n+1}dx}{\sqrt[3]{(1-x^3)^2}} = \frac{2B'C}{3n+2} = \frac{1}{3n+2}\int \frac{xdx}{\sqrt[3]{(1-x^3)^2}}.$$

Coroll. 2.

343. Quia nunc ratio exponentium ad ternarium non amplius in computum ingreditur, erit gene-

generaliter:

$$\int \frac{x^{\lambda-1}dx}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{x^{\lambda+1}dx}{\sqrt[3]{(1-x^3)^2}} = \frac{1}{\lambda}\int \frac{dx}{\sqrt[3]{(1-x^3)}}$$

$$\int \frac{x^{\lambda}dx}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{x^{\lambda-1}dx}{\sqrt[3]{(1-x^3)^2}} = \frac{1}{\lambda}\int \frac{xdx}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{dx}{\sqrt[3]{(1-x^3)^2}}$$

$$\int \frac{x^{\lambda}dx}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{x^{\lambda-1}dx}{\sqrt[3]{(1-x^3)^2}} = \frac{1}{\lambda}\int \frac{xdx}{\sqrt[3]{(1-x^3)^2}}$$

quare ex binis poſtremis conſequimur:

$$\int \frac{xdx}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{dx}{\sqrt[3]{(1-x^3)^2}} = \int \frac{xdx}{\sqrt[3]{(1-x^3)^2}}.$$

Coroll. 3.

344. Ponatur $x = z^n$ et $\lambda n = m$, et noſtra Theoremata ſequentes induent formas:

$$\int \frac{z^{m-1}dz}{\sqrt[3]{(1-z^{3n})}} \cdot \int \frac{z^{m+2n-1}dz}{\sqrt[3]{(1-z^{3n})^2}} = \frac{1}{m}\int \frac{z^{n-1}dz}{\sqrt[3]{(1-z^{3n})}}$$

$$\int \frac{z^{m+n-1}dz}{\sqrt[3]{(1-z^{3n})}} \cdot \int \frac{z^{m-1}dz}{\sqrt[3]{(1-z^{3n})^2}} = \frac{n}{m}\int \frac{z^{2n-1}dz}{\sqrt[3]{(1-z^{3n})}} \cdot \int \frac{z^{n-1}dz}{\sqrt[3]{(1-z^{3n})^2}}$$

$$= \frac{1}{m}\int \frac{z^{2n-1}dz}{\sqrt[3]{(1-z^{3n})^2}}.$$

Problema 41.

345. Dato integrali $\int \frac{x^{m-1}dx}{(1-x^n)^{\frac{n-k}{n}}}$, assignare integrale huius formulae $\int \frac{x^{m+\lambda n-1}dx}{(1-x^n)^{\frac{n-k}{n}}}$ posito $x=1$.

Solutio.

Vt integrale sit finitum necesse est, vt m et k sint numeri positiui. Cum igitur per reductionem generalem sit

$$\int x^{m+n-1}dx(1-x^n)^{\frac{\mu}{\nu}} = \frac{m\nu}{m\nu+n(\mu+\nu)} \int x^{m-1}dx(1-x^n)^{\frac{\mu}{\nu}}$$

ponatur $\nu=n$ et $\mu=k-n$, vt sit $\mu+\nu=k$ erit

$$\int \frac{x^{m+n-1}dx}{(1-x^n)^{\frac{n-k}{n}}} = \frac{m}{m+k} \int \frac{x^{m-1}dx}{(1-x^n)^{\frac{n-k}{n}}}.$$

Ponatur ergo huius formulae valor, quia datur, $=$ A, haecque reductio repetita continuo dabit, posito breuitatis gratia P pro $(1-x^n)^{\frac{n-k}{n}}$

$$\int \frac{x^{m-1}dx}{P} = A$$

$$\int \frac{x^{m+n-1}dx}{P} = \frac{m}{m+k} A$$

$$\int \frac{x^{m+2n-1}dx}{P} = \frac{m(m+n)}{(m+k)(m+n+k)} A$$

$$\int \frac{x^{m+3n-1}dx}{P} = \frac{m(m+n)(m+2n)}{(m+k)(m+n+k)(m+2n+k)} A$$

et generaliter

$$\int \frac{x^{m+\alpha n-1}dx}{P} = \frac{m(m+n)(m+2n) \quad \ldots . (m+(\alpha-1)n)}{(m+k)(m+n+k)(m+2n+k) \ldots . (m+(\alpha-1)n+k)} A.$$

Coroll. 1.

346. Si simili modo alia formula sit $\int \frac{x^{p-1}dx}{(1-x^n)^{\frac{n-q}{n}}} = B$ posito $x = 1$, at breuitatis gratia scribatur Q pro $(1-x^n)^{\frac{n-q}{n}}$ habebimus

$$\int \frac{x^{p+\alpha n-1}dx}{Q} = \frac{p(p+n)(p+2n) \quad \ldots . (p+(\alpha-1)n)}{(p+q)(p+n+q)(p+2n+q) \ldots . (p+(\alpha-1)n+q)} B$$

quae totidem atque illa continet factores.

Coroll. 2.

347. Statuatur nunc $p = m+k$, vt posterior, numerator aequalis fiat priori denominatori, et productum harum duarum formularum est

$$\frac{m(m+n)(m+2n) \quad \ldots . (m+(\alpha-1)n)}{(m+k+q)(m+n+k+q)(m+2n+k+q) \ldots . (m+(\alpha-1)n+k+q)} AB$$

fiat porro $m+k+q = m+n$ seu $q = n-k$, erit hoc productum $= \frac{m}{m+\alpha n} AB$; ideoque

$$\int \frac{x^{m+\alpha n-1}dx}{(1-x^n)^{\frac{n-k}{n}}} \cdot \int \frac{x^{m+k+\alpha n-1}dx}{(1-x^n)^{\frac{k}{n}}} = \frac{m}{m+\alpha n} \int \frac{x^{m-1}dx}{(1-x^n)^{\frac{n-k}{n}}} \cdot \int \frac{x^{m+k-1}dx}{(1-x^n)^{\frac{k}{n}}}$$

 quod

quod eſt Theorema omni attentione dignum, cum hic non amplius opus ſit, vt α ſit numerus integer.

Coroll. 3.

348. Quare loco $m+\alpha n$ ſcribamus μ, erit:

$$\mu\int\frac{x^{\mu-1}dx}{(1-x^n)^{\frac{n-k}{n}}}\int\frac{x^{\mu+k-1}dx}{(1-x^n)^{\frac{k}{n}}}=m\int\frac{x^{m-1}dx}{(1-x^n)^{\frac{n-k}{n}}}\cdot\int\frac{x^{m+k-1}dx}{(1-x^n)^{\frac{k}{n}}}.$$

Hinc ſi ſumamus $m+k=n$ ſeu $m=n-k$, ob $\int\frac{x^{n-1}dx}{(1-x^n)^{\frac{k}{n}}}=\frac{1-(1-x^n)^{\frac{n-k}{n}}}{n-k}=\frac{1}{n-k}$ poſito $x=1$, erit

$$\int\frac{x^{\mu-1}dx}{(1-x^n)^{\frac{n-k}{n}}}\cdot\int\frac{x^{\mu+k-1}dx}{(1-x^n)^{\frac{k}{n}}}=\frac{1}{\mu}\int\frac{x^{n-k-1}dx}{(1-x^n)^{\frac{n-k}{n}}}=\frac{\pi}{\mu n\,\text{ſin.}\frac{k\pi}{n}}.$$

Ac poſito $x=z^{\nu}$, tum vero $\mu n=p$, $\nu n=q$, et $k=\lambda n$ habebitur:

$$\int\frac{z^{p-1}dz}{(1-z^q)^{1-\lambda}}\cdot\int\frac{z^{p+\lambda q-1}dz}{(1-z^q)^{\lambda}}=\frac{1}{p}\int\frac{z^{(1-\lambda)q-1}dz}{(1-z^q)^{1-\lambda}}.$$

Scholion 1.

349. Theoremata particuliaria, quae hinc conſequuntur, ita ſe habebunt:

I. $n=2$; $k=1$; $\int\frac{x^{\mu-1}dx}{\sqrt{(1-xx)}}\cdot\int\frac{x^{\mu}dx}{\sqrt{(1-xx)}}=\frac{1}{\mu}\int\frac{dx}{\sqrt{(1-xx)}}=\frac{\pi}{2\mu}$

II.

II. $n=3;\ k=1;\ \int\frac{x^{\mu-1}dx}{\sqrt[3]{(1-x^3)^2}}\cdot\int\frac{x^{\mu}dx}{\sqrt[3]{(1-x^3)}}=\frac{1}{\mu}\int\frac{xdx}{\sqrt[3]{(1-x^3)^2}}=\frac{2\pi}{3\mu\sqrt{3}}$

$n=3;\ k=2;\ \int\frac{x^{\mu-1}dx}{\sqrt[3]{(1-x^3)}}\cdot\int\frac{x^{\mu+1}dx}{\sqrt[3]{(1-x^3)^2}}=\frac{1}{\mu}\int\frac{dx}{\sqrt[3]{(1-x^3)}}=\frac{2\pi}{3\mu\sqrt{3}}$

III. $n=4;\ k=1;\ \int\frac{x^{\mu-1}dx}{\sqrt[4]{(1-x^4)^3}}\cdot\int\frac{x^{\mu}dx}{\sqrt[4]{(1-x^4)}}=\frac{1}{\mu}\int\frac{xxdx}{\sqrt[4]{(1-x^4)^3}}=\frac{\pi}{2\mu\sqrt{2}}$

$n=4;\ k=2;\ \int\frac{x^{\mu-1}dx}{\sqrt{(1-x^4)}}\cdot\int\frac{x^{\mu+1}dx}{\sqrt{(1-x^4)}}=\frac{1}{\mu}\int\frac{xdx}{\sqrt{(1-x^4)}}=\frac{\pi}{4\mu}$

$n=4;\ k=3;\ \int\frac{x^{\mu-1}dx}{\sqrt[4]{(1-x^4)}}\cdot\int\frac{x^{\mu+2}dx}{\sqrt[4]{(1-x^4)^3}}=\frac{1}{\mu}\int\frac{dx}{\sqrt[4]{(1-x^4)}}=\frac{\pi}{2\mu\sqrt{2}}$

etc.

Vbi notandum eſt formulam $\int\frac{x^{n-k-1}dx}{(1-x^n)^{\frac{n-k}{n}}}$ ad rationalitatem reduci poſſe. Ponatur enim $\frac{x^n}{1-x^n}=z^n$ ſeu $x^n=\frac{z^n}{1+z^n}$, vnde $\frac{dx}{x}=\frac{dz}{z(1+z^n)}$. Quare cum formula noſtra ſit $=\int(\frac{x^n}{1-x^n})^{\frac{n-k}{n}}\cdot\frac{dx}{x}$ euadet ea $=\int\frac{z^{n-k-1}dz}{1+z^n}$ cuius integrale ita determinari debet, vt euaneſcat poſito $x=0$ ideoque $z=0$: tum vero poſito $x=1$, hoc eſt $z=\infty$ dabit valorem, quo hic vtimur. Mox autem oſtendemus valorem huius integralis $\int\frac{z^{n-k-1}dz}{1+z^n}$ poſito

$z=\infty$,

$z = \infty$, ideoque et huius $\int \frac{x^{n-k-1}dx}{(1-x^n)^{\frac{n-k}{n}}}$ per angulos exprimi posse: quorum valores hic statim apposui. Deinde etiam notari meretur formulae $\int \frac{x^{m-1}dx}{(1-x^n)^{\frac{n-k}{n}}}$ haec transformatio oriunda, posito $1 - x^n = z^n$, quae praebet: $-\int \frac{z^{k-1}dz}{(1-z^n)^{\frac{n-m}{n}}}$, ita integranda, vt euanescat posito $x = 0$ seu $z = 1$, tum vero statui debet $x = 1$ seu $z = 0$. Quod eodem redit, ac si mutato signo haec formula $\int \frac{z^{k-1}dz}{(1-z^n)^{\frac{n-m}{n}}}$ ita integretur, vt euanescat, posito $z = 0$, tum vero ponatur $z = 1$. Cum iam nihil impediat quo minus loco z scribamus x, habebimus hoc insigne Theorema:

$$\int \frac{x^{m-1}dx}{(1-x^n)^{\frac{n-k}{n}}} = \int \frac{x^{k-1}dx}{(1-x^n)^{\frac{n-m}{n}}}$$

ita vt in huiusmodi formula exponentes m et k inter se commutare liceat, pro casu scilicet $x = 1$. Ita pro praecedente formula ad rationalitatem reducibili, vbi $m = n - k$ erit

$$\int \frac{x^{n-k-1}dx}{(1-x^n)^{\frac{n-k}{n}}} = \int \frac{x^{k-1}dx}{(1-x^n)^{\frac{k}{n}}}$$

vnde

vnde sequitur etiam fore posito $z = \infty$

$$\int \frac{z^{n-k-1}\,dz}{1+z^n} = \int \frac{z^{k-1}\,dz}{1+z^n}.$$

Scholion 2.

350. Hinc etiam formularum magis compositarum integralia pro casu $x=1$; per series concinnas exprimi possunt. Cum enim in reductione superiori posito $m+k=\mu$ seu $k=\mu-m$ sit

$$\int \frac{x^{m+n-1}\,dx}{(1-x^n)^{\frac{m+n-\mu}{n}}} = \frac{m}{\mu}\int \frac{x^{m-1}\,dx}{(1-x^n)^{\frac{m+n-\mu}{n}}}$$

si habeatur huiusmodi formula differentialis

$$dy = \frac{x^{m-1}\,dx}{(1-x^n)^{\frac{m+n-\mu}{n}}}(A + Bx^n + Cx^{2n} + Dx^{3n} + \text{etc.})$$

quam ita integrari oporteat, vt y euanescat posito $x=0$, ac requiratur valor ipsius y casu $x=1$, erit si hoc casu fieri ponamus $\int \frac{x^{m-1}\,dx}{(1-x^n)^{\frac{m+n-\mu}{n}}} = O$ iste valor

$$= O\left(A + \frac{m}{\mu}B + \frac{m(m+n)}{\mu(+n)}C + \frac{m(m+n)(m+2n)}{\mu(\mu+n)(\mu+2n)}D + \text{etc.}\right)$$

Vicissim ergo proposita hac serie

$$A + \frac{m}{\mu}B + \frac{m(m+n)}{\mu(\mu+n)}C + \frac{m(m+n)(m+2n)}{\mu(\mu+n)(\mu+2n)}D + \text{etc.}$$

eius summa aequabitur huic formulae integrali

$$\frac{1}{O}\int \frac{x^{m-1}\,dx}{(1-x^n)^{\frac{m+n-\mu}{n}}}(A + Bx^n + Cx^{2n} + Dx^{3n} + \text{etc.})$$

si

ſi poſt integrationem ponatur $x = 1$. Quod ſi ergo eueniat, vt huius ſeriei $A + Bx^n + Cx^{2n} +$ etc. ſumma aſſignari, indeque integratio abſolui queat, obtinebitur ſumma illius ſeriei.

Problema 42.

351. Integralis huius formulae $\frac{x^{m-1}dx}{1+x^n}$ ita determinatum, vt poſito $x = 0$ euaneſcat, valorem caſu $x = \infty$ aſſignare.

Solutio.

Huius formulae integrale iam ſupra §. 77. exhibuimus, et quidem ita determinatum, vt poſito $x = 0$ euaneſcat, quod poſito breuitatis gratia $\frac{\pi}{n} = \omega$ ita ſe habet:

$$-\tfrac{2}{n}\operatorname{cof.} m\omega\, l\sqrt{(1 - 2x\operatorname{cof.}\omega + xx)} + \tfrac{2}{n}\operatorname{fin.} m\omega.\operatorname{Arc.tang.}\frac{x\operatorname{fin.}\omega}{1 - x\operatorname{cof.}\omega}$$

$$-\tfrac{2}{n}\operatorname{cof.} 3m\omega\, l\sqrt{(1 - 2x\operatorname{cof.}3\omega + xx)} + \tfrac{2}{n}\operatorname{fin.} 3m\omega.\operatorname{Arc.tang.}\frac{x\operatorname{fin.}3\omega}{1 - x\operatorname{cof.}3\omega}$$

$$-\tfrac{2}{n}\operatorname{cof.} 5m\omega\, l\sqrt{(1 - 2x\operatorname{cof.}5\omega + xx)} + \tfrac{2}{n}\operatorname{fin.} 5m\omega.\operatorname{Arc.tang.}\frac{x\operatorname{fin.}5\omega}{1 - x\operatorname{cof.}5\omega}$$

$$\vdots$$

$$-\tfrac{2}{n}\operatorname{cof.} \lambda m\omega\, l\sqrt{(1 - 2x\operatorname{cof.}\lambda\omega + xx)} + \tfrac{2}{n}\operatorname{fin.} \lambda m\omega.\operatorname{Arc.tang.}\frac{x\operatorname{fin.}\lambda\omega}{1 - x\operatorname{cof.}\lambda\omega}$$

vbi λ denotat maximum numerum imparem exponente n minorem, ac ſi n fuerit ipſe numerus impar,

par, insuper accedit pars $\pm\frac{1}{n}l(1+x)$, prout m fuerit vel numerus impar, vel par; illo scilicet casu signum $+$, hoc vero signum $-$ valet. Hic igitur quaeritur istius integralis valor, qui prodit posito $x=\infty$. Primo ergo partes logarithmos implicantes expendamus, et quia ob $x=\infty$ est $l\sqrt{(1-2x\operatorname{cos}.\lambda\omega+xx)}=l(x-\operatorname{cos}.\lambda\omega)=lx+l(1-\frac{\operatorname{cos}.\lambda\omega}{x})=lx$, ob $\frac{\operatorname{cos}.\lambda\omega}{x}=0$; vnde partes logarithmicae praebent:

$$-\frac{2lx}{n}(\operatorname{cos}.m\omega+\operatorname{cos}.3m\omega+\operatorname{cos}.5m\omega+\ldots+\operatorname{cos}.\lambda m\omega)$$
$$[\pm\frac{lx}{n}\text{ si } n \text{ impar }].$$

Ponamus hanc seriem cosinuum

$$\operatorname{cos} m\omega+\operatorname{cos}.3m\omega+\operatorname{cos}.5m\omega+\ldots+\operatorname{cos}\lambda m\omega=s$$

eritque per $2\operatorname{sin} m\omega$ multiplicando

$$\begin{aligned}2s\operatorname{sin}.m\omega=&\operatorname{sin}.2m\omega+\operatorname{sin}.4m\omega+\operatorname{sin}.6m\omega\ldots+\operatorname{sin}.(\lambda+1)m\omega\\&-\operatorname{sin}.2m\omega-\operatorname{sin}.4m\omega-\operatorname{sin}.6m\omega\end{aligned}$$

vnde fit $s=\frac{\operatorname{sin}.(\lambda+1)m\omega}{2\operatorname{sin}.m\omega}$. Quare si n sit numerus par erit $\lambda=n-1$, sicque partes logarithmicae fiunt

$$-\frac{lx}{n}\cdot\frac{\operatorname{sin}.nm\omega}{\operatorname{sin}.m\omega}=-\frac{lx}{n}\cdot\frac{\operatorname{sin}.m\pi}{\operatorname{sin}.m\omega}\text{ ob } n\omega=\pi.$$

At propter m numerum integrum est $\operatorname{sin}.m\pi=0$, vnde hae partes euanescunt. Sin autem sit n numerus impar, est $\lambda=n-2$, et summa partium logarithmicarum fit

$$-\frac{lx}{n}\cdot\frac{\operatorname{sin}.(n-1)m\omega}{\operatorname{sin}.m\omega}\pm\frac{lx}{n}$$

at $\operatorname{sin}.(n-1)m\omega=\operatorname{sin}.(m\pi-m\omega)=\pm\operatorname{sin}.m\omega$, vbi signum superius valet, si m sit numerus impar, con-

tra vero inferius, quod idem de altera ambiguitate eſt tenendum, ita vt habeamus $\mp \frac{lx}{n} \cdot \frac{\sin. m\omega}{\sin. m\omega} \pm \frac{lx}{n} = 0$. Perpetuo ergo partes logarithmicae ſe mutuo tollunt; quod etiam inde eſt perſpicuum, quod alioquin integrale foret infinitum, cum tamen manifeſto debeat eſſe finitum.

Relinquuntur ergo ſoli anguli, quos in vnam ſummam colligamus; conſideretur ergo Arc. tang. $\frac{x \sin. \lambda\omega}{1 - x\cos. \lambda\omega}$, qui arcus caſu $x = 0$ euaneſcit, tum vero caſu $x = \frac{1}{\cos. \lambda\omega}$ fit quadrans, vlterius ergo aucta x quadrantem ſuperabit, donec facto $x = \infty$ eius tangens fiat $= -\frac{\sin. \lambda\omega}{\cos. \lambda\omega} = -\text{tang.} \lambda\omega = \text{tang.}(\pi - \lambda\omega)$ ideoque ipſe arcus $= \pi - \lambda\omega$, ex quo hi arcus iunctim ſumti dabunt:

$$\tfrac{2}{n}((\pi-\omega)\sin. m\omega + (\pi-3\omega)\sin. 3m\omega + (\pi-5\omega)\sin. 5m\omega + \ldots (\pi-\lambda\omega)\sin. \lambda m\omega)$$

vnde duas ſeries adipiſcimur:

$$\tfrac{2\pi}{n}(\sin. m\omega + \sin. 3m\omega + \sin. 5m\omega + \ldots + \sin. \lambda m\omega) = \tfrac{2\pi}{n} p$$

$$\tfrac{-2\omega}{n}(\sin. m\omega + 3\sin. 3m\omega + 5\sin. 5m\omega + \ldots + \lambda\sin. \lambda m\omega) = \tfrac{-2\omega}{n} q,$$

quas ſeorſim inueſtigemus, ac pro poſteriori quidem cum ante habuiſſemus:

$$\cos. m\omega + \cos. 3m\omega + \cos. 5m\omega + \ldots \cos. \lambda m\omega = s = \frac{\sin. (\lambda+1) m\omega}{2\sin. m\omega}$$

ſi angulum ω vt variabilem ſpectemus, differentiatio praebet:

$$-m d\omega(\sin. m\omega + 3\sin. 3m\omega + 5\sin. 5m\omega + \ldots + \lambda\sin. \lambda m\omega) = \frac{(\lambda+1) m d\omega \cos. (\lambda+1) m\omega}{2\sin. m\omega} - \frac{m d\omega \sin. (\lambda+1) m\omega \cos. m\omega}{2\sin. m\omega^2}$$

ergo

ergo

$$-q = \frac{(\lambda+1)\,\mathrm{cof.}\,(\lambda+1)m\omega}{2\,\mathrm{fin.}\,m\omega} - \frac{\mathrm{fin.}\,(\lambda+1)m\omega\,\mathrm{cof.}\,m\omega}{2\,\mathrm{fin.}\,m\omega^2}$$

feu

$$-q = \frac{\lambda\,\mathrm{cof.}\,(\lambda+1)m\omega}{2\,\mathrm{fin.}\,m\omega} - \frac{\mathrm{fin.}\,\lambda m\omega}{2\,\mathrm{fin.}\,m\omega^2}.$$

Pro altera ferie

$$p = \mathrm{fin}\,m\omega + \mathrm{fin.}\,3m\omega + \mathrm{fin.}\,5m\omega + \ldots + \mathrm{fin.}\,\lambda m\omega$$

multiplicemus vtrinque per $2\,\mathrm{fin.}\,m\omega$, fietque

$$2p\,\mathrm{fin.}\,m\omega = 1 - \mathrm{cof.}\,2m\omega - \mathrm{cof.}\,4m\omega - \mathrm{cof.}\,6m\omega \ldots\ldots - \mathrm{cof.}\,(\lambda+1)m\omega$$
$$+ \mathrm{cof.}\,2m\omega + \mathrm{cof.}\,4m\omega + \mathrm{cof.}\,6m\omega$$

ficque erit $p = \frac{1 - \mathrm{cof.}\,(\lambda+1)m\omega}{2\,\mathrm{fin.}\,m\omega}$.

Quodfi iam fuerit n numerus par, erit $\lambda = n-1$, indeque

$\mathrm{cof.}\,(\lambda+1)m\omega = \mathrm{cof.}\,nm\omega = \mathrm{cof.}\,m\pi$ et $\mathrm{fin.}\,(\lambda+1)m\omega = \mathrm{fin.}\,m\pi = 0$ ergo $p = \frac{1-\mathrm{cof.}\,m\pi}{2\,\mathrm{fin.}\,m\omega}$ et $-q = \frac{n\,\mathrm{cof.}\,m\pi}{2\,\mathrm{fin.}\,m\omega}$; hincque omnes arcus iunctim fumti $\frac{2\pi}{n}.\frac{(1-\mathrm{cof.}\,m\pi)}{2\,\mathrm{fin.}\,m\omega} + \frac{2\omega}{n}.\frac{n\,\mathrm{cof.}\,m\pi}{2\,\mathrm{fin.}\,m\omega} = \frac{\pi}{n\,\mathrm{fin.}\,m\omega}$ ob. $n\omega = \pi$.

Sit nunc n numerus impar, erit $\lambda = n-2$, indeque:

$\mathrm{cof.}\,(\lambda+1)m\omega = \mathrm{cof.}\,(m\pi - m\omega)$, et $\mathrm{fin.}\,(\lambda+1)m\omega = \mathrm{fin.}\,(m\pi - m\omega.)$ feu

$\mathrm{cof.}\,(\lambda+1)m\omega = \mathrm{cof.}\,m\pi\,\mathrm{cof.}\,m\omega$, et $\mathrm{fin.}\,(\lambda+1)m\omega = -\mathrm{cof.}\,m\pi\,\mathrm{fin.}\,m\omega$. ergo

$p = \frac{1-\mathrm{cof.}\,m\pi.\,\mathrm{cof.}\,m\omega}{2\,\mathrm{fin.}\,m\omega}$ et $-q = \frac{(n-1)\,\mathrm{cof.}\,m\pi.\,\mathrm{cof.}\,m\omega}{2\,\mathrm{fin.}\,m\omega} + \frac{\mathrm{cof.}\,m\pi\,\mathrm{cof.}\,m\omega}{2\,\mathrm{fin.}\,m\omega}$

vnde fumma omnium angulorum

$$\frac{\pi(1-\mathrm{cof.}\,m\pi.\,\mathrm{cof.}\,m\omega)}{n\,\mathrm{fin.}\,m\omega} + \frac{\omega(n-1)\,\mathrm{cof.}\,m\pi.\,\mathrm{cof.}\,m\omega}{n\,\mathrm{fin.}\,m\omega} + \frac{\omega\,\mathrm{cof.}\,m\pi\,\mathrm{cof.}\,m\omega}{n\,\mathrm{fin.}\,m\omega}$$

quae ob $n\omega = \pi$ reducitur ad $\frac{\pi}{n\,\text{fin.}\,m\omega}$.

Siue ergo exponens n fit positiuus siue negatiuus, posito $x = \infty$ habemus

$$\int \frac{x^{m-1}dx}{1+x^n} = \frac{\pi}{n\,\text{fin.}\,m\omega} = \frac{\pi}{n\,\text{fin.}\frac{m\pi}{n}}.$$

Coroll. 1.

352. Hinc ergo erit formula supra memorata (349)

$$\int \frac{z^{n-k-1}dz}{1+z^n} = \int \frac{z^{k-1}dz}{1+z^n} = \frac{\pi}{n\,\text{fin.}\frac{(n-k)\pi}{n}} = \frac{\pi}{n\,\text{fin.}\frac{k\pi}{n}} \text{ posito } z = \infty.$$

Vnde sequitur fore etiam formulam, cui hanc aequari ostendimus:

$$\int \frac{x^{n-k-1}dx}{(1-x^n)^{\frac{n-k}{n}}} = \int \frac{x^{k-1}dx}{(1-x^n)^{\frac{k}{n}}} = \frac{\pi}{n\,\text{fin.}\frac{k\pi}{n}} \text{ posito } x = 1.$$

Coroll. 2.

353. Percurramus casus simpliciores, pro vtroque formularum genere, posito $z = \infty$ et $x = 1$:

$$\int \frac{dz}{1+zz} = \int \frac{dx}{\sqrt{(1-xx)}} = \frac{\pi}{2\,\text{fin.}\frac{1}{2}\pi} = \frac{\pi}{2}$$

$$\int \frac{dz}{1+z^3} = \int \frac{zdz}{1+z^3} = \int \frac{dx}{\sqrt[3]{(1-x^3)}} = \int \frac{xdx}{\sqrt[3]{(1+x^3)^2}} = \frac{\pi}{3\,\text{fin.}\frac{1}{3}\pi} = \frac{2\pi}{3\sqrt{3}}$$

$$\int \frac{dz}{1+z^4} = \int \frac{zzdz}{1+z^4} = \int \frac{dx}{\sqrt[4]{(1-x^4)}} = \int \frac{xxdx}{\sqrt[4]{(1-x^4)^3}} = \frac{\pi}{4\,\text{fin.}\frac{1}{4}\pi} = \frac{\pi}{2\sqrt{2}}$$

$$\int \frac{dz}{1+z^6} = \int \frac{z^4dz}{1+z^6} = \int \frac{dx}{\sqrt[6]{(1-x^6)}} = \int \frac{x^4dx}{\sqrt[6]{(1-x^6)^5}} = \frac{\pi}{6\,\text{fin.}\frac{1}{6}\pi} = \frac{\pi}{3}.$$

Coroll. 3.

Coroll. 3.

354. Cum fit

$$\frac{1}{(1-x^n)^{\frac{k}{n}}}=1+\frac{k}{n}x^n+\frac{k(k+n)}{n.2n}x^{2n}+\frac{k(k+n)(k+2n)}{n.2n.3n}x^{3n}\text{ etc.}$$

erit per $x^{k-1}dx$ multiplicando, tum integrando, ac $x=1$ ponendo

$$\frac{\pi}{n\sin.\frac{k\pi}{n}}=\frac{1}{k}+\frac{k}{n(k+n)}+\frac{k(k+n)}{n.2n(k+2n)}+\frac{k(k+n)(k+2n)}{n.2n.3n.(k+3n)}\text{ etc.}$$

et loco k fcribendo $n-k$ erit quoque

$$\frac{\pi}{n\sin.\frac{k\pi}{n}}=\frac{1}{n-k}+\frac{n-k}{n(2n-k)}+\frac{(n-k)(2n-k)}{n.2n.(3n-k)}+\frac{(n-k)(2n-k)(3n-k)}{n.2n.3n(4n-k)}\text{ etc.}$$

Scholion.

355. Pro formulis quantitates tranfcendentes continentibus fupra iam praecipuos valores, quos integralia dum variabili certus quidam valor tribuitur, recipiunt, euoluimus, ita vt non opus fit huiusmodi formulas hic denuo examinare. Hinc autem intelligitur eos valores integralis $\int Xdx$ prae reliquis effe notatu dignos, ac plerumque multo fuccinctius exprimi poffe, qui eiusmodi valoribus variabilis x refpondent, quibus functio X vel fit infinita vel in nihilum abit. Ita integralia formularum $\int\frac{x^{m-1}dx}{(1-x^n)^{\frac{\mu}{\nu}}}$ et $\int\frac{z^{m-1}dz}{1+z^n}$, valores prae reliquis memorabiles recipiunt, fi fiat $x=1$ et $z=\infty$, vbi illius denominator euanefcit, huius vero fit infinitus.

 Caete-

Caeterum omni attentione dignum eſt, quod hic oſtendimus, formulae integralis $\int \frac{z^{m-1}dz}{1+z^n}$ valorem caſu $z=\infty$ tam concinne exprimi, vt ſit $\frac{\pi}{n\,\text{ſin.}\frac{m}{n}\pi}$, cuius demonſtratio cum per tot ambages ſit adſtructa, merito ſuſpicionem excitat, eam via multo faciliori confici poſſe, etiamſi modus nondum perſpiciatur. Id quidem manifeſtum eſt, hanc demonſtrationem ex ratione ſinuum angulorum multiplorum peti oportere; et quoniam in introductione ſin. $\frac{m}{n}\pi$ per productum infinitorum factorum expreſſi, mox videbimus, inde eandem veritatem multo facilius deduci poſſe, etiamſi ne hanc quidem viam pro maxime naturali haberi velim. Sequens autem caput huiusmodi inueſtigationi deſtinaui, quo valores integralium, quos vti in hoc capite certo quodam caſu recipiunt, per producta infinita ſeu ex innumeris factoribus conſtantia exprimere docebo; quandoquidem hinc inſignia ſubſidia in Analyſin redundant, pluraque alia incrementa inde expectari poſſunt.

CAPVT IX.

CAPVT IX.

DE EVOLVTIONE INTEGRALIVM PER PRODVCTA INFINITA.

Problema 43.

356.

Valorem huius integralis $\int\frac{dx}{\sqrt{(1-xx)}}$, quem casu $x=1$ recipit, in productum infinitum euolvere.

Solutio.

Quemadmodum supra formulas altiores ad simplicem reduximus, ita hic formulam $\int\frac{dx}{\sqrt{(1-xx)}}$ continuo ad altiores perducamus. Ita cum posito $x=1$ sit

$$\int\frac{x^{m-1}dx}{\sqrt{(1-xx)}}=\frac{m+1}{m}\int\frac{x^{m+1}dx}{\sqrt{(1-xx)}} \text{ erit}$$

$$\int\frac{dx}{\sqrt{(1-xx)}}=\frac{2}{1}\int\frac{xxdx}{\sqrt{(1-xx)}}=\frac{2.4}{1.3}\int\frac{x^4dx}{\sqrt{(1-xx)}}=\frac{2.4.6}{1.3.5}\int\frac{x^6dx}{\sqrt{(1-xx)}} \text{ etc.}$$

vnde concludimus fore indefinite:

$$\frac{dx}{\sqrt{(1-xx)}}=\frac{1.3.5.7\ldots\ldots 2i}{2.4.6.8\ldots\ldots(2i-1)}\int\frac{x^{2i}dx}{\sqrt{(1-xx)}}$$

atque adeo etiam si pro i sumatur numerus infinitus. Nunc

Nunc simili modo a formula $\int \frac{x\,dx}{\sqrt{(1-xx)}}$ ascendamus reperiemusque

$$\int \frac{x\,dx}{\sqrt{(1-xx)}} = \frac{3.5.7.9\ldots\ldots(2i+1)}{2.4.6.8\ldots\ldots 2i} \int \frac{x^{2i+1}\,dx}{\sqrt{(1-xx)}}$$

atque obseruo si i sit numerus infinitus, formulas istas $\int \frac{x^{2i}\,dx}{\sqrt{(1-xx)}}$ et $\int \frac{x^{2i+1}\,dx}{\sqrt{(1-xx)}}$ rationem aequalitatis esse habituras. Ex reductione enim principali perspicuum est, si m sit numerus infinitus, fore $\int \frac{x^{m-1}\,dx}{\sqrt{(1-xx)}} = \int \frac{x^{m+1}\,dx}{\sqrt{(1-xx)}} = \int \frac{x^{m+3}\,dx}{\sqrt{(1-xx)}}$, atque adeo in genere $\int \frac{x^{m+\mu}\,dx}{\sqrt{(1-xx)}} = \int \frac{x^{m+\nu}\,dx}{\sqrt{(1-xx)}}$, quantumuis magna fuerit differentia inter μ et ν, modo finita. Cum igitur sit $\int \frac{x^{2i}\,dx}{\sqrt{(1-xx)}} = \int \frac{x^{2i+1}\,dx}{\sqrt{(1-xx)}}$, si ponamus:

$$\frac{2.4.6.\ldots\ldots 2i}{1.3.5.\ldots\ldots(2i-1)} = M \quad \text{et} \quad \frac{3.5.7.\ldots\ldots(2i+1)}{2.4.6.\ldots\ldots 2i} = N$$

erit $\int \frac{dx}{\sqrt{(1-xx)}} : \int \frac{x\,dx}{\sqrt{(1-xx)}} = M:N = \frac{M}{N}:1$; posito $x=1$.

At est $\int \frac{x\,dx}{\sqrt{(1-xx)}} = 1$ et $\int \frac{dx}{\sqrt{(1-xx)}} = \frac{\pi}{2}$, vnde colligitur $\int \frac{dx}{\sqrt{(1-xx)}} = \frac{M}{N}$, quia producta M et N ex aequali factorum numero constant, si primum factorem $\frac{2}{1}$ producti M per primum factorem $\frac{3}{2}$ producti N, secundum $\frac{4}{3}$ illius, per secundum $\frac{5}{4}$ huius et ita porro diuidamus, fiet

$$\frac{M}{N} = \frac{2.2}{1.3} \cdot \frac{4.4}{3.5} \cdot \frac{6.6}{5.7} \cdot \frac{8.8}{7.9} \text{ etc.}$$

vnde

vnde obtinemus pro casu $x = 1$ per productum infinitum

$$\int \frac{dx}{\sqrt{(1-xx)}} = \frac{2.2}{1.3} \cdot \frac{4.4}{3.5} \cdot \frac{6.6}{5.7} \cdot \frac{8.8}{7.9} \cdot \text{ etc.} = \frac{\pi}{2}.$$

Coroll. 1.

357. Pro valore ergo ipsius π idem productum infinitum elicuimus, quod olim iam Wallisius inuenerat, et cuius veritatem in Introductione confirmauimus, diuersissimis viis incedentes, erit itaque

$$\pi = 2 \cdot \frac{2.2}{1.3} \cdot \frac{4.4}{3.5} \cdot \frac{6.6}{5.7} \cdot \frac{8.8}{7.9} \text{ etc.}$$

Coroll. 2.

358. Nihil interest, quonam ordine singuli factores in hoc producto disponantur, dummodo nulli relinquantur. Ita aliquot ab initio seorsim sumendo; reliqui ordine debito disponi possunt, veluti

$$\frac{\pi}{2} = \frac{2}{1} \times \frac{2.4}{3.3} \cdot \frac{4.6}{5.5} \cdot \frac{6.8}{7.7} \cdot \frac{8.10}{9.9} \text{ etc. vel}$$

$$\frac{\pi}{2} = \frac{2.4}{1.3} \times \frac{2.6}{3.5} \cdot \frac{4.8}{5.7} \cdot \frac{6.10}{7.9} \cdot \frac{8.12}{9.11} \text{ etc. vel}$$

$$\frac{\pi}{2} = \frac{2}{1} \times \frac{2.4}{1.5} \cdot \frac{4.6}{3.7} \cdot \frac{6.8}{5.9} \cdot \frac{8.10}{7.11} \text{ etc. vel}$$

$$\frac{\pi}{2} = \frac{2.4}{3.5} \cdot \frac{2.6}{1.7} \cdot \frac{4.8}{3.9} \cdot \frac{6.10}{5.11} \cdot \frac{8.12}{7.13} \text{ etc.}$$

Scholion.

359. Fundamentum ergo huius euolutionis in hoc consistit, quod valor integralis $\int \frac{x^{i+\alpha}dx}{\sqrt{(1-xx)}}$ denotante i numerum infinitum idem sit, vtcunque numerus finitus α varietur. Atque hoc quidem ex reductione $\int \frac{x^{i-1}dx}{\sqrt{(1-xx)}} = \frac{i+1}{i} \int \frac{x^{i+1}dx}{\sqrt{(1-xx)}}$ manifestum est, si pro α valores binario differentes assumantur. Deinde autem nullum est dubium, quin hoc integrale $\int \frac{x^{i+1}dx}{\sqrt{(1-xx)}}$ inter haec $\int \frac{x^{i}dx}{\sqrt{(1-xx)}}$ et $\int \frac{x^{i+2}dx}{\sqrt{(1-xx)}}$ quasi limites contineatur, qui cum sint inter se aequales, necesse est omnes formulas intermedias iisdem quoque esse aequales. Atque hoc latius patet ad formulas magis complicatas, ita vt denotante i numerum infinitum sit

$$\int \frac{x^{i+\alpha}dx}{(1-x^n)^k} = \int \frac{x^{i}dx}{(1-x^n)^k}.$$

Cum enim sit

$$\int \frac{x^{m+n-1}dx}{(1-x^n)^{\frac{n-k}{n}}} = \frac{m}{m+k} \int \frac{x^{m-1}dx}{(1-x^n)^{\frac{n-k}{n}}}$$

hae formulae posito $m=\infty$ sunt aequales; vnde illarum quoque aequalitas casibus, quibus $\alpha=n$, vel $\alpha=2n$, vel $\alpha=3n$ etc. perspicitur; sin autem α medium quempiam valorem teneat formulae ipsius quoque

quoque valor medium quoddam tenere debet inter valores aequales, ideoque ipsis erit aequalis. Hoc igitur principio stabilito sequens problema resoluere poterimus.

Problema 44.

360. Rationem horum duorum integralium $\int x^{m-1}dx(1-x^n)^{\frac{k-n}{n}}$ et $\int x^{\mu-1}dx(1-x^n)^{\frac{k-n}{n}}$ casu $x=1$ per productum infinitorum factorum exprimere.

Solutio.

Cum sit $\int x^{m-1}dx(1-x^n)^{\frac{k-n}{n}} = \frac{m+k}{m}\int x^{m+n-1}dx(1-x^n)^{\frac{k-n}{n}}$ casu $x=1$, valor istius integralis ad integrale infinite remotum reducetur hoc modo:

$$\int x^{m-1}dx(1-x^n)^{\frac{k-n}{n}} = \frac{(m+k)(m+k+n)(m+k+2n)\ldots\ldots(m+k+in)}{m\,(m+n)\,(m+2n)\ldots\ldots(m+in)}\int x^{m+in+n-1}dx(1-x^n)^{\frac{k-n}{n}}$$

vbi i numerum infinitum denotare assumimus. Simili autem modo pro altera formula proposita erit

$$\int x^{\mu-1}dx(1-x^n)^{\frac{k-n}{n}} = \frac{(\mu+k)(\mu+k+n)(\mu+k+2n)\ldots\ldots(\mu+k+in)}{\mu\,(\mu+n)\,(\mu+2n)\ldots\ldots(\mu+in)}\int x^{\mu+in+n-1}dx(1-x^n)^{\frac{k-n}{n}}$$

atque hae postremae formulae integrales ob exponentes infinitos, aequales erunt, non obstante inaequalitate numerorum m et μ: tum vero bina haec producta infinita pari factorum numero constant. Quare si singuli per singulos, hoc est primus per primum, secundus per secundum diuidantur, ratio binorum integralium propositorum ita exprimetur:

$$\frac{\int x^{m-1}dx(1-x^n)^{\frac{k-n}{n}}}{\int x^{\mu-1}dx(1-x^n)^{\frac{k-n}{n}}} = \frac{\mu(m+k)}{m(\mu+k)}\cdot\frac{(\mu+n)(m+k+n)}{(m+n)(\mu+k+n)}\cdot\frac{(\mu+2n)(m+k+2n)}{(m+2n)(\mu+k+2n)} \text{ etc.}$$

 si

ſi quidem ambo integralia ita determinentur, vt poſito $x = 0$ euaneſcant, tum vero ſtatuatur $x = 1$; litteris autem m, μ, n, k numeros poſitiuos denotari neceſſe eſt.

Coroll. 1.

361. Si differentia numerorum m et μ aequetur multiplo ipſius n, in producto inuento infiniti factores ſe deſtruunt, relinqueturque factorum numerus finitus, vti ſi $\mu = m + n$ habebitur:

$$\frac{(m+n)(m+k)}{m(m+k+n)} \cdot \frac{(m+2n)(m+k+n)}{(m+n)(m+k+2n)} \cdot \frac{(m+3n)(m+k+3n)}{(m+3n)(m+k+3n)} \text{ etc.}$$

quod reducitur ad $\frac{m+k}{m}$.

Coroll. 2.

362. Valor autem illius producti neceſſario eſt finitus, id quod tam ex formulis integralibus, quarum rationem exprimit, patet, quam inde, quod in ſingulis factoribus numeratores et denominatores ſunt alternatim maiores et minores.

Coroll. 3.

263. Si ponamus $m = 1$, $\mu = 3$, $n = 4$ et $k = 2$, erit

$$\frac{\int \frac{dx}{\sqrt{(1-x^4)}}}{\int \frac{xx\,dx}{\sqrt{(1-x^4)}}} = \frac{3.3}{1.5} \cdot \frac{7.7}{5.9} \cdot \frac{11.11}{9.13} \cdot \frac{15.15}{13.17} \text{ etc.}$$

ſupra autem inuenimus productum harum binarum formularum eſſe $= \frac{\pi}{4}$.

Pro-

Problema 45.

364. Valorem huius integralis $\int x^{m-1}dx(1-x^n)^{\frac{k-n}{n}}$, quem posito $x=1$ recipit, per productum infinitum exprimere.

Solutio.

Cum in probl. praeced. ratio huius integralis ad hoc alterum $\int x^{\mu-1}dx(1-x^n)^{\frac{k-n}{n}}$ per productum infinitum sit assignata, in hoc exponens μ ita accipiatur, vt integrale exhiberi possit. Capiatur ergo $\mu=n$, et integrale fit $=C-\frac{1}{k}(1-x^n)^{\frac{k}{n}}=\frac{1-(1-x^n)^{\frac{k}{n}}}{k}$ ita determinatum, vt posito $x=0$ euanescat, ponatur nunc, vt conditio postulat, $x=1$, et quia hoc integrale erit $=\frac{1}{k}$, habebimus formulae propositae integrale casu $x=1$ ita expressum

$$\int x^{m-1}dx(1-x^n)^{\frac{k-n}{n}}=\frac{1}{k}\cdot\frac{n(m+k)}{m(k+n)}\cdot\frac{2n(m+k+n)}{(m+n)(k+2n)}\cdot\frac{3n(m+k+2n)}{(m+2n)(k+3n)}\text{ etc.}$$

quod singulos factores partiendo ita repraesentari potest

$$\int x^{m-1}dx(1-x^n)^{\frac{k-n}{n}}=\frac{n}{mk}\,\frac{2n(m+k)}{(m+n)(k+n)}\cdot\frac{3n(m+k+n)}{(m+2n)(k+2n)}\cdot\frac{4n(m+k+2n)}{(m+3n)(k+3n)}\text{ etc.}$$

Coroll. 1.

365. Cum in hac expressione litterae m et k sint permutabiles, sequitur etiam haec integralia posito

fi quidem ambo integralia ita determinentur, vt pofito $x = 0$ euanefcant, tum vero ftatuatur $x = 1$; litteris autem m, μ, n, k numeros pofitiuos denotari neceffe eft.

Coroll. 1.

361. Si differentia numerorum m et μ aequetur multiplo ipfius n, in producto inuento infiniti factores fe deftruunt, relinqueturque factorum numerus finitus, vti fi $\mu = m + n$ habebitur:

$$\frac{(m+n)(m+k)}{m(m+k+n)} \cdot \frac{(m+2n)(m+k+n)}{(m+n)(m+k+2n)} \cdot \frac{(m+3n)(m+k+2n)}{(m+2n)(m+k+3n)} \text{ etc.}$$

quod reducitur ad $\frac{m+k}{m}$.

Coroll. 2.

362. Valor autem illius producti neceffario eft finitus, id quod tam ex formulis integralibus, quarum rationem exprimit, patet, quam inde, quod in fingulis factoribus numeratores et denominatores funt alternatim maiores et minores.

Coroll. 3.

263. Si ponamus $m = 1$, $\mu = 3$, $n = 4$ et $k = 2$, erit

$$\frac{\int \frac{dx}{\sqrt{(1 - x^4)}}}{\int \frac{xxdx}{\sqrt{(1 - x^4)}}} = \frac{3 \cdot 3}{1 \cdot 5} \cdot \frac{7 \cdot 7}{5 \cdot 9} \cdot \frac{11 \cdot 11}{9 \cdot 13} \cdot \frac{15 \cdot 15}{13 \cdot 17} \text{ etc.}$$

fupra autem inuenimus productum harum binarum formularum effe $= \frac{\pi}{4}$.

Pro-

Problema 45.

364. Valorem huius integralis $\int x^{m-1}dx(1-x^n)^{\frac{k-n}{n}}$, quem posito $x=1$ recipit, per productum infinitum exprimere.

Solutio.

Cum in probl. praeced. ratio huius integralis ad hoc alterum $\int x^{\mu-1}dx(1-x^n)^{\frac{k-n}{n}}$ per productum infinitum sit assignata, in hoc exponens μ ita accipiatur, vt integrale exhiberi possit. Capiatur ergo $\mu=n$, et integrale fit $=C-\frac{1}{k}(1-x^n)^{\frac{k}{n}}=\frac{1-(1-x^n)^{\frac{k}{n}}}{k}$ ita determinatum, vt posito $x=0$ euanescat, ponatur nunc, vt conditio postulat, $x=1$, et quia hoc integrale erit $=\frac{1}{k}$, habebimus formulae propositae integrale casu $x=1$ ita expressum

$$\int x^{m-1}dx(1-x^n)^{\frac{k-n}{n}}=\frac{1}{k}\cdot\frac{n(m+k)}{m(k+n)}\cdot\frac{2n(m+k+n)}{(m+n)(k+2n)}\cdot\frac{3n(m+k+2n)}{(m+2n)(k+3n)}\text{ etc.}$$

quod singulos factores partiendo ita repraesentari potest

$$\int x^{m-1}dx(1-x^n)^{\frac{k-n}{n}}=\frac{n}{mk}\frac{2n(m+k)}{(m+n)(k+n)}\cdot\frac{3n(m+k+n)}{(m+2n)(k+2n)}\cdot\frac{4n(m+k+2n)}{(m+3n)(k+3n)}\text{ etc.}$$

Coroll. 1.

365. Cum in hac expressione litterae m et k sint permutabiles, sequitur etiam haec integralia po-

fito $x=1$ inter fe effe aequalia:

$$\int x^{m-1}dx(1-x^n)^{\frac{k-n}{n}}=\int^{k-1}dx(1-x^n)^{\frac{m-n}{n}}$$

quam aequalitatem iam fupra §. 348. elicuimus.

Coroll. 2.

366. Cum formulae noftrae valor fi $m=n-k$ aequalis fit valori huius $\int\frac{z^{k-1}dz}{1+z^n}$ pofito $z=\infty$, fi ob $m+k=n$ ftatuamus $m=\frac{n+\alpha}{2}$ et $k=\frac{n-\alpha}{2}$, habebimus:

$$\int\frac{x^{m-1}dx}{(1-x^n)^{\frac{n+\alpha}{2n}}}=\frac{x^{k-1}dx}{(1-x^n)^{\frac{n-\alpha}{2n}}}=\int\frac{z^{k-1}dz}{1+z^n}=\int\frac{z^{m-1}dz}{1+z^n}$$

$$=\frac{4n}{nn-\alpha\alpha}\cdot\frac{2.4nn}{9nn-\alpha\alpha}\cdot\frac{4.6nn}{25nn-\alpha\alpha}\cdot\frac{6.8nn}{49nn-\alpha\alpha}\text{ etc.}$$

Quod productum etiam hoc modo exponi poteft

$$\frac{2}{n-\alpha}\cdot\frac{2n.2n}{(n+\alpha)(3n-\alpha)}\cdot\frac{4n.4n}{(3n+\alpha)(5n-\alpha)}\cdot\frac{6n.6n}{(5n+\alpha)(7n-\alpha)}\text{ etc.}$$

quod ergo etiam exprimit valorem ipfius $\frac{\pi}{n\,\text{fin.}\frac{m\pi}{n}}$ $=\frac{\pi}{n\,\text{cof.}\frac{\alpha\pi}{2n}}$ per §. 350.

Coroll. 3.

Coroll. 3.

367. Vel ſi ſimpliciter ponamus $k = n - m$, fiet

$$\int \frac{x^{m-1}dx}{(1-x^n)^{\frac{m}{n}}} = \int \frac{x^{n-m-1}dx}{(1-x^n)^{\frac{n-m}{n}}} = \int \frac{z^{m-1}dz}{1+z^n} = \int \frac{z^{n-m-1}dz}{1+z^n}$$

$$= \frac{1}{n-m} \cdot \frac{nn}{m(2n-m)} \cdot \frac{4nn}{(n+m)(3n-m)} \cdot \frac{9nn}{(2n+m)(4n-m)} \text{ etc.}$$

quae ex forma primum inuenta oritur. Haec ergo aequalitas ſubſiſtit, ſi ponatur $x = 1$ et $z = \infty$.

Scholion. 1.

368. In introductione autem pro multiplicatione angulorum inueneram

$$\text{ſin.} \frac{m\pi}{n} = \frac{m\pi}{n}\left(1 - \frac{mm}{nn}\right)\left(1 - \frac{mm}{4nn}\right)\left(1 - \frac{mm}{9nn}\right)\left(1 - \frac{mm}{16nn}\right) \text{ etc.}$$

et cum $\text{ſin.} \frac{(n-m)\pi}{n} = \text{ſin.} \frac{m\pi}{n}$, ob $n - m = k$ erit etiam

$$\text{ſin} \frac{m\pi}{n} = \frac{k\pi}{n}\left(1 - \frac{kk}{nn}\right)\left(1 - \frac{kk}{4nn}\right)\left(1 - \frac{kk}{9nn}\right)\left(1 - \frac{kk}{16nn}\right)$$

quae reducitur ad hanc formam

$$\text{ſin.} \frac{m\pi}{n} = \frac{k\pi}{n} \cdot \frac{(n-k)(n+k)}{nn} \cdot \frac{(2n-k)(2n+k)}{4nn} \cdot \frac{(3n-k)(3n+k)}{9nn} \text{ etc.}$$

et pro k ſuo valore reſtituto

$$\text{ſin.} \frac{m\pi}{n} = \frac{\pi}{n}(n-m) \cdot \frac{m(2n-m)}{nn} \cdot \frac{(n+m)(3n-m)}{4nn} \cdot \frac{(2n+m)(4n-m)}{9nn} \text{ etc.}$$

vnde manifeſto pro $\frac{\pi}{n \,\text{ſin.} \frac{m\pi}{n}}$ idem reperitur productum, quod valorem noſtrorum integralium exprimit,

mit, ficque nouam habemus demonftrationem pro Theoremate illo eximio fupra per multas ambages euicto, effe

$$\int \frac{x^{m-1}\,dx}{(1-x^n)^{\frac{m}{n}}} = \int \frac{x^{n-m-1}\,dx}{(1-x^n)^{\frac{n-m}{n}}} = \int \frac{z^{m-1}\,dz}{1+z^n} = \int \frac{z^{n-m-1}\,dz}{1+z^n}$$

$$= \frac{\pi}{n \sin.\frac{m\pi}{n}}.$$

Scholion 2.

369. Quo noftra formula latius pateat, ponamus $\frac{k}{n} = \frac{\mu}{\nu}$ feu $k = \frac{\mu n}{\nu}$, et nancifcemur $\int x^{m-1}dx(1-x^n)^{\frac{\mu}{\nu}-1}$

$$= \frac{\nu}{m\mu} \cdot \frac{2(m\nu+n\mu)}{(m+n)(\mu+\nu)} \cdot \frac{3(m\nu+n(\mu+\nu))}{(m+2n)(\mu+2\nu)} \cdot \frac{4(m\nu+n(\mu+2\nu))}{(m+3n)(\mu+3\nu)} \cdot \text{etc.}$$

$$= \frac{\nu}{m\mu} \cdot \frac{2(m\nu+n\mu)}{(m+n)(\mu+\nu)} \cdot \frac{3(m\nu+n\mu+n\nu)}{(m+2n)(\mu+2\nu)} \cdot \frac{4(m\nu+n\mu+2n\nu)}{(m+3n)(\mu+3\nu)} \cdot \frac{5(m\nu+n\mu+3n\nu)}{(m+4n)(\mu+4\nu)} \cdot \text{etc.}$$

in qua expreffione litterae m, n et μ, ν funt permutabiles praeterquam in primo factore, qui cum reliquis lege continuitatis non connectitur; ac fi per n multiplicemus, permutabilitas erit perfecta, vnde concludimus fore

$$n\int x^{m-1}dx(1-x^n)^{\frac{\mu}{\nu}-1} = \nu\int x^{\mu-1}dx(1-x^\nu)^{\frac{m}{n}-1}$$

quae aequalitas cafu $\nu = n$ ad fupra obferuatam reducitur. Caeterum iuuabit cafus praecipuos perpendiffe, quos ex valoribus μ et ν defumamus.

Exem-

Exemplum 1.

370. *Sit* $\mu = 1$ et $\nu = 2$, *fietque*

$$\int \frac{x^{m-1}dx}{\sqrt{(1-x^n)}} = \frac{2}{m} \cdot \frac{2(2m+n)}{3(m+n)} \cdot \frac{3(2m+3n)}{5(m+2n)} \cdot \frac{4(2m+5n)}{7(m+3n)} \text{ etc.} = \frac{2}{n} \int \frac{dx}{\sqrt[n]{(1-x^2)^{n-m}}}$$

quae expreſſio ita commodius repraeſentatur:

$$\int \frac{x^{m-1}dx}{\sqrt{(1-x^n)}} = \frac{2}{m} \cdot \frac{4(2m+n)}{3(2m+2n)} \cdot \frac{6(2m+3n)}{5(2m+4n)} \cdot \frac{8(2m+5n)}{7(2m+6n)} \text{ etc.}$$

vnde ſequentes caſus ſpecialiſſimi deducuntur:

$$\int \frac{dx}{\sqrt{(1-xx)}} = 2 \cdot \frac{2 \cdot 4}{3 \cdot 3} \cdot \frac{4 \cdot 6}{5 \cdot 5} \cdot \frac{6 \cdot 8}{7 \cdot 7} \text{ etc.} = \int \frac{dx}{\sqrt{(1-xx)}}$$

$$\int \frac{dx}{\sqrt{(1-x^3)}} = 2 \cdot \frac{4 \cdot 5}{3 \cdot 8} \cdot \frac{6 \cdot 11}{5 \cdot 14} \cdot \frac{8 \cdot 17}{7 \cdot 20} \cdot \frac{10 \cdot 23}{9 \cdot 26} \text{ etc.} = \frac{2}{3} \int \frac{dx}{\sqrt[3]{(1-x^2)^2}}$$

$$\int \frac{x\,dx}{\sqrt{(1-x^3)}} = 1 \cdot \frac{4 \cdot 7}{3 \cdot 10} \cdot \frac{6 \cdot 13}{5 \cdot 16} \cdot \frac{8 \cdot 19}{7 \cdot 22} \cdot \frac{10 \cdot 25}{9 \cdot 28} \text{ etc.} = \frac{2}{3} \int \frac{dx}{\sqrt[3]{(1-x^2)}}$$

$$\int \frac{dx}{\sqrt{(1-x^4)}} = 2 \cdot \frac{4 \cdot 3}{3 \cdot 5} \cdot \frac{6 \cdot 7}{5 \cdot 9} \cdot \frac{8 \cdot 11}{7 \cdot 13} \cdot \frac{10 \cdot 15}{9 \cdot 17} \text{ etc.} = \frac{1}{2} \int \frac{dx}{\sqrt[4]{(1-x^2)^3}}$$

$$\int \frac{x\,dx}{\sqrt{(1-x^4)}} = 1 \cdot \frac{4 \cdot 4}{3 \cdot 6} \cdot \frac{6 \cdot 8}{5 \cdot 10} \cdot \frac{8 \cdot 12}{7 \cdot 14} \cdot \frac{10 \cdot 16}{9 \cdot 18} \text{ etc.} = \frac{1}{2} \int \frac{dx}{\sqrt[4]{(1-x^2)}}$$

$$= 1 \cdot \frac{2 \cdot 4}{3 \cdot 3} \cdot \frac{4 \cdot 6}{5 \cdot 5} \cdot \frac{6 \cdot 8}{7 \cdot 7} \cdot \frac{8 \cdot 10}{9 \cdot 9} \text{ etc.}$$

$$\int \frac{xx\,dx}{\sqrt{(1-x^4)}} = \frac{2}{3} \cdot \frac{4 \cdot 5}{3 \cdot 7} \cdot \frac{6 \cdot 9}{5 \cdot 11} \cdot \frac{8 \cdot 13}{7 \cdot 15} \cdot \frac{10 \cdot 17}{9 \cdot 19} \text{ etc.}$$

$$\int \frac{x^3\,dx}{\sqrt{(1-x^4)}} = \frac{2}{4} \cdot \frac{4 \cdot 6}{3 \cdot 8} \cdot \frac{6 \cdot 10}{5 \cdot 12} \cdot \frac{8 \cdot 14}{7 \cdot 16} \cdot \frac{10 \cdot 18}{9 \cdot 20} \text{ etc.} = \frac{1}{2}.$$

Exemplum 2.

271. *Sit* $\mu = 1$ *et* $\nu = 3$, *fietque*

$$\int \frac{x^{m-1}dx}{\sqrt[3]{(1-x^n)^2}} = \frac{3}{m} \cdot \frac{2(3m+n)}{4(m+n)} \cdot \frac{3(3m+4n)}{7(m+2n)} \cdot \frac{4(3m+7n)}{10(m+3n)} \text{ etc.} = \frac{3}{n} \int \frac{dx}{\sqrt[n]{(1-x^3)^{n-m}}}$$

vnde ſequentes caſus ſpecialiſſimi deducuntur:

$$\int \frac{dx}{\sqrt[3]{(1-x^2)^2}} = \frac{3}{1} \cdot \frac{2 \cdot 5}{4 \cdot 3} \cdot \frac{3 \cdot 11}{7 \cdot 5} \cdot \frac{4 \cdot 17}{10 \cdot 7} \cdot \frac{5 \cdot 23}{13 \cdot 9} \text{ etc.} = \frac{3}{2} \int \frac{dx}{\sqrt{(1-x^3)}}$$

$$\int \frac{dx}{\sqrt[3]{(1-x^3)^2}} = \frac{3}{1} \cdot \frac{2 \cdot 6}{4 \cdot 4} \cdot \frac{3 \cdot 15}{7 \cdot 7} \cdot \frac{4 \cdot 24}{10 \cdot 10} \cdot \frac{5 \cdot 33}{13 \cdot 13} \text{ etc.} = \int \frac{dx}{\sqrt[3]{(1-x^3)^2}}$$

$$= \frac{3}{1} \cdot \frac{2 \cdot 6}{4 \cdot 4} \cdot \frac{5 \cdot 9}{7 \cdot 7} \cdot \frac{8 \cdot 12}{10 \cdot 10} \cdot \frac{11 \cdot 15}{13 \cdot 13} \text{ etc.}$$

$$\int \frac{x\,dx}{\sqrt[3]{(1-x^3)^2}} = \frac{3}{2} \cdot \frac{2 \cdot 9}{4 \cdot 5} \cdot \frac{3 \cdot 18}{7 \cdot 8} \cdot \frac{4 \cdot 27}{10 \cdot 11} \cdot \frac{5 \cdot 36}{13 \cdot 14} \text{ etc.} = \int \frac{dx}{\sqrt[3]{(1-x^3)}}$$

ſiue $= \frac{3}{2} \cdot \frac{5 \cdot 6}{4 \cdot 5} \cdot \frac{6 \cdot 9}{7 \cdot 8} \cdot \frac{9 \cdot 12}{10 \cdot 11} \cdot \frac{12 \cdot 15}{13 \cdot 14}$ etc.

$$\int \frac{dx}{\sqrt[3]{(1-x^4)^2}} = \frac{3}{1} \cdot \frac{2 \cdot 7}{4 \cdot 5} \cdot \frac{3 \cdot 19}{7 \cdot 9} \cdot \frac{4 \cdot 31}{10 \cdot 13} \cdot \frac{5 \cdot 43}{13 \cdot 17} \text{ etc.} = \frac{3}{4} \int \frac{dx}{\sqrt[4]{(1-x^3)^3}}$$

$$\int \frac{xx\,dx}{\sqrt[3]{(1-x^4)^2}} = 1 \cdot \frac{2 \cdot 13}{4 \cdot 7} \cdot \frac{3 \cdot 25}{7 \cdot 11} \cdot \frac{4 \cdot 37}{10 \cdot 15} \cdot \frac{5 \cdot 49}{13 \cdot 19} \text{ etc.} = \frac{3}{4} \int \frac{dx}{\sqrt[4]{(1-x^3)}}.$$

Exemplum 3.

372. *Sit* $\mu = 2$ *et* $\nu = 3$, *fietque*

$$\int \frac{x^{m-1}dx}{\sqrt[3]{(1-x^n)}} = \frac{3}{2m} \cdot \frac{2(3m+2n)}{5(m+n)} \cdot \frac{3(3m+5n)}{8(m+2n)} \cdot \frac{4(3m+8n)}{11(m+3n)} \text{ etc.} = \frac{3}{n} \int \frac{x\,dx}{\sqrt[n]{(1-x^3)^{n-m}}}$$

vnde

vnde sequentes casus speciales deducuntur:

$$\int\frac{dx}{\sqrt[3]{(1-x^2)}}=\frac{3}{2}\cdot\frac{2\cdot 7}{5\cdot 3}\cdot\frac{3\cdot 13}{8\cdot 5}\cdot\frac{4\cdot 19}{11\cdot 7}\cdot\frac{5\cdot 25}{14\cdot 9}\text{ etc.}=\frac{3}{4}\int\frac{x\,dx}{\sqrt{(1-x^3)}}$$

$$\int\frac{dx}{\sqrt[3]{(1-x^3)}}=\frac{3}{2}\cdot\frac{2\cdot 9}{5\cdot 4}\cdot\frac{3\cdot 18}{8\cdot 7}\cdot\frac{4\cdot 27}{11\cdot 10}\cdot\frac{5\cdot 36}{14\cdot 13}\text{ etc.}=\int\frac{x\,dx}{\sqrt[3]{(1-x^3)^2}}$$

$$\text{siue}=\frac{3\cdot 3}{2\cdot 4}\cdot\frac{6\cdot 6}{5\cdot 7}\cdot\frac{9\cdot 9}{8\cdot 10}\cdot\frac{12\cdot 12}{11\cdot 13}\text{ etc.}$$

$$\int\frac{x\,dx}{\sqrt[3]{(1-x^3)}}=\frac{3}{4}\cdot\frac{2\cdot 12}{5\cdot 5}\cdot\frac{3\cdot 21}{8\cdot 8}\cdot\frac{4\cdot 30}{11\cdot 11}\cdot\frac{5\cdot 39}{14\cdot 14}\text{ etc.}=\int\frac{x\,dx}{\sqrt[3]{(1-x^6)}}$$

$$\text{siue}=\frac{3}{4}\cdot\frac{4\cdot 6}{5\cdot 5}\cdot\frac{7\cdot 9}{8\cdot 8}\cdot\frac{10\cdot 12}{11\cdot 11}\cdot\frac{13\cdot 15}{14\cdot 14}\text{ etc.}$$

$$\int\frac{dx}{\sqrt[3]{(1-x^4)}}=\frac{3}{2}\cdot\frac{2\cdot 11}{5\cdot 5}\cdot\frac{3\cdot 23}{8\cdot 9}\cdot\frac{4\cdot 35}{11\cdot 13}\cdot\frac{5\cdot 47}{14\cdot 17}\text{ etc.}=\frac{3}{4}\int\frac{x\,dx}{\sqrt[4]{(1-x^3)^3}}$$

$$\int\frac{x^2\,dx}{\sqrt[3]{(1-x^4)}}=\frac{1}{2}\cdot\frac{2\cdot 17}{5\cdot 7}\cdot\frac{3\cdot 29}{8\cdot 11}\cdot\frac{4\cdot 41}{11\cdot 15}\cdot\frac{5\cdot 53}{14\cdot 19}\text{ etc.}=\frac{3}{4}\int\frac{x\,dx}{\sqrt[4]{(1-x^3)}}.$$

Exemplum 4.

373. *Sit* $\mu=1$ *et* $\nu=4$, *fietque*

$$\int\frac{x^{m-1}dx}{\sqrt[4]{(1-x^n)^1}}=\frac{4}{m}\cdot\frac{2(4m+n)}{5(m+n)}\cdot\frac{3(4m+5n)}{9(m+2n)}\cdot\frac{4(4m+9n)}{13(m+3n)}\text{etc.}=\frac{4}{n}\int\frac{dx}{\sqrt[n]{(1-x^4)^{n-m}}}$$

vnde sequentes casus speciales prodeunt:

$$\int\frac{dx}{\sqrt[4]{(1-x^2)^3}}=\frac{4}{1}\cdot\frac{2\cdot 6}{5\cdot 3}\cdot\frac{3\cdot 14}{5\cdot 5}\cdot\frac{4\cdot 22}{9\cdot 7}\cdot\frac{5\cdot 30}{13\cdot 9}\text{ etc.}=2\int\frac{dx}{\sqrt{(1-x^4)}}$$

$$\text{seu}=\frac{4}{3}\cdot\frac{4\cdot 7}{3\cdot 5}\cdot\frac{6\cdot 7}{5\cdot 9}\cdot\frac{8\cdot 11}{7\cdot 9}\cdot\frac{10\cdot 15}{9\cdot 13}\text{ etc.}$$

$$\int\frac{dx}{\sqrt[4]{(1-x^3)^3}}=\frac{4}{1}\cdot\frac{2\cdot 7}{5\cdot 4}\cdot\frac{3\cdot 10}{9\cdot 7}\cdot\frac{4\cdot 13}{13\cdot 10}\cdot\frac{5\cdot 16}{17\cdot 13}\text{etc.}=\frac{4}{3}\int\frac{dx}{\sqrt[3]{(1-x^4)^2}}$$

$$\int\frac{x\,dx}{\sqrt[4]{(1-x^4)^3}} = \frac{2}{1}\cdot\frac{2\cdot 11}{5\cdot 5}\cdot\frac{3\cdot 23}{9\cdot 8}\cdot\frac{4\cdot 35}{13\cdot 12}\cdot\frac{5\cdot 47}{17\cdot 14}\text{ etc.} = \frac{4}{3}\int\frac{dx}{\sqrt[3]{(1-x^3)}}$$

$$\int\frac{dx}{\sqrt[4]{(1-x^4)^3}} = \frac{4}{1}\cdot\frac{2\cdot 8}{5\cdot 5}\cdot\frac{3\cdot 24}{9\cdot 9}\cdot\frac{4\cdot 40}{13\cdot 13}\cdot\frac{5\cdot 56}{17\cdot 17}\text{ etc.} = \int\frac{dx}{\sqrt[4]{(1-x^4)^3}}$$

$$\text{seu } = \frac{4}{1}\cdot\frac{4\cdot 4}{5\cdot 5}\cdot\frac{6\cdot 12}{9\cdot 9}\cdot\frac{8\cdot 20}{13\cdot 13}\cdot\frac{10\cdot 28}{17\cdot 17}\text{ etc.}$$

$$\text{seu } = \frac{4}{1}\cdot\frac{2\cdot 8}{5\cdot 5}\cdot\frac{6\cdot 12}{9\cdot 9}\cdot\frac{10\cdot 16}{13\cdot 13}\cdot\frac{14\cdot 20}{17\cdot 17}\text{ etc.}$$

$$\int\frac{xx\,dx}{\sqrt[4]{(1-x^4)^3}} = \frac{4}{3}\cdot\frac{2\cdot 16}{5\cdot 7}\cdot\frac{3\cdot 32}{9\cdot 11}\cdot\frac{4\cdot 48}{13\cdot 15}\cdot\frac{5\cdot 64}{17\cdot 19}\text{ etc.} = \int\frac{dx}{\sqrt[4]{(1-x^4)}}$$

$$\text{seu } = \frac{4}{3}\cdot\frac{4\cdot 8}{5\cdot 7}\cdot\frac{6\cdot 16}{9\cdot 11}\cdot\frac{8\cdot 24}{13\cdot 15}\cdot\frac{10\cdot 32}{17\cdot 19}\text{ etc.}$$

$$\text{seu } = \frac{4}{3}\cdot\frac{4\cdot 8}{5\cdot 7}\cdot\frac{8\cdot 12}{9\cdot 11}\cdot\frac{12\cdot 16}{13\cdot 15}\cdot\frac{16\cdot 20}{17\cdot 19}\text{ etc.}$$

Atque in his et praecedentibus iam casus $\mu = 3$ et $\nu = 4$ est contentus.

Scholion.

374. Caeterum hae formulae, in quas litteras μ et ν introduxi, latius non patent quam primum consideratae, series enim pendent a binis fractionibus $\frac{m}{n}$ et $\frac{\mu}{\nu}$ quae cum semper ad communem denominatorem reuocari queant, formulas $\int\frac{x^{m-1}\,dx}{\sqrt[n]{(1-x^n)^{n-k}}} = \int\frac{x^{k-1}\,dx}{\sqrt[n]{(1-x^n)^{n-m}}}$ perpendisse sufficiet. Cum igitur earum valor casu $x = 1$ aequetur huic producto:

$$\frac{1}{k}\cdot\frac{n(m+k)}{m(k+n)}\cdot\frac{2n(m+k+n)}{(m+n)(k+2n)}\cdot\frac{3n(m+k+2n)}{(m+2n)(k+3n)}\text{ etc.}$$

fi

si in singulis membris factores numeratorum permutemus, et membra aliter partiamur, idem productum hanc induet formam:

$$\frac{m+k}{mk} \cdot \frac{m(m+k+n)}{(m+n)(k+n)} \cdot \frac{2n(m+k+2n)}{(m+2n)(k+2n)} \cdot \frac{3n(m+k+3n)}{(m+3n)(k+3n)} \text{ etc.}$$

quae ad memoriam magis accommodata videtur. Simili modo cum sit

$$\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}} = \int \frac{x^{q-1}dx}{\sqrt[n]{(1-x^n)^{n-}}} = \frac{p+q}{pq} \cdot \frac{n(p+q+n)}{(p+n)(q+n)} \cdot \frac{2n(p+q+2n)}{(p+2n)(q+2n)} \cdot \frac{3n(p+q+3n)}{(p+3n)(q+3n)} \text{ etc.}$$

illam formam per hanc diuidendo erit

$$\frac{\int x^{m-1}dx(1-x^n)^{\frac{k-n}{n}}}{\int x^{p-1}dx(1-x^n)^{\frac{q-n}{n}}} = \frac{pq(m+k)}{mk(p+q)} \cdot \frac{(p+n)(q+n)(m+k+n)}{(m+n)(k+n)(p+q+n)} \cdot \frac{(p+2n)(q+2n)(m+k+2n)}{(m+2n)(k+2n)(p+q+2n)} \text{ etc.}$$

cuius omnia membra eadem lege continentur. Hinc autem eximiae comparationes huiusmodi formularum deduci possunt, quae quo facilius commemorari queant, breuitatis causa sequenti scriptionis compendio vtar.

Definitio.

475. Formulae integralis $\int x^{p-1}dx(1-x^n)^{\frac{q-n}{n}}$ valorem, quem posito $x=1$ recipit, breuitatis gratia hoc signo $\left(\frac{p}{q}\right)$ indicemus, vbi quidem exponentem n, quem in comparatione plurium huiusmodi formularum, eundem esse assumo subintelligi oportet.

 Coroll. 1.

Coroll. 1.

376. Primum igitur patet esse $\left(\frac{p}{q}\right)=\left(\frac{q}{p}\right)$, et vtramque formulam esse

$$=\frac{p+q}{pq}\cdot\frac{n(p+q+n)}{(p+n)(q+n)}\cdot\frac{2n(p+q+2n)}{(p+2n)(q+2n)}\text{ etc.}$$

quorum membrorum progressio est manifesta, dum singuli factores tam numeratoris quam denominatoris continuo eodem numero n augentur, ita vt ex cognito primo membro sequentia facile formentur.

Coroll. 2.

377. Deinde si sit $p=n$ ob formulam integrabilem, liquet esse $\left(\frac{n}{q}\right)=\left(\frac{q}{n}\right)=\frac{1}{q}$, item $\left(\frac{p}{n}\right)=\left(\frac{n}{p}\right)=\frac{1}{p}$. Porro cum $\int x^{p-1}dx(1-x^n)^{-\frac{p}{n}}=\frac{\pi}{n\sin.\frac{p\pi}{n}}$, ob $q-n=-p$ seu $p+q=n$ erit $\left(\frac{p}{n-p}\right)=\left(\frac{n-p}{p}\right)=\frac{\pi}{n\sin.\frac{p\pi}{n}}$. Quare valor formulae $\left(\frac{p}{q}\right)$ absolute assignari potest, quoties fuerit vel $p=n$, vel $q=n$, vel $p+q=n$.

Coroll. 3.

378. Quia etiam inuenimus hanc reductionem

$$\int x^{p+n-1}dx(1-x^n)^{\frac{q-n}{n}}=\frac{p}{p+q}\int x^{p-1}dx(1-x^n)^{\frac{q-n}{n}}$$

sequitur fore $\left(\frac{p+n}{q}\right)=\frac{p}{p+q}\left(\frac{p}{q}\right)$, hincque $\left(\frac{p}{q}\right)=\left(\frac{q}{p}\right)=\frac{p-n}{p+q-n}\left(\frac{p-n}{q}\right)=\frac{q-n}{p+q-n}\left(\frac{p}{q-n}\right)$ tum vero etiam

etiam $(\frac{p}{q}) = \frac{(p-n)(q-n)}{(p+q-n)(p+q-2n)} \cdot [\frac{p-n}{q-n}]$ vnde ſemper numeri p et q infra n deprimi poſſunt.

Problema 46.

379. Inuenire diuerſa producta ex binis huiusmodi formulis, quae inter ſe ſiant aequalia.

Solutio.

Quaerantur ergo numeri a, b, c, d, et p, q, r, s, vt fiat $(\frac{a}{b})(\frac{c}{d}) = (\frac{p}{q})(\frac{r}{s})$, quod, cum ſit

$$(\frac{a}{b}) = \frac{a+b}{ab} \cdot \frac{n(a+b+n)}{(a+n)(b+n)} \text{ etc. } (\frac{c}{d}) = \frac{c+d}{cd} \cdot \frac{n(c+d+n)}{(c+n)(b+n)} \text{ etc.}$$

$$(\frac{p}{q}) = \frac{p+q}{pq} \cdot \frac{n(p+q+n)}{(p+n)(q+n)} \text{ etc. } (\frac{r}{s}) = \frac{r+s}{rs} \cdot \frac{n(r+s+n)}{(r+n)(s+n)} \text{ etc.}$$

eueniet, ſi fuerit

$$\frac{(a+b)(c+d)}{abcd} = \frac{(p+q)(r+s)}{pqrs} \text{ ſeu}$$

$$abcd(p+q)(r+s) = pqrs(a+b)(c+d)$$

ita vt, cum vtrinque ſex ſint factores, ſinguli ſingulis ſint aequales. Ex quaternis ergo $abcd$ et $pqrs$ binos ad minimum aequales eſſe oportet: ſit itaque $s = d$ efficique oportet

$$abc(p+q)(r+d) = pqr(a+b)(c+d).$$

I. Sumatur alter factor r qui cum ipſi c aequari nequeat, quia alioquin fieret $(\frac{c}{d}) = (\frac{r}{s})$, ſtatuatur $r = b$ vt fiat

$$ac(p+q)(b+d) = pq(a+b)(c+d),$$

hic

hic neque p neque q ipsi $p+q$ aequari potest, poni ergo debet

1) vel $p+q=a+b$ vt sit $ac(b+d)=pq(c+d)$, quia neque c neque $(b+d)$ ipsi $c+d$ aequari potest, fieret enim vel $d=0$, vel $b=c$ et $(\frac{r}{s})=(\frac{c}{d})$ relinquitur $a=c+d$, et $pq=c(b+d)$ ideoque $p=b+d$ et $q=c$, vnde conficitur:

$$(\frac{c+d}{b})(\frac{c}{d})=(\frac{b+d}{c})(\frac{b}{d}).$$

2) Vel $p+q=c+d$ ergo $ac(b+d)=pq(a+b)$, hic c neque ipsi p neque q aequari potest, fieret enim $(\frac{p}{q})=(\frac{c}{d})$ vnde fiat $c=a+b$, vt sit $pq=a(b+d)$ ergo $p=a$; $q=b+d$; $r=b$; $s=d$ consequenter

$$(\frac{a}{b})(\frac{a+b}{d})=(\frac{b+d}{a})(\frac{b}{d}).$$

II. Quia $r=b$ non differt a praecedenti, ob a et b permutabiles, statuatur $r=p+q$, fietque

$$abc(d+p+q)=pq(a+b)(c+d).$$

Quoniam r ipsi c aequari nequit, factor $d+p+q$ neque ipsi p neque q neque $c+d$ aequalis poni potest, relinquitur ergo $d+p+q=a+b$, et $abc=pq(c+d)$, vbi quia c ipsi $c+d$ aequari nequit, ac p et q pari conditione gaudent, fiat $p=c$, erit $q=a+b-c-d$, et $ab=(c+d)(a+b-c-d)$ vnde $a=c+d$; $q=b$; $p=c$; $r=b+c$; $s=d$; sicque conficitur:

$$(\frac{c+d}{b})(\frac{c}{d})=(\frac{c}{b})(\frac{b+c}{d}).$$

Coroll. 1.

Coroll. 1.

380. Hae solutiones eodem fere redeunt, indeque tria producta binarum formularum, aequalia eruuntur:

$$\left(\frac{c}{d}\right)\left(\frac{c+d}{b}\right)=\left(\frac{c}{b}\right)\left(\frac{b+c}{d}\right)=\left(\frac{b}{d}\right)\left(\frac{b+d}{c}\right)$$

vel in literis p, q, r

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\left(\frac{q}{r}\right)\left(\frac{q+r}{p}\right)=\left(\frac{r}{p}\right)\left(\frac{p+r}{q}\right).$$

Coroll. 2.

381. Si hae formulae in producta infinita euoluantur reperietur

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\frac{p+q+r}{pqr}\cdot\frac{nn(p+q+r+n)}{(p+n)(q+n)(r+n)}\cdot\frac{4nn(p+q+r+2n)}{(p+2n)(q+2n)(r+2n)}\text{ etc.}$$

vnde patet tres litteras p, q, r vtcunque inter se permutari posse, atque hinc ternas illas formulas concludere licet.

Coroll. 3.

382. Restituamus ipsas formulas integrales, et sequentia tria producta erunt inter se aequalia

$$\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}\cdot\int\frac{x^{p+q-1}dx}{\sqrt[n]{(1-x^n)^{n-r}}}=$$

$$\int\frac{x^{q-1}dx}{\sqrt[n]{(1-x^n)^{n-r}}}\cdot\int\frac{x^{q+r-1}dx}{\sqrt[n]{(1-x^n)^{n-p}}}=$$

$$\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-r}}}\cdot\int\frac{x^{p+r-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}.$$

Coroll. 4.

383. Hic casus notatu dignus, quo $p+q=n$, tum enim ob $\left(\frac{p+q}{r}\right)=\left(\frac{n}{r}\right)=\frac{1}{r}$ et $\left(\frac{p}{q}\right)=\frac{\pi}{n\,\text{fin.}\frac{p\pi}{n}}$, haec tria producta fient $=\frac{\pi}{nr\,\text{fin.}\frac{p\pi}{n}}$. Erit scilicet

$$\int\frac{x^{n-p-1}dx}{\sqrt[n]{(1-x^n)^{n-r}}}\cdot\int\frac{x^{n-p+r-1}dx}{\sqrt[n]{(1-x^n)^{n-p}}}=\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-r}}}\cdot\int\frac{x^{p+r-1}dx}{\sqrt[n]{(1-x^n)^{p}}}$$
$$=\frac{\pi}{nr\,\text{fin.}\frac{p\pi}{n}}.$$

Scholion.

384. Triplex ista proprietas productorum ex binis formulis maxime est notatu digna, ac pro variis numeris loco p, q, r substituendis obtinebuntur sequentes aequalitates speciales:

p	q	r	
1	1	2	$\left(\frac{1}{1}\right)\left(\frac{2}{2}\right)=\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)$
1	2	2	$\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)=\left(\frac{2}{2}\right)\left(\frac{4}{1}\right)$
1	2	3	$\left(\frac{2}{1}\right)\left(\frac{3}{3}\right)=\left(\frac{3}{2}\right)\left(\frac{5}{1}\right)=\left(\frac{3}{1}\right)\left(\frac{4}{2}\right)$
1	1	3	$\left(\frac{1}{1}\right)\left(\frac{3}{2}\right)=\left(\frac{3}{1}\right)\left(\frac{4}{1}\right)$
2	2	3	$\left(\frac{2}{2}\right)\left(\frac{4}{3}\right)=\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)$
1	3	3	$\left(\frac{3}{1}\right)\left(\frac{4}{3}\right)=\left(\frac{3}{3}\right)\left(\frac{6}{1}\right)$
2	3	3	$\left(\frac{3}{2}\right)\left(\frac{5}{3}\right)=\left(\frac{3}{3}\right)\left(\frac{6}{2}\right)$
1	1	4	$\left(\frac{1}{1}\right)\left(\frac{4}{2}\right)=\left(\frac{4}{1}\right)\left(\frac{5}{1}\right)$
1	2	4	$\left(\frac{2}{1}\right)\left(\frac{4}{3}\right)=\left(\frac{4}{2}\right)\left(\frac{6}{1}\right)=\left(\frac{4}{1}\right)\left(\frac{5}{2}\right)$
1	3	4	$\left(\frac{3}{1}\right)\left(\frac{4}{4}\right)=\left(\frac{4}{1}\right)\left(\frac{5}{3}\right)=\left(\frac{4}{3}\right)\left(\frac{7}{1}\right)$
1	4	4	$\left(\frac{4}{1}\right)\left(\frac{5}{4}\right)=\left(\frac{4}{4}\right)\left(\frac{8}{1}\right)$

p	q	r	
2	2	4	$\left(\frac{2}{2}\right)\left(\frac{4}{4}\right)=\left(\frac{4}{2}\right)\left(\frac{6}{2}\right)$
2	3	4	$\left(\frac{3}{2}\right)\left(\frac{5}{4}\right)=\left(\frac{4}{3}\right)\left(\frac{7}{2}\right)=\left(\frac{4}{2}\right)\left(\frac{6}{3}\right)$
2	4	4	$\left(\frac{4}{2}\right)\left(\frac{6}{4}\right)=\left(\frac{4}{4}\right)\left(\frac{8}{2}\right)$
3	3	4	$\left(\frac{3}{3}\right)\left(\frac{6}{4}\right)=\left(\frac{4}{3}\right)\left(\frac{7}{3}\right)$
3	4	4	$\left(\frac{4}{3}\right)\left(\frac{7}{4}\right)=\left(\frac{4}{4}\right)\left(\frac{8}{3}\right)$.

Quae formulae pro omnibus numeris n valent, ac si numeri maiores quam n occurrant, eos ad minores reduci posse supra vidimus.

Problema 47.

385. Inuenire producta diuersa ex ternis huiusmodi formulis, quae inter se sint aequalia.

Solutio.

Consideretur productum $\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right)$, quod euolutum praebet:

$$\frac{p+q+r+s}{pqrs}\cdot\frac{n^3(p+q+r+s+n)}{(p+n)(q+n)(r+n)(s+n)} \text{ etc.}$$

quod eundem valorem retinere euidens est, quomodocunque quatuor litterae inter se commutentur. Tum vero eadem euolutio prodit ex hoc producto: $\left(\frac{p}{q}\right)\left(\frac{r}{s}\right)\left(\frac{p+q}{r+s}\right)$, vbi eadem permutatio locum habet. Aequalia ergo sunt inter se omnia haec producta:

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right);\ \left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right)\left(\frac{p+q+r}{s}\right);\ \left(\frac{p}{s}\right)\left(\frac{p+s}{q}\right)\left(\frac{p+q+s}{r}\right)$$

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+q+s}{r}\right);\ \left(\frac{p}{r}\right)\left(\frac{p+r}{s}\right)\left(\frac{p+r+s}{q}\right);\ \left(\frac{p}{s}\right)\left(\frac{p+s}{r}\right)\left(\frac{p+r+s}{q}\right)$$

$$\left(\frac{q}{r}\right)\left(\frac{q+r}{p}\right)\left(\frac{p+q+r}{s}\right);\ \left(\frac{q}{s}\right)\left(\frac{q+s}{p}\right)\left(\frac{p+q+s}{r}\right);\ \left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right)\left(\frac{q+r+s}{q}\right)$$

$$\left(\frac{q}{r}\right)\left(\frac{q+r}{s}\right)\left(\frac{q+r+s}{p}\right);\ \left(\frac{q}{s}\right)\left(\frac{q+s}{r}\right)\left(\frac{q+r+s}{p}\right);\ \left(\frac{r}{s}\right)\left(\frac{r+s}{q}\right)\left(\frac{q+r+s}{p}\right).$$

Producta alterius formae ope praecedentis proprietatis hinc ſponte fluunt: eſt enim

$$\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p+q}\right).$$

Deinde vero etiam hoc productum $\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+r}{s}\right)$ euolutum pro primo membro dat: $\frac{(p+q+r)(p+r+s)}{pqrs(p+r)}$, in quo tam p et r, quam q et s inter ſe permutare licet, ita vt ſit

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right)\left(\frac{p+r}{q}\right)$$

Scholion.

386. Quantumuis late haec patere videantur, tamen nullas nouas comparationes ſuppeditant, quae non iam in praecedenti contineantur. Poſtrema enim aequalitas $\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right)\left(\frac{p+r}{q}\right)$ oritur ex multiplicatione harum

$$\begin{cases}\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right)\\ \left(\frac{p}{r}\right)\left(\frac{p+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right).\end{cases}$$

Priorum vero formatio ex hoc exemplo patebit aequalitas $\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right)\left(\frac{p+r+s}{q}\right)$ oritur ex multiplicatione harum

$$\begin{cases}\left(\frac{p}{q}\right)\left(\frac{p+q}{r+s}\right)=\left(\frac{r+s}{p}\right)\left(\frac{p+r+s}{q}\right)\\ \left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p+q}\right)\end{cases}$$

Iſtae autem comparationes praecipue vtiles ſunt ad valores diuerſarum formularum eiusdem ordinis ſeu pro

pro dato numero n inuicem reducendos, vt integratio ad paucissimas reuocetur, quibus datis reliquae per eas definiri queant.

Problema 48.

387. Formulas simplicissimas exhibere, ad quas integratio omnium casuum in forma $\left(\frac{p}{q}\right) = \int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$ contentorum reduci queat.

Solutio.

Primo est $\left(\frac{n}{p}\right) = \frac{1}{p}$, vnde habentur hi casus

$$\left(\frac{n}{1}\right) = 1;\ \left(\frac{n}{2}\right) = \frac{1}{2};\ \left(\frac{n}{3}\right) = \frac{1}{3};\ \left(\frac{n}{4}\right) = \frac{1}{4};\ \left(\frac{n}{5}\right) = \frac{1}{5} \text{ etc.}$$

Deinde est $\left(\frac{p}{n-p}\right) = \frac{\pi}{n \sin.\frac{p\pi}{n}}$, vnde omnium harum formularum valores sunt cogniti quas indicemus:

$$\left(\frac{n-1}{1}\right) = \alpha;\ \left(\frac{n-2}{2}\right) = \beta;\ \left(\frac{n-3}{3}\right) = \gamma;\ \left(\frac{n-4}{4}\right) = \delta \text{ etc.}$$

Verum hi non sufficiunt ad reliquos omnes expediendos, praeterea tanquam cognitos spectari oportet hos:

$$\left(\frac{n-2}{1}\right) = A;\ \left(\frac{n-3}{2}\right) = B;\ \left(\frac{n-4}{3}\right) = C;\ \left(\frac{n-5}{4}\right) = D \text{ etc.}$$

atque ex his reliqui omnes determinari poterunt ope aequationum supra demonstratarum; vnde potissimum has notasse iuuabit:

$$\left(\frac{n-a}{a}\right)\left(\frac{n}{b}\right)=\left(\frac{n-a}{b}\right)\left(\frac{n-a+b}{a}\right)$$

$$\left(\frac{n-a}{a}\right)\left(\frac{n-a-b}{b}\right)=\left(\frac{n-b}{b}\right)\left(\frac{n-a-b}{a}\right)$$

$$\left(\frac{n-a}{a}\right)\left(\frac{n-b-1}{b}\right)\left(\frac{n-a-b}{a-1}\right)=\left(\frac{n-b}{b}\right)\left(\frac{n-a}{a-1}\right)\left(\frac{n-a-b}{a}\right).$$

Ex harum prima posito $a=b+1$ inuenitur

$$\left(\frac{n-1}{a}\right)=\left(\frac{n-a}{a}\right)\left(\frac{n}{a-1}\right):\left(\frac{n-a}{a-1}\right) \text{ vbi } \left(\frac{n}{a-1}\right)=\frac{1}{a-1}$$

ideoque per formulas assumtas definitur $\left(\frac{n-1}{a}\right)$.

Ex secunda posito $b=1$ inuenitur

$$\left(\frac{n-a-1}{1}\right)=\left(\frac{n-1}{1}\right)\left(\frac{n-a-1}{a}\right):\left(\frac{n-a}{a}\right).$$

Ex tertia posito $b=1$ deducitur

$$\left(\frac{n-a-1}{a-1}\right)=\left(\frac{n-1}{1}\right)\left(\frac{n-a}{a-1}\right)\left(\frac{n-a-1}{a}\right):\left(\frac{n-a}{a}\right)\left(\frac{n-2}{1}\right)$$

sicque reperiuntur omnes formulae $\left(\frac{n-a-2}{a}\right)$, et ex his porro ponendo $b=2$ in tertia:

$$\left(\frac{n-a-2}{a-1}\right)=\left(\frac{n-2}{2}\right)\left(\frac{n-a}{n-1}\right)\left(\frac{n-a-2}{a}\right):\left(\frac{n-a}{a}\right)\left(\frac{n-3}{2}\right)$$

vnde reperiuntur formae $\left(\frac{n-a-3}{a}\right)$ et ita porro omnes $\left(\frac{n-a-b}{a}\right)$, quippe quae forma omnes complectitur. Labor autem per priores aequationes non mediocriter contrahitur. Inuenta enim $\left(\frac{n-a-2}{a}\right)$ ex pr ma colligitur:

$$\left(\frac{n-2}{a+2}\right)=\left(\frac{n-a-2}{a+2}\right)\left(\frac{n}{a}\right):\left(\frac{n-a-1}{a}\right) \text{ ex secunda vero}$$

$$\left(\frac{n-a-2}{a}\right)=\left(\frac{n-2}{2}\right)\left(\frac{n-a-2}{a}\right):\left(\frac{n-a}{a}\right)$$

simili-

similique modo ex inuentis formulis $\left(\frac{n-a-1}{a}\right)$ derivantur hae

$$\left(\frac{n-1}{a+1}\right)=\left(\frac{n-a-1}{a+1}\right)\left(\frac{n}{a}\right):\left(\frac{n-a-1}{a}\right)$$

$$\left(\frac{n-a-1}{1}\right)=\left(\frac{n-1}{1}\right)\left(\frac{n-a-1}{a}\right):\left(\frac{n-a}{a}\right).$$

Coroll. 1.

388. Ex aequatione $\left(\frac{n-1}{a}\right)=\frac{1}{a-1}\left(\frac{n-a}{a}\right):\frac{n-a}{a-1}$) definiuntur

$\left(\frac{n-1}{1}\right)=\frac{\beta}{1A}$; $\left(\frac{n-1}{2}\right)=\frac{\gamma}{2B}$; $\left(\frac{n-1}{3}\right)=\frac{\delta}{3C}$; $\left(\frac{n-1}{4}\right)=\frac{\varepsilon}{4D}$ etc.

Ex aequatione vero $\left(\frac{n-a-1}{1}\right)=\left(\frac{n-1}{1}\right)\left(\frac{n-a-1}{a}\right):\left(\frac{n-a}{a}\right)$ hae formulae

$\left(\frac{n-2}{1}\right)=\frac{\alpha A}{\alpha}$; $\left(\frac{n-3}{1}\right)=\frac{\alpha B}{\beta}$; $\left(\frac{n-4}{1}\right)=\frac{\alpha C}{\gamma}$; $\left(\frac{n-5}{1}\right)=\frac{\alpha D}{\delta}$ etc.

Coroll. 2.

389. Aequatio $\left(\frac{n-a-1}{a-1}\right)=\left(\frac{n-1}{1}\right)\left(\frac{n-a}{a-1}\right)\left(\frac{n-a-1}{a}\right):\left(\frac{n-a}{a}\right)\left(\frac{n-2}{1}\right)$ praebet

$\left(\frac{n-3}{1}\right)=\frac{\alpha AB}{\beta A}$; $\left(\frac{n-4}{2}\right)=\frac{\alpha BC}{\gamma A}$; $\left(\frac{n-5}{3}\right)=\frac{\alpha CD}{\delta A}$; $\left(\frac{n-6}{4}\right)=\frac{\alpha DE}{\varepsilon A}$ etc.

vnde reperiuntur $\left(\frac{n-2}{a+2}\right)=\left(\frac{n-a-2}{a+2}\right)\left(\frac{n}{a}\right):\left(\frac{n-a-2}{a}\right)$ istae formulae

$\left(\frac{n-2}{3}\right)=\frac{\gamma\beta A}{1\alpha AB}$; $\left(\frac{n-2}{4}\right)=\frac{\delta\gamma A}{2\alpha BC}$; $\left(\frac{n-2}{5}\right)=\frac{\varepsilon\delta A}{3\alpha CD}$; $\left(\frac{n-2}{6}\right)=\frac{\zeta\varepsilon A}{4\alpha DE}$ etc.

atque etiam istae $\left(\frac{n-a-2}{a}\right)=\left(\frac{n-2}{2}\right)\left(\frac{n-a+2}{a}\right):\left(\frac{n-a}{a}\right)$ quae sunt

$\left(\frac{n-3}{2}\right)=\frac{\beta\alpha AB}{\alpha\beta A}$; $\left(\frac{n-4}{2}\right)=\frac{\beta\alpha BC}{\beta\gamma A}$; $\left(\frac{n-5}{2}\right)=\frac{\beta\alpha CD}{\gamma\delta A}$; $\left(\frac{n-6}{2}\right)=\frac{\beta\alpha DE}{\delta\varepsilon A}$ etc.

Coroll. 3.

Coroll. 3.

390. Tum aequatio $\left(\frac{n-a-1}{a-1}\right)=\left(\frac{n-2}{2}\right)\left(\frac{n-a}{a-1}\right)\left(\frac{n-a-2}{a}\right):\left(\frac{n-a}{a}\right)\left(\frac{n-1}{2}\right)$ dat

$\left(\frac{n-4}{1}\right)=\frac{\alpha\beta ABC}{\beta\gamma AB}$; $\left(\frac{n-5}{2}\right)=\frac{\alpha\beta\gamma CD}{\gamma\delta AB}$; $\left(\frac{n-6}{3}\right)=\frac{\alpha\beta CDE}{\delta\varepsilon AB}$; $\left(\frac{n-7}{4}\right)=\frac{\alpha\beta\gamma DEF}{\varepsilon\zeta AB}$

hinc $\left(\frac{n-3}{a+3}\right)=\left(\frac{n-a-3}{a+3}\right)\left(\frac{n}{a}\right):\left(\frac{n-a-3}{a}\right)$ praebet

$$\left(\frac{n-1}{4}\right)=\frac{\beta\gamma\delta AB}{1\alpha\beta ABC};\ \left(\frac{n-2}{5}\right)=\frac{\gamma\delta\varepsilon AB}{2\alpha\beta BCD};\ \left(\frac{n-3}{6}\right)=\frac{\delta\varepsilon\zeta AB}{3\alpha\beta CDE}\ \text{etc.}$$

atque ex $\left(\frac{n-a-3}{3}\right)=\left(\frac{n-3}{3}\right)\left(\frac{n-a-3}{a}\right):\left(\frac{n-3}{a}\right)$ deducuntur

$\left(\frac{n-5}{4}\right)=\frac{\alpha\beta\gamma BCD}{\beta\gamma\delta AB}$; $\left(\frac{n-6}{5}\right)=\frac{\alpha\beta\gamma CDE}{\gamma\delta\varepsilon AB}$; $\left(\frac{n-7}{6}\right)=\frac{\alpha\beta\gamma DEF}{\delta\varepsilon\zeta AB}$ etc.

Exemplum 1.

391. *Casus in hac forma* $\int\frac{x^{p-1}dx}{\sqrt[2]{(1-x^2)^{2-q}}}=\left(\frac{p}{q}\right)$ *contentos, vbi* $n=2$ *euoluere, vbi est* $\left(\frac{p+2}{q}\right)=\frac{p}{p+q}\left(\frac{p}{q}\right)$.

Manifestum est has formulas omnes vel algebraice vel per angulos expediri, his tamen regulis vtentes, quia numeri p et q binarium superare non debent, vnam formulam a circulo pendentem habemus $\left(\frac{1}{1}\right)=\frac{\pi}{2\sin\frac{\pi}{2}}=\frac{\pi}{2}=\alpha$, vnde nostri casus erunt:

$\left(\frac{1}{2}\right)=1$; $\left(\frac{2}{2}\right)=\frac{1}{2}$

$\left(\frac{1}{1}\right)=\alpha$.

Exem-

Exemplum 2.

392. *Casus in hac forma* $\int \frac{x^{p-1}dx}{\sqrt[3]{(1-x^3)^{3-q}}} = \left(\frac{p}{q}\right)$ *contentos*, *vbi* $n=3$, *euoluere*, *vbi est* $\left(\frac{p+3}{q}\right)=\frac{p}{p+q}\left(\frac{p}{q}\right)$.

Hic casus principales, ad quos ceteri reducuntur, sunt $\left(\frac{2}{1}\right)=\frac{\pi}{3\,\text{sin.}\frac{\pi}{3}}=\frac{2\pi}{3\sqrt{3}}=\alpha$ et $\left(\frac{1}{1}\right)=A=\int\frac{dx}{\sqrt[3]{(1-x^3)^2}}$, qua concessa erunt reliqui:

$\left(\frac{3}{1}\right)=1;\quad \left(\frac{3}{2}\right)=\frac{1}{2};\quad \left(\frac{3}{3}\right)=\frac{1}{3}$

$\left(\frac{2}{1}\right)=\alpha;\quad \left(\frac{2}{2}\right)=\frac{\alpha}{A}$

$\left(\frac{1}{1}\right)=A.$

Exemplum 3.

393. *Casus in hac forma* $\int \frac{x^{p-1}dx}{\sqrt[4]{(1-x^4)^{4-q}}} = \left(\frac{p}{q}\right)$ *contentos*, *vbi* $n=4$ *euoluere*, *vbi est* $\left(\frac{p+4}{q}\right)=\frac{p}{p+q}\left(\frac{p}{q}\right)$.

A circulo pendent hae duae

$\left(\frac{3}{1}\right)=\frac{\pi}{4\,\text{sin.}\frac{\pi}{4}}=\frac{\pi}{2\sqrt{2}}=\alpha$ et $\left(\frac{2}{2}\right)=\frac{\pi}{4\,\text{sin.}\frac{2\pi}{4}}=\frac{\pi}{4}=\beta$,

praeterea vero vna transcendente singulari opus est $\left(\frac{2}{1}\right)=A$, vnde reliquae ita determinantur:

$\left(\frac{4}{1}\right)=1;\ \left(\frac{4}{2}\right)=\frac{1}{2};\ \left(\frac{4}{3}\right)=\frac{1}{3};\ \left(\frac{4}{4}\right)=\frac{1}{4}$

$\left(\frac{3}{1}\right)=\alpha;\ \left(\frac{3}{2}\right)=\frac{\beta}{A};\ \left(\frac{3}{3}\right)=\frac{\alpha}{2A}$

$\left(\frac{2}{1}\right)=A;\ \left(\frac{2}{2}\right)=\beta$

$\left(\frac{1}{1}\right)=\frac{\alpha A}{\beta}.$

Exemplum 4.

394. *Casus in hac forma* $\int \frac{x^{p-1}dx}{\sqrt[5]{(1-x^5)^{5-q}}} = \left(\frac{p}{q}\right)$ *contentos, vbi* $n=5$ *euoluere, vbi est* $\left(\frac{p+5}{q}\right) = \frac{p}{p+q}\left(\frac{p}{q}\right)$.

A circulo pendent hae duae formulae:

$$\left(\frac{4}{1}\right) = \frac{\pi}{5\,\text{sin.}\frac{\pi}{5}} = \alpha \text{ et } \left(\frac{3}{2}\right) = \frac{\pi}{5\,\text{sin.}\frac{2\pi}{5}} = \beta$$

praeter quas duas nouas transcendentes assumi oportet

$$\left(\frac{3}{1}\right) = A \text{ et } \left(\frac{2}{2}\right) = B$$

per quas omnes sequenti modo determinantur

$$\left(\frac{5}{1}\right) = 1;\ \left(\frac{5}{2}\right) = \frac{1}{2};\ \left(\frac{5}{3}\right) = \frac{1}{3};\ \left(\frac{5}{4}\right) = \frac{1}{4};\ \left(\frac{5}{5}\right) = \frac{1}{5}$$

$$\left(\frac{4}{1}\right) = \alpha;\ \left(\frac{4}{2}\right) = \frac{\beta}{A};\ \left(\frac{4}{3}\right) = \frac{\beta}{2B};\ \left(\frac{4}{4}\right) = \frac{\alpha}{3A}$$

$$\left(\frac{3}{1}\right) = A;\ \left(\frac{3}{2}\right) = \beta;\ \left(\frac{3}{3}\right) = \frac{\beta\beta}{\alpha B}$$

$$\left(\frac{2}{1}\right) = \frac{\alpha B}{\beta};\ \left(\frac{2}{2}\right) = B$$

$$\left(\frac{1}{1}\right) = \frac{\alpha A}{\beta}.$$

Exemplum 5.

395. *Casus in hac forma* $\int \frac{x^{p-1}dx}{\sqrt[6]{(1-x^6)^{6-q}}} = \left(\frac{p}{q}\right)$ *contentos vbi* $n=6$, *euoluere.*

A circulo pendent hae tres formulae:

$$\left(\frac{5}{1}\right) = \frac{\pi}{6\,\text{sin.}\frac{\pi}{6}} = \frac{\pi}{3} = \alpha;\ \left(\frac{4}{2}\right) = \frac{\pi}{6\,\text{sin.}\frac{2\pi}{6}} = \frac{\pi}{3\sqrt{3}} = \beta;$$

$$\left(\frac{3}{3}\right) = \frac{\pi}{6\,\text{sin.}\frac{3\pi}{6}} = \frac{\pi}{6} = \gamma$$

tum

tum vero assumantur hae duae transcendentes:

$$\left(\tfrac{4}{1}\right) = A \text{ et } \left(\tfrac{3}{2}\right) = B$$

atque per has omnes sequenti modo determinantur

$$\left(\tfrac{6}{1}\right) = 1;\ \left(\tfrac{6}{2}\right) = \tfrac{1}{2};\ \left(\tfrac{6}{3}\right) = \tfrac{1}{3};\ \left(\tfrac{6}{4}\right) = \tfrac{1}{4};\ \left(\tfrac{6}{5}\right) = \tfrac{1}{5};\ \left(\tfrac{6}{6}\right) = \tfrac{1}{6}$$

$$\left(\tfrac{5}{1}\right) = \alpha;\ \left(\tfrac{5}{2}\right) = \tfrac{\beta}{A};\ \left(\tfrac{5}{3}\right) = \tfrac{\gamma}{2B};\ \left(\tfrac{5}{4}\right) = \tfrac{\beta}{3B};\ \left(\tfrac{5}{5}\right) = \tfrac{\alpha}{4A}$$

$$\left(\tfrac{4}{1}\right) = A;\ \left(\tfrac{4}{2}\right) = \beta;\ \left(\tfrac{4}{3}\right) = \tfrac{\beta\gamma}{\alpha B};\ \left(\tfrac{4}{4}\right) = \tfrac{\beta\gamma A}{2\alpha BB}$$

$$\left(\tfrac{3}{1}\right) = \tfrac{\alpha B}{\beta};\ \left(\tfrac{3}{2}\right) = B;\ \left(\tfrac{3}{3}\right) = \gamma$$

$$\left(\tfrac{2}{1}\right) = \tfrac{\alpha B}{\gamma};\ \left(\tfrac{2}{2}\right) = \tfrac{\alpha BB}{\gamma A}$$

$$\left(\tfrac{1}{1}\right) = \tfrac{\alpha A}{\beta}.$$

Scholion.

396. Has determinationes quousque libuerit, continuare licet, in quibus praecipue notari debent casus nouas transcendentium species introducentes; quorum primus occurrit si $n = 3$, estque $\left(\tfrac{1}{1}\right) = \int \frac{dx}{\sqrt[3]{(1-x^3)^2}}$, cuius valorem per productum infinitum supra vidimus esse

$$= \frac{1}{1} \cdot \frac{2 \cdot 6}{4 \cdot 4} \cdot \frac{5 \cdot 9}{7 \cdot 7} \cdot \frac{8 \cdot 12}{10 \cdot 10} \text{ etc.}$$

quod ex formula $\left(\tfrac{1}{1}\right)$ ob $n = 3$ etiam est

$$\frac{2}{1 \cdot 1} \cdot \frac{3 \cdot 5}{4 \cdot 4} \cdot \frac{6 \cdot 8}{7 \cdot 7} \cdot \frac{9 \cdot 11}{10 \cdot 10} \cdot \frac{12 \cdot 14}{13 \cdot 13} \text{ etc.}$$

Deinde ex classe $n = 4$ nascitur haec noua forma transcendens:

$$\left(\tfrac{2}{1}\right) = \int \frac{x\,dx}{\sqrt[4]{(1-x^4)^3}} = \int \frac{dx}{\sqrt[4]{(1-x^4)^3}} = \int \frac{dx}{\sqrt{(1-x^4)}},$$

quae aequatur huic producto infinito

$$\frac{1}{1 \cdot 2} \cdot \frac{4 \cdot 7}{5 \cdot 6} \cdot \frac{8 \cdot 11}{9 \cdot 10} \cdot \frac{12 \cdot 15}{13 \cdot 14} \cdot \frac{16 \cdot 19}{17 \cdot 18} \text{ etc.} = \frac{1}{2} \cdot \frac{2 \cdot 7}{5 \cdot 3} \cdot \frac{4 \cdot 11}{9 \cdot 5} \cdot \frac{6 \cdot 15}{13 \cdot 7} \cdot \frac{8 \cdot 19}{17 \cdot 9} \text{ etc.}$$

 Ex

Ex classe $n=5$ impetramus duas nouas formulas transcendentes

$$\left(\frac{3}{1}\right)=\int\frac{x^2\,dx}{\sqrt[5]{(1-x^5)^4}}=\int\frac{dx}{\sqrt[5]{(1-x^5)^3}}=\frac{4}{1\cdot3}\cdot\frac{5\cdot9}{6\cdot8}\cdot\frac{10\cdot14}{11\cdot13}\cdot\frac{15\cdot19}{16\cdot18}\text{ etc. et}$$

$$\left(\frac{2}{2}\right)=\int\frac{x\,dx}{\sqrt[5]{(1-x^5)^3}}=\frac{4}{2\cdot2}\cdot\frac{5\cdot9}{7\cdot7}\cdot\frac{10\cdot14}{12\cdot12}\cdot\frac{15\cdot19}{17\cdot17}\text{ etc.}$$

ita vt sit

$$\left(\frac{3}{1}\right):\left(\frac{2}{2}\right)=\frac{2\cdot2}{1\cdot3}\cdot\frac{7\cdot7}{5\cdot9}\cdot\frac{12\cdot12}{10\cdot14}\cdot\frac{17\cdot17}{16\cdot18}\text{ etc.}$$

Classis $n=6$ has duas formulas transcendentes suppeditat:

1) $\left(\frac{4}{1}\right)=\int\frac{x^3\,dx}{\sqrt[6]{(1-x^6)^5}}=\int\frac{dx}{\sqrt[3]{(1-x^6)}}=\frac{1}{2}\int\frac{y\,dy}{\sqrt[6]{(1-y^3)^5}}$ posito $xx=y$ et

2) $\left(\frac{3}{2}\right)=\int\frac{x^2\,dx}{\sqrt[3]{(1-x^6)^2}}=\int\frac{x\,dx}{\sqrt{(1-x^4)}}=\frac{1}{2}\int\frac{dy}{\sqrt{(1-y^4)}}=\frac{1}{3}\int\frac{dz}{\sqrt[3]{(1-zz)^2}}$

sumto $y=xx$ et $z=x^3$. Notandum autem est inter has et primam $\int\frac{dx}{\sqrt[3]{(1-x^3)^2}}=2\int\frac{y\,dy}{\sqrt[6]{(1-y^6)^4}}=2\left(\frac{2}{2}\right)$ relationem dari, quae est $2\gamma\left(\frac{4}{1}\right)\left(\frac{2}{2}\right)=\alpha\left(\frac{3}{2}\right)\left(\frac{1}{1}\right)$ ita vt prima admissa hic altera sufficiat.

CALCV-

CALCVLI INTEGRALIS
LIBER PRIOR.

PARS PRIMA

SEV

METHODVS INVESTIGANDI FVNCTIONES VNIVS VARIABILIS EX DATA RELATIONE QVACVNQVE DIFFERENTIALIVM PRIMI GRADVS.

SECTIO SECVNDA

DE

INTEGRATIONE AEQVATIONVM DIFFERENTIALIVM.

CAPVT I.

DE SEPARATIONE VARIABILIVM.

Definitio.

397.

In aequatione differentiali *ſeparatio variabilium* locum habere dicitur, cum aequationem ita in duo membra diſpeſcere licet, vt in vtroque vnica tantum variabilis cum ſuo differentiali inſit.

Coroll. 1.

398. Quando igitur aequatio differentialis ita eſt comparata, vt ad hanc formam $Xdx = Ydy$ reduci poſſit, in qua X functio ſit ſolius x et Y ſolius y, tum ea aequatio ſeparationem variabilium admittere dicitur.

Coroll. 2.

Coroll. 2.

399. Quodsi P et X functiones ipsius x tantum, at Q et Y functiones ipsius y tantum denotent, haec aequatio $PYdx = QXdy$ separationem variabilium admittit, nam per XY diuisa abit in $\frac{Pdx}{X} = \frac{Qdy}{Y}$ in qua variabiles sunt separatae.

Coroll. 3.

400. In forma ergo generali $\frac{dy}{dx} = V$, separatio variabilium locum habet, si V eiusmodi fuerit functio ipsarum x et y, vt in duos factores resolui possit, quorum alter solam variabilem x, alter solam y contineat. Si enim sit $V = XY$, inde prodit aequatio separata $\frac{dy}{Y} = Xdx$.

Scholion.

401. Posita differentialium ratione $\frac{dy}{dx} = p$, in hac sectione eiusmodi relationem inter x, y et p considerare instituimus, qua p aequetur functioni cuicunque ipsarum x et y. Hic igitur primum eum casum contemplamur, quo ista functio in duos factores resoluitur, quorum alter est functio tantum ipsius x et alter ipsius y, ita vt aequatio ad hanc formam reduci possit $Xdx = Ydy$, in qua binae variabiles a se inuicem separatae esse dicuntur. Atque in hoc casu formulae simplices ante tractatae continentur, quando $Y = 1$, vt sit $dy = Xdx$, et $y = \int Xdx$, vbi totum negotium ad integrationem formulae

mulae Xdx reuocatur. Haud maiorem autem habet difficultatem aequatio feparata $Xdx = Ydy$ quam perinde ac formulas fimplices tractare licet, id quod in fequente problemate oftendemus.

Problema 49.

402. Aequationem differentialem, in qua variabiles funt feparatae, integrare, feu aequationem inter ipfas variabiles inuenire.

Solutio.

Aequatio feparationem variabilium admittens femper ad hanc formam $Ydy = Xdx$ reducitur; vbi Xdx tanquam differentiale functionis cuiusdam ipfius x et Ydy tanquam differentiale functionis cuiusdam ipfius y fpectari poteft, cum igitur differentialia fint aequalia eorum integralia quoque aequalia effe, vel quantitate conftante differre neceffe eft. Integrentur ergo per praecepta fuperioris fectionis feorfim ambae formulae, feu quaerantur integralia $\int Ydy$ et $\int Xdx$, quibus inuentis erit vtique $\int Ydy = \int Xdx + \text{Conft.}$ qua aequatione relatio finita inter quantitates x et y exprimitur.

Coroll. 1.

403. Quoties ergo aequatio differentialis feparationem variabilium admittit, toties integratio per eadem praecepta, quae fupra de formulis fimplicibus funt tradita, abfolui poteft.

Coroll. 2.

404. In aequatione integrali $\int Y dy = \int X dx +$ Const. vel ambae functiones $\int Y dy$ et $\int X dx$ sunt algebraicae, vel altera algebraica, altera vero transcendens, vel ambae transcendentes, sicque relatio inter x et y vel erit algebraica, vel transcendens.

Scholion.

405. In separatione variabilium a nonnullis totum fundamentum resolutionis aequationum differentialium constitui solet, ita vt cum aequatio proposita separationem variabilium non admittat, idonea substitutio sit inuestiganda, cuius beneficio nouae variabiles introductae separationem patiantur. Totum ergo negotium huc reducitur, vt proposita aequatione differentiali quacunque eiusmodi substitutio seu nouarum variabilium introductio doceatur, vt deinceps separatio variabilium locum sit habitura. Optandum vtique esset, vt huiusmodi methodus pro quouis casu idoneam substitutionem inueniendi aperiretur; sed nihil omnino certi in hoc negotio est compertum, dum pleraeque substitutiones, quae adhuc in vsu fuerunt, nullis certis principiis innituntur. Deinde autem variabilium separatio non tanquam verum fundamentum omnis integrationis spectari potest, propterea quod in aequationibus differentialibus secundi altiorisue gradus nullum vsum praestat; infra autem aliud principium latissime patens sum

sum expositurus. In hoc capite interim praecipuas integrationes ope separationis variabilium administratas exponere operae pretium videtur; quandoquidem in hoc arduo negotio, quam plurimas methodos cognoscere, plurimum interest.

Problema 50.

406. Aequationem differentialem $Pdx = Qdy$, in qua P et Q sint functiones homogeneae eiusdem dimensionum numeri ipsarum x et y, ad separationem variabilium reducere, eiusque integrale inuenire.

Solutio.

Cum P et Q sint functiones homogeneae ipsarum x et y eiusdem dimensionum numeri, erit $\frac{P}{Q}$ functio homogenea nullius dimensionis, quae ergo posito $y = ux$ abit in functionem ipsius u. Ponatur igitur $y = ux$, abeatque $\frac{P}{Q}$ in U functionem ipsius u, ita vt sit $dy = Udx$. Sed ob $y = ux$, fit $dy = udx + xdu$, qua substitutione nostra aequatio induet hanc formam $udx + xdu = Udx$, inter binas variabiles x et u, quae manifesto sunt separabiles. Nam dispositis terminis dx continentibus ad vnam partem habetur:

$$xdu = (U - u)dx \text{ ideoque } \frac{dx}{x} = \frac{du}{U - u}$$

quae integrata dat $lx = \int \frac{du}{U - u}$, ita vt iam ex variabili u determinetur x, vnde porro cognoscitur $y = ux$.

Coroll. 1.

407. Quodſi ergo integrale $\int\frac{du}{U-u}$ etiam per logarithmos exprimi poſſit, ita vt lx aequetur logarithmo functionis cuiuspiam ipſius u, habebitur aequatio algebraica inter x et u, ideoque pro u, poſito valore $\frac{y}{x}$ aequatio algebraica inter x et y.

Coroll. 2.

408. Cum ſit $y=ux$ erit $ly=lu+lx$, ideoque cum ſit $lx=\int\frac{du}{U-u}$ erit $ly=lu+\int\frac{du}{U-u}=\int\frac{du}{u}+\int\frac{du}{U-u}$, quibus integralibus in vnum reductis fit $ly=\int\frac{Udu}{u(U-u)}$. Verum hic notandum eſt, non in vtraque integratione pro lx et ly conſtantem arbitrariam adiicere licere; ſtatim enim atque alteri integrali eſt adiecta, ſimul conſtans alteri adiicienda definitur, cum eſſe debeat $ly=lx+lu$.

Coroll. 3.

409. Cum ſit $\int\frac{du}{U-u}=\int\frac{du-dU+dU}{U-u}=\int\frac{dU}{U-u}-\int\frac{dU-du}{U-u}$ ob hoc poſterius membrum per logarithmos integrabile, erit $lx=\int\frac{dU}{U-u}-l(U-u)$ ſeu $lx(U-u)=\int\frac{dU}{U-u}$. Perinde ergo eſt, ſiue haec formula $\int\frac{du}{U-u}$ ſiue $\int\frac{dU}{U-u}$ integretur.

Scholion.

Scholion.

410. Quoniam haec methodus ad omnes aequationes homogeneas patet, neque etiam ob irrationalitatem, quae forte in functionibus P et Q inest impeditur, imprimis est aestimanda, plurimumque aliis methodis anteferenda, quae tantum ad aequationes nimis speciales sunt accommodatae. Atque hinc etiam discimus omnes aequationes, quae ope cuiusdam substitutionis ad homogeneitatem reuocari possunt, per eandem methodum tractari posse. Veluti si proponatur haec aquatio $dz + zzdx = \frac{adx}{xx}$, statim patet posito $z = \frac{1}{y}$ eam ad hanc homogeneam $-\frac{dy}{yy} + \frac{dx}{yy} = \frac{adx}{xx}$ seu $xxdy = dx(xx - ayy)$ reduci. Caeterum non difficulter perspicitur, vtrum aequatio proposita huiusmodi substitutione ad homogeneitatem perduci queat? Plerumque, quoties quidem fieri potest, sufficit has positiones $x = u^m$ et $y = v^n$ tentasse, vbi facile iudicabitur, num exponentes m et n ita assumere liceat, vt vbique idem dimensionum numerus prodeat, magis enim complicatis substitutionibus in hoc genere vix locus conceditur, nisi forte quasi sponte se prodant. Methodum autem integrandi hic expositam aliquot exemplis illustrasse iuuabit.

Exemplum 1.

411. *Proposita aequatione differentiali homogenea* $xdx + ydy = mydx$, *eius integrale inuenire.*

Cum ergo hinc fit $\frac{dy}{dx} = \frac{my-x}{y}$ posito $y = ux$ fit $\frac{my-x}{y} = \frac{mu-1}{u}$, ideoque ob $dy = udx + xdu$ erit $udx + xdu = \frac{(mu-1)}{u}dx$ hincque

$$\frac{dx}{x} = \frac{udu}{mu-1-uu} = \frac{-udu}{1-mu+uu} \text{ seu}$$

$$\frac{dx}{x} = \frac{-udu+\frac{1}{2}mdu}{1-mu+uu} - \frac{\frac{1}{2}mdu}{1-mu+uu}$$

vnde integrando

$$lx = -\tfrac{1}{2}l(1-mu+uu) - \tfrac{1}{2}m\int\frac{du}{1-mu+uu} + \text{Const.}$$

vbi tres casus sunt considerandi prout $m > 2$, vel $m < 2$ vel $m = 2$.

1) Sit $m > 2$ et $1-mu+uu$ huiusmodi formam habebit

$(u-a)(u-\frac{1}{a})$ vt fit $m = a + \frac{1}{a} = \frac{aa+1}{a}$ et ob

$$\frac{du}{(u-a)(u-\frac{1}{a})} = \frac{a}{aa-1}\cdot\frac{du}{u-a} - \frac{a}{aa-1}\cdot\frac{du}{u-\frac{1}{a}} \text{ fiet}$$

$$lx = -\tfrac{1}{2}l(1-mu+uu) - \frac{(aa+1)}{2(aa-1)}l\frac{u-a}{u-\frac{1}{a}} + C \text{ seu}$$

$$lx\sqrt{(1-mu+uu)} + \frac{aa+1}{2(aa-1)}l\frac{au-aa}{au-1} = lc$$

et restituto valore $u = \frac{y}{x}$ aequatio integralis erit

$$l\sqrt{(xx-mxy+yy)} + \frac{aa+1}{2(aa-1)}l\frac{ay-aax}{ay-x} = lc \text{ seu}$$

$$\left(\frac{ay-aax}{ay-x}\right)^{\frac{aa+1}{2(aa-1)}}\sqrt{(xx-mxy+yy)} = C.$$

2) Sit

2) Sit $m < 2$ seu $m = 2\cos.\alpha$ erit

$$\int \frac{du}{1 - 2u\cos.\alpha + uu} = \frac{1}{\sin.\alpha}\text{Ang. tang.}\frac{u\sin.\alpha}{1 - u\cos.\alpha}$$

vnde

$$lx\sqrt{(1 - mu + uu)} = C - \frac{\cos.\alpha}{\sin.\alpha}\text{Ang. tang.}\frac{u\sin.\alpha}{1 - u\cos.\alpha}$$

seu $$l\sqrt{(xx - mxy + yy)} = C - \frac{\cos.\alpha}{\sin.\alpha}\text{Ang. tang.}\frac{y\sin.\alpha}{x - y\cos.\alpha}.$$

3) Sit $m = 2$ erit $\int \frac{du}{(1-u)^2} = \frac{1}{1-u}$, hincque

$$lx(1-u) = C - \frac{1}{1-u} \text{ seu } l(x-y) = B - \frac{x}{x-y}.$$

Exemplum 2.

412. *Proposita aequatione differentiali homogenea* $dx(\alpha x + \beta y) = dy(\gamma x + \delta y)$ *eius integrale inuenire.*

Posito $y = ux$ erit $udx + xdu = dx.\frac{\alpha + \beta u}{\gamma + \delta u}$, ideoque

$$\frac{dx}{x} = \frac{du(\gamma + \delta u)}{\alpha + \beta u - \gamma u - \delta uu} = \frac{du(\delta u + \frac{1}{2}\gamma - \frac{1}{2}\beta) + du(\frac{1}{2}\gamma + \frac{1}{2}\beta)}{\alpha + (\beta - \gamma)u - \delta uu}$$

vnde integrando

$$lx = C - l\sqrt{(\alpha + (\beta - \gamma)u - \delta uu)} + \tfrac{1}{2}(\beta + \gamma)\int \frac{du}{\alpha + (\beta - \gamma)u - \delta uu}$$

vbi iidem casus, qui ante, sunt considerandi, prout scilicet denominator $\alpha + (\beta - \gamma)u - \delta uu$ vel duos factores habet reales et inaequales, vel aequales, vel imaginarios.

Exem-

Exemplum 3.

413. *Proposita aequatione differentiali homogenea* $xdx + ydy = xdy - ydx$ *eius integrale inuenire.*

Cum hinc sit $\frac{dy}{dx} = \frac{x+y}{x-y}$ posito $y = ux$ fit $udx + xdu = \frac{1+u}{1-u}dx$, seu $xdu = \frac{1+uu}{1-u}dx$, vnde colligitur $\frac{dx}{x} = \frac{du - udu}{1+uu}$, et integrando

$lx = \text{Ang tang.}\,u - l\sqrt{(1+uu)} + C$ seu

$l\sqrt{(xx+yy)} = C + \text{Ang. tang.}\,\frac{y}{x}$.

Exemplum 4.

414. *Proposita aequatione differentiali homogenea* $xxdy = (xx - ayy)dx$ *eius integrale inuenire.*

Hic ergo est $\frac{dy}{dx} = \frac{xx - ayy}{xx}$, et posito $y = ux$ prodit $udx + xdu = (1 - auu)dx$ ideoque $\frac{dx}{x} = \frac{du}{1-u-auu}$. et $lx = \int \frac{du}{1-u-auu}$, cuius euolutioni non opus est immorari.

Exemplum 5.

415. *Proposita aequatione differentiali homogenea* $xdy - ydx = dx\sqrt{(xx+yy)}$ *eius integrale inuenire.*

Erit ergo $\frac{dy}{dx} = \frac{y+\sqrt{(xx+yy)}}{x}$, vnde posito $y = ux$ fit $udx + xdu = (u + \sqrt{(1+uu)})dx$ seu $xdu = dx\sqrt{(1+uu)}$ ita vt sit $\frac{dx}{x} = \frac{du}{\sqrt{(1+uu)}}$, cuius integrale est $lx = la + l(u + \sqrt{(1+uu)}) = la + l(\frac{y+\sqrt{(xx+yy)}}{x})$ seu $lx = la + l\frac{x}{\sqrt{(xx+yy)} - y}$, vnde colligitur $x = \frac{ax}{\sqrt{(xx+yy)} - y}$, seu $\sqrt{(xx+yy)} = a + y$ hincque $xx = aa + 2ay$.

Scholion.

Scholion.

416. Huc etiam functiones transcendentes numerari possunt, modo afficiant functiones nullius dimensionis ipsarum x et y, quia posito $y = ux$ simul in functiones ipsius u abeunt. Ita si in aequatione $Pdx = Qdy$, praeterquam quod P et Q sunt functiones homogeneae eiusdem dimensionum numeri, insint huiusmodi formulae $l\frac{\sqrt{(xx+yy)}}{x}$, $e^{y:x}$, Ang. sin. $\frac{x}{\sqrt{(xx+yy)}}$; cos. $\frac{nx}{y}$ etc. methodus exposita pari successu adhiberi potest, quia posito $y = ux$ ratio $\frac{dy}{dx}$ aequatur functioni solius nouae variabilis u.

Problema 51.

417. Aequationem differentialem primi ordinis

$$dx(\alpha + \beta x + \gamma y) = dy(\delta + \varepsilon x + \zeta y)$$

ad separationem variabilium reuocare et integrare.

Solutio.

Ponatur $\alpha + \beta x + \gamma y = t$ et $\delta + \varepsilon x + \zeta y = u$, vt fiat $t\,dx = u\,dy$. At inde colligimus

$$x = \frac{\zeta t - \gamma u - \alpha\zeta + \gamma\delta}{\beta\zeta - \gamma\varepsilon} \text{ et } y = \frac{\beta u - \varepsilon t + \alpha\varepsilon - \beta\delta}{\beta\zeta - \gamma\varepsilon}$$

hincque $dx : dy = \zeta dt - \gamma du : \beta du - \varepsilon dt$, vnde nanciscimur hanc aequationem

$$\zeta t\,dt - \gamma t\,du = \beta u\,du - \varepsilon u\,dt \text{ seu}$$

$$dt(\zeta t + \varepsilon u) = du(\beta u + \gamma t)$$

quae cum sit homogenea et cum exemplo §. 412. conueniat, integratio iam est expedita.

Verum tamen casus existit, quo haec reductio ad homogeneitatem locum non habet, cum fuerit $\beta\zeta - \gamma\varepsilon = 0$ quoniam tum introductio nouarum variabilium t et u tollitur. Hic ergo casus peculiarem requirit solutionem, quae ita instituatur; quoniam tum aequatio proposita eiusmodi formam est habitura

$$\alpha dx + (\beta x + \gamma y) dx = \delta dy + n(\beta x + \gamma y) dy$$

ponamus $\beta x + \gamma y = z$, erit $\frac{dy}{dx} = \frac{\alpha + z}{\delta + nz}$. At $dy = \frac{dz - \beta dx}{\gamma}$ ergo $\frac{dz - \beta dx}{\gamma} = \frac{\alpha + z}{\delta + nz} dx$, vbi variabiles manifesto sunt separabiles, fit enim $dx = \frac{dz(\delta + nz)}{\alpha\gamma + \beta\delta + (\gamma + n\beta)z}$ cuius integratio logarithmos inuoluit, nisi sit $\gamma + n\beta = 0$ quo casu algebraice dat $x = \frac{2\delta z + nzz}{2(\alpha\gamma + \beta\delta)} + C$.

Coroll. 1.

418. Aequatio ergo differentialis primi ordinis, vti vocatur, in genere ad homogenietatem reduci nequit, sed casus, quibus $\beta\zeta = \gamma\varepsilon$, inde excipi debent, qui etiam ad aequationem separatam omnino diuersam deducunt.

Coroll. 2.

419. Si in his casibus exceptis sit $n = 0$, seu haec proposita sit aequatio $dy = dx(\alpha + \beta x + \gamma y)$, posito $\beta x + \gamma y = z$ ob $\delta = 1$ haec oritur aequatio

tio $dx = \frac{dz}{\alpha\gamma + \beta + \gamma z}$, cuius integrale est

$$\gamma x = l\frac{\beta + \alpha\gamma + \gamma z}{C} = l\frac{\beta + \alpha\gamma + \beta\gamma x + \gamma\gamma y}{C}$$ seu

$$\beta + \gamma(\alpha + \beta x + \gamma y) = Ce^{\gamma x}.$$

Problema 52.

420. Proposita aequatione differentiali huiusmodi:

$$dy + Pydx = Qdx$$

in qua P et Q sint functiones quaecunque ipsius x; altera autem variabilis y cum suo differentiali nusquam plus vna habeat dimensionem, eam ad separationem variabilium perducere et integrare.

Solutio.

Quaeratur eiusmodi functio ipsius x, quae sit X, vt facta substitutione $y = Xu$ aequatio prodeat separabilis: Tum autem oritur

$$\begin{aligned} Xdu + udX &= Qdx \\ + PXudx & \end{aligned}$$

quam aequationem separationem admittere euidens est, si fuerit $dX + PXdx = 0$, seu $\frac{dX}{X} = -Pdx$, vnde integratio dat $lX = -\int Pdx$ et $X = e^{-\int Pdx}$; hac ergo pro X sumta functione, aequatio nostra transformata erit: $Xdu = Qdx$, seu $du = \frac{Qdx}{X} = e^{\int Pdx}Qdx$

 vnde

vnde cum P et Q sint functiones datae ipsius x, erit $u = \int e^{\int P dx} Q dx = \frac{y}{X}$. Quocirca aequationis propositae integrale est $y = e^{-\int P dx} \int e^{\int P dx} Q dx$.

Coroll. 1.

421. Resolutio ergo huius aequationis $dy + Pydx = Qdx$ duplicem requirit integrationem alteram formulae $\int P dx$, alteram formulae $\int e^{\int P dx} Q dx$. Sufficit autem in posteriori constantem arbitrariam adiecisse, cum valor ipsius y plus vna non recipiat. Etiamsi enim in priori loco $\int P dx$ scribatur $\int P dx + C$, formula pro y manet eadem.

Coroll. 2.

422. Dum ergo formula $P dx$ integratur, sufficit eius integrale particulare sumi, ideoque constanti ingredienti eiusmodi valorem tribui conuenit, vt integralis forma fiat simplicissima.

Scholion.

423. En ergo aliud aequationum genus non minus late patens quam praecedens homogenearum, quod ad separationem variabilium perduci, hocque modo integrari potest. Inde autem in Analysin maxima vtilitas redundat, cum hic litterae P et Q functiones quascunque ipsius x denotent. Hoc ergo modo manifestum est tractari posse hanc aequationem $R dy + P y dx = Q dx$, si etiam R functionem quam-

quamcunque ipsius x denotet, facta enim diuisione per R forma proposita prodit, modo loco P et Q scribatur $\frac{P}{R}$ et $\frac{Q}{R}$, ita vt integrale futurum sit

$$y = e^{-\int \frac{P dx}{R}} \int \frac{e^{\int \frac{P dx}{R}} Q dx}{R}.$$

Ad huius problematis illustrationem quaedam exempla adiiciamus.

Exemplum 1.

424 *Proposita aequatione differentiali* $dy + y dx = a x^n dx$ *eius integrale inuenire.*

Cum hic sit $P = 1$ et $Q = a x^n$ erit $\int P dx = x$, et aequatio integralis fiet

$$y = e^{-x} \int e^x x^n dx$$

quae si n sit numerus integer positiuus euadet

$$y = e^{-x}(e^x(x^n - n x^{n-1} + n(n-1) x^{n-2} - \text{etc.}) + C)$$

qua euoluta prodit

$$y = C e^{-x} + x^n - n x^{n-1} + n(n-1) x^{n-2} - n(n-2)(n-1) x^{n-3} + \text{etc.}$$

vnde pro simplioribus valoribus ipsius n

si $n = 0$; erit $y = C e^{-x} + 1$
si $n = 1$; erit $y = C e^{-x} + x - 1$
si $n = 2$; erit $y = C e^{-x} + x^2 - 2x + 2.1$
si $n = 3$; erit $y = C e^{-x} + x^3 - 3x^2 + 3.2x - 3.2.1$
etc.

Coroll. 1.

425. Si ergo constans C sumatur $=0$, habebitur integrale particulare

$$y=x^n-nx^{n-1}+n(n-1)x^{n-2}-n(n-1)(n-2)x^{n-3}+\text{etc.}$$

quod ergo est algebraicum, dummodo n sit numerus integer positiuus.

Coroll. 2.

426. Si integrale ita determinari debeat, vt posito $x=0$, valor ipsius y euanescat, constans C aequalis sumi debet vltimo termino constanti signo mutato, vnde id semper erit transcendens.

Exemplum 2.

427. *Proposita aequatione differentiali* $(1-xx)dy+xydx=adx$ *eius integrale inuenire.*

Aequatio ista per $1-xx$ diuisa ad hanc formam reducitur $dy+\frac{xydx}{1-xx}=\frac{adx}{1-xx}$, ita vt sit $P=\frac{x}{1-xx}$; $Q=\frac{a}{1-xx}$; hinc $\int Pdx=-l\sqrt{(1-xx)}$ et $e^{\int Pdx}=\frac{1}{\sqrt{(1-xx)}}$, ex quo integrale reperitur:

$$y=\sqrt{(1-xx)}\int\frac{adx}{(1-xx)^{\frac{3}{2}}}=(\frac{ax}{\sqrt{(1-xx)}}+C)\sqrt{(1-xx)}$$

quocirca integrale quaesitum erit

$$y=ax+c\sqrt{(1-xx)}$$

quod

quod si ita determinari debeat, vt posito $x=0$, sumi oportet $c=0$, eritque $y=ax$.

Exemplum 3.

428. *Proposita aequatione differentiali* $dy+\frac{nydx}{\sqrt{(1+xx)}}=adx$ *eius integrale inuenire.*

Cum hic sit $P=\frac{n}{\sqrt{(1+xx)}}$ et $Q=a$ erit $\int Pdx=nl(x+\sqrt{(1+xx)})$ et $e^{\int Pdx}=(x+\sqrt{(1+xx)})^n$, et $e^{-\int Pdx}=(\sqrt{(1+xx)}-x)^n$ vnde integrale quaesitum erit

$$y=(\sqrt{(1+xx)}-x)^n\int adx(x+\sqrt{(1+xx)})^n$$

ad quod euoluendum ponatur $x+\sqrt{(1+xx)}=u$, et fiet $x=\frac{uu-1}{2u}$, hinc $dx=\frac{du(1+uu)}{2uu}$, ergo

$$\int u^n dx=\frac{u^{n-1}}{2(n-1)}+\frac{u^{n+1}}{2(n+1)}+C$$

Nunc quia $(\sqrt{(1+xx)}-x)^n=u^{-n}$ erit

$$y=Cu^{-n}+\frac{au^{-1}}{2(n-1)}+\frac{au}{2(n+1)} \text{ siue}$$

$$y=C(\sqrt{(1+xx)}-x)^n+\frac{a}{2(n-1)}(\sqrt{(1+xx)}-x)+\frac{a}{2(n+1)}(\sqrt{1+xx}+x)$$

quae expressio ad hanc formam reducitur:

$$y=C(\sqrt{(1+xx)}-x)^n+\frac{na}{nn-1}\sqrt{(1+xx)}-\frac{ax}{nn-1}$$

si

si integrale ita determinari debeat, vt posito $x=0$ fiat $y=0$, sumi oportet $C=-\frac{na}{nn-1}$.

Problema 53.

429. Proposita aequatione differentiali

$$dy+Pydx=Qy^{n+1}dx$$

vbi P et Q denotent functiones quascunque ipsius x, eam ad separationem variabilium reducere et integrare.

Solutio.

Haec aequatio posito $\frac{1}{y^n}=z$ statim ad formam modo tractatam reducitur, nam ob $\frac{dy}{y}=-\frac{dz}{nz}$, aequatio nostra per y diuisa, scilicet $\frac{dy}{y}+Pdx=Qy^n dx$ statim abit in $-\frac{dz}{nz}+Pdx=\frac{Qdx}{z}$ seu $dz-nPzdx=-nQdx$ cuius integrale est

$$z=-e^{n\int Pdx}\int e^{-n\int Pdx}nQdx \text{ ideoque}$$

$$\frac{1}{y^n}=-ne^{n\int Pdx}\int e^{-n\int Pdx}Qdx.$$

Tractari autem potest vt praecedens quaerendo eiusmodi functionem X, vt facta substitutione $y=Xu$ prodeat aequatio separabilis: prodit autem

$$Xdu+udX+PXudx=X^{n+1}u^{n+1}Qdx.$$

Fiat

Fiat ergo $dX + PXdx = 0$ seu $X = e^{-\int Pdx}$ eritque

$$\frac{du}{u^{n+1}} = X^n Q dx = e^{-n\int Pdx} Q dx$$

et integrando:

$$-\frac{1}{nu^n} = \int e^{-n\int Pdx} Q dx.$$

Iam quia $u = \frac{y}{X} = e^{\int Pdx} y$ habebitur vt ante

$$\frac{1}{y^n} = -ne^{+n\int Pdx} \int e^{-n\int Pdx} Q dx.$$

Scholion.

430. Hic ergo casus a praecedente non differre est censendus, ita vt hic nihil noui sit praestitum. Atque haec duo genera sunt fere sola, quae quidem aliquanto latius pateant, in quibus separatio variabilium obtineri queat. Caeteri casus, qui ope cuiusdam substitutionis ad variabilium separationem praeparari possunt, plerumque sunt nimis speciales, quam vt insignis vsus inde expectari possit. Interim tamen aliquot casus prae caeteris memorabiles hic exponamus.

Problema. 54.

431. Proposita hac aequatione differentiali:

$$\alpha y dx + \beta x dy + x^m y^n (\gamma y dx + \delta x dy) = 0$$

eam ad separationem variabilium reducere et integrare.

 Solutio.

Solutio.

Tota aequatione per xy diuisa nanciscimur hanc formam

$$\frac{\alpha dx}{x}+\frac{\beta dy}{y}+x^m y^n\left(\frac{\gamma dx}{x}+\frac{\delta dy}{y}\right)=0$$

vnde statim has substitutiones $x^\alpha y^\beta = t$ et $x^\gamma y^\delta = u$ insigni vsu non esse carituras colligimus: inde enim fit

$$\frac{\alpha dx}{x}+\frac{\beta dy}{y}=\frac{dt}{t} \text{ et } \frac{\gamma dx}{x}+\frac{\delta dy}{y}=\frac{du}{u}$$

hincque aequatio nostra $\frac{dt}{t}+x^m y^n.\frac{du}{u}=0.$
At ex substitutione sequitur:

$$x^{\alpha\delta-\beta\gamma}=t^\delta u^{-\beta}, \text{ et } y^{\alpha\delta-\beta\gamma}=u^\alpha t^{-\gamma}$$

ideoque

$$x=t^{\frac{\delta}{\alpha\delta-\beta\gamma}}u^{\frac{-\beta}{\alpha\delta-\beta\gamma}} \text{ et } y=t^{\frac{-\gamma}{\alpha\delta-\beta\gamma}}u^{\frac{\alpha}{\alpha\delta-\beta\gamma}}$$

quibus substitutis fit

$$\frac{dt}{t}+t^{\frac{\delta m-\gamma n}{\alpha\delta-\beta\gamma}}u^{\frac{\alpha n-\beta m}{\alpha\delta-\beta\gamma}}.\frac{du}{u}=0 \text{ ideoque}$$

$$t^{\frac{\gamma n-\delta m}{\alpha\delta-\beta\gamma}-1}dt+u^{\frac{\alpha n-\beta m}{\alpha\delta-\beta\gamma}-1}du=0$$

cuius aequationis integrale est:

$$\frac{t^{\frac{\gamma n-\delta m}{\alpha\delta-\beta\gamma}}}{\gamma n-\delta m}+\frac{u^{\frac{\alpha n-\beta m}{\alpha\delta-\beta\gamma}}}{\alpha n-\beta m}=C$$

vbi tantum superest vt restituantur valores $t=x^\alpha y^\beta$ et $u=x^\gamma y^\delta$. Caeterum notetur, si fuerit vel $\gamma n-\delta m=0$ vel $\alpha n-\beta m=0$, loco illorum membrorum vel lt vel lu scribi debere.

Scholion.

Scholion.

432. Ad aequationem propositam ducit quaestio, qua eiusmodi relatio inter variabiles x et y quaeritur, ut fiat

$$\int y\,dx = axy + b x^{m+1} y^{n+1}$$

ad hanc enim resoluendam differentialia sumi debent, quo prodit

$$y\,dx = ax\,dy + ay\,dx + bx^m y^n ((m+1) y\,dx + (n+1) x\,dy)$$

qua aequatione cum nostra forma comparata, est

$$\alpha = a - 1;\ \beta = a;\ \gamma = (m+1) b, \text{ et } \delta = (n+1) b$$

ergo

$$\alpha\delta - \beta\gamma = (n-m) ab - (n+1) b$$
$$\alpha n - \beta m = (n-m) a - n, \text{ et } \gamma n - \delta m = (n-m) b$$

unde aequatio integralis fit manifesta.

Problema 55.

433. Proposita hac aequatione differentiali:

$$y\,dy + dy(a + bx + nxx) = y\,dx(c + nx)$$

eam ad separationem variabilium reducere, et integrare.

Solutio.

Cum hinc sit $\frac{dy}{dx} = \frac{y(c+nx)}{y+a+bx+nxx}$, tentetur haec substitutio $\frac{y(c+nx)}{y+a+bx+nxx} = u$, seu $y = \frac{u(a+bx+nxx)}{c+nx-u}$ fierique debet $dy = u\,dx$, seu $\frac{dy}{y} = \frac{u\,dx}{y} = \frac{dx(c+nx-u)}{a+bx+nxx}$; at ex logarithmis colligitur

$$\frac{dy}{y} = \frac{du}{u} + \frac{dx(b+2nx)}{a+bx+nxx} - \frac{n\,dx + du}{c+nx-u} = \frac{dx(c+nx-u)}{a+bx+nxx}$$

quae contrahitur in

$$\frac{du(c+nx)-nu\,dx}{u(c+nx-u)}=\frac{dx(c-b-nx-u)}{a+bx+nxx} \text{ ſeu}$$

$$\frac{du(c+nx)}{u(c+nx-u)}=\frac{dx(na+cc-bc+(b-2c)u+uu)}{(c+nx-u)(a+bx+nxx)}$$

quae per $c+nx-u$ multiplicata manifeſto eſt ſeparabilis, prodiitque

$$\frac{dx}{(a+bx+nxx)(c+nx)}=\frac{du}{u(na+cc-bc+(b-2c)u+uu)}.$$

cuius ergo integratio per logarithmos et angulos abſolui poteſt. Caſu autem hic vix praeuidendo euenit, vt haec ſubſtitutio ad votum ſucceſſerit, neque hoc problema magnopere iuuabit.

Problema 56.

434. Propoſitam hac aequationem differentialem

$$(y-x)dy=\frac{n\,dx(1+yy)\sqrt{(1+yy)}}{\sqrt{(1+xx)}}$$

ad ſeparationem variabilium reducere et integrare.

Solutio.

Ob irrationalitatem duplicem vix vllo modo patet, cuiusmodi ſubſtitutione vti conueniat. Eiusmodi certe quaeri conuenit, qua eidem ſigno radicali non ambae variabiles ſimul implicentur. Ad hunc ſcopum commoda videtur haec ſubſtitutio $y=\frac{x-u}{1+xu}$, qua fit $y-x=\frac{-u(1+xx)}{1+xu}$, $1+yy=\frac{(1+xx)(1+uu)}{(1+xu)^2}$ et $dy=\frac{dx(1+uu)-du(1+xx)}{(1+xu)^2}$ atque his valoribus in noſtra aequatione ſubſtitutis, prodit

$$-udx(1+uu)+udu(1+xx)=ndx(1+uu)\sqrt{(1+uu)}$$

quae

quae manifesto separationem variabilium admittit: colligitur scilicet

$$\frac{dx}{1+xx} = \frac{u\,du}{(1+uu)(n\sqrt{(1+uu)}+u)}$$

quae aequatio posito $1+uu=tt$ concinnior redditur

$$\frac{dx}{1+xx} = \frac{dt}{t(nt+\sqrt{(tt-1)})}$$

et ope positionis $t=\frac{1+ss}{2s}$ sublata irrationalitate

$$\frac{dx}{1+xx} = -\frac{2ds(1-ss)}{(1+ss)(n+1+(n-1)ss)} = \frac{2ds}{1+ss} - \frac{2nds}{n+1+(n-1)ss}$$

cuius integratio nulla amplius laborat difficultate.

Scholion.

435. In hoc casu praecipue substitutio $y=\frac{x-u}{1+xu}$ notari meretur, qua duplex irrationalitas tollitur: vnde operae pretium erit videre, quid hac substitutione generaliori praestari possit: $y=\frac{\alpha x+u}{1+\beta xu}$; inde autem fit

$$\alpha-\beta yy=\frac{(\alpha-\beta uu)(1-\alpha\beta xx)}{(1+\beta xu)^2};\quad y-\alpha x=\frac{u(1-\alpha\beta xx)}{1+\beta xu}$$

et $$dy=\frac{dx(\alpha-\beta uu)+du(1-\alpha\beta xx)}{(1+\beta xu)^2}$$

ac iam facile perspicitur, in cuiusmodi aequationibus haec substitutio vsum afferre possit; eius scilicet benificio haec duplex irrationalitas $\frac{\sqrt{(\alpha-\beta yy)}}{\sqrt{(1-\alpha\beta xx)}}$ reducitur ad hanc simplicem $\frac{\sqrt{(\alpha-\beta uu)}}{1+\beta xu}$ quam porro facile rationalem reddere licet. Atque hic fere sunt casus, in quibus reductio ad separabilitatem locum inuenit, quibus probe perpensis aditus facile patebit ad

ad reliquos casus, qui quidem etiamnum sunt tractati; vnicam vero adhuc inuestigationem apponam circa casus, quibus haec aequatio $dy + yydx = ax^m dx$ separationem variabilium admittit quandoquidem ad huiusmodi aequationes frequenter peruenitur, atque haec ipsa aequatio olim inter Geometras omni studio est agitata.

Problema 57.

436. Pro aequatioe $dy + yydx = ax^m dx$ valores exponentis m definire, quibus eam ad separationem variabilium reducere licet.

Solutio.

Primo haec aequatio sponte est separabilis casu $m = 0$, tum enim ob $dy = dx(a - yy)$ fit $dx = \frac{dy}{a-yy}$. Omnis ergo inuestigatio in hoc versatur, vt ope substitutionum alii casus ad hunc reducantur.

Ponamus $y = \frac{b}{z}$, et fit $-bdz + bbdx = ax^m zzdx$, quae forma vt propositae similis euadat, statuatur $x^{m+1} = t$, vt fit $x^m dx = \frac{dt}{m+1}$, et $dx = \frac{t^{\frac{-m}{m+1}}dt}{m+1}$, eritque

$$bdz + \frac{azzdt}{m+1} = \frac{bb}{m+1} t^{\frac{-m}{m+1}} dt$$

quae sumto $b = \frac{a}{m+1}$ ad similitudinem propositae propius accedit, vt fit $dz + zzdt = \frac{a}{(m+1)^2} t^{\frac{-m}{m+1}} dt$. Si ergo haec esset separabilis, ipsa proposita ista substitu-

stitutione separabilis fieret et vicissim; vnde concludimus, si aequatio proposita separationem admittat casu $m=n$, eam quoque esse admissuram casu $m=\frac{-n}{n+1}$. Hinc autem ex casu $m=0$ alius non reperitur.

Ponamus $y=\frac{1}{x}-\frac{z}{xx}$, vt sit $dy=-\frac{dx}{xx}-\frac{dz}{xx}+\frac{2zdx}{x^3}$, et $yydx=\frac{dx}{xx}-\frac{2zdx}{x^3}+\frac{zzdx}{x^4}$, vnde prodit

$$-\frac{dz}{xx}+\frac{zzdx}{x^4}=ax^m dx \text{ seu } dz-\frac{zzdx}{xx}=-ax^{m+2}dx$$

fit nunc $x=\frac{1}{t}$ et fit $dz+zzdt=at^{-m-4}dt$, quae cum propositae sit similis, discimus, si separatio succedat casu $m=n$, etiam succedere casu $m=-n-4$.

Ex vno ergo casu $m=n$ consequimur duos scilicet $m=-\frac{n}{n+1}$ et $m=-n-4$. Cum igitur constet casus $m=0$, hinc formulae alternatim adhibitae praebent sequentes

$m=-4$; $m=-\frac{4}{3}$; $m=-\frac{8}{3}$; $m=-\frac{8}{5}$; $m=-\frac{12}{5}$; $m=-\frac{12}{7}$; $m=-\frac{16}{7}$ etc.

qui casus omnes in hac formula $m=\frac{-4i}{2i\pm1}$ continentur.

Coroll. 1.

437. Quodsi ergo fuerit vel $m=\frac{-4i}{2i+1}$ vel $m=\frac{-4i}{2i-1}$ aequatio $dy+yydx=ax^m dx$ per aliquot substitutiones repetitas tandem ad formam $du+uudv=cdv$, cuius separatio et integratio constat, reduci potest.

Coroll. 2.

438. Scilicet si fuerit $m=\frac{-4i}{2i+1}$, aequatio $dy+yydx=ax^m dx$ per substitutiones $x=t^{\frac{1}{m+1}}$ et $y=\frac{a}{(m+1)z}$ reducitur ad hanc $dz+zzdt=\frac{a}{(m+1)^2}t^n dt$, vt sit $n=\frac{-4i}{2i-1}$, qui casus vno gradu inferior est censendus.

Coroll. 3.

439. Sin autem fuerit $m=\frac{-4i}{2i-1}$, aequatio $dy+yydx=ax^m dx$ per has substitutiones $x=\frac{1}{t}$ et $y=\frac{1}{x}-\frac{z}{xx}$ seu $y=t-ttz$ reducitur ad hanc $dz+zzdt=at^n dt$, in qua est $n=\frac{-4(i-1)}{2i-1}=\frac{-4(i-1)}{2(i-1)+1}$, qui casus denuo vno gradu inferior est.

Coroll. 4.

440. Omnes ergo casus separabiles hoc modo inuenti, pro exponente m dant numeros negatiuos intra limites 0 et -4 contentos, ac si i sit numerus infinitus prodit casus $m=-2$ qui autem per se constat, cum aequatio $dy+yydx=\frac{a\,dx}{xx}$ posito $y=\frac{1}{z}$ fiat homogenea.

Scholion 1.

441. Aequatio haec $dy+yydx=ax^m dx$ vocari solet Riccatiana ab Auctore Comite *Riccati*, qui primus casus separabiles proposuit. Hic quidem eam in forma simplicissima exhibui, cum eo haec $dy+Ayyt^\mu dt=Bt^\lambda dt$ ponendo $At^\mu dt=dx$ et $At^{\mu+1}=(\mu+1)x$ statim reducatur. Caeterum etsi binae

binae fubftitutiones, quibus hic fum vfus, funt fimpliciffimae, tamen magis compofitis adhibendis nulli alii cafus feparabiles deteguntur: ex quo hoc omnino memorabile eft vifum, hanc aequationem rariffime feparationem admittere, tametfi numerus cafuum, quibus hoc praeftari queat, reuera fit infinitus. Caeterum haec inueftigatio ab exponente ad fimplicem coefficientem traduci poteft; pofito enim $y = x^{\frac{m}{2}} z$, prodit $dz + \frac{mzdx}{2x} + x^{\frac{m}{2}} zzdx = ax^{\frac{m}{2}} dx$ vbi fi fiat $x^{\frac{m}{2}} dx = dt$, et $x^{\frac{m+2}{2}} = \frac{m+2}{2} t$, erit $\frac{dx}{x} = \frac{2dt}{(m+2)t}$ hincque $dz + \frac{mzdt}{(m+2)t} + zzdt = adt$, quae ergo aequatio, quoties fuerit $\frac{m}{m+2} = \pm 2i$ feu numerus par tam pofitiuus, quam negatiuus, feparabilis reddi poteft, ita vt haec aequatio $dz \pm \frac{2izdt}{t} + zzdt = adt$ femper fit integrabilis. Si praeterea ponatur $z = u - \frac{m}{2(m+2)t}$ oritur $du + uudt = adt - \frac{m(m+4)dt}{4(m+2)^2 tt}$ et pro cafibus feparabilitatis $m = \frac{-4i}{2i \pm 1}$ habetur $du + uudt = adt + \frac{i(i \pm 1)dt}{tt}$. Vberiorem autem huius aequationis euolutionem, quandoquidem eft maximi momenti, in fequentibus docebo; vbi integratione aequationum differentialium per feries infinitas fum acturus, hinc enim facilius cafus feparabiles eruemus, fimulque integralia affignare poterimus.

Scholion 2.

442. Ampliora praecepta circa feparationem variabilium, quae quidem vfum fint habitura, vix

 tradi

tradi poſſe videntur, vnde intelligitur in pauciſſimis aequationibus differentialibus hanc methodum adhiberi poſſe. Progrediar igitur ad aliud principium explicandum, vnde integrationes haurire liceat, quod multo latius patet, dum etiam ad aequationes differentiales altiorum graduum accommodari poteſt, ita vt in eo verus ac naturalis fons omnium integrationum contineri videatur. Iſtud autem principium in hoc conſiſtit, quod propoſita quacunque aequatione differentiali inter duas variabiles, ſemper detur functio quaedam, per quam aequatio multiplicata fiat integrabilis; aequationis ſcilicet omnia membra ad eandem partem diſponi oportet, vt talem formam obtineat $Pdx + Qdy = 0$; ac tum dico ſemper dari functionem quandam variabilium puta V, vt facta multiplicatione formula $VPdx + VQdy$ integrabilis exiſtat, ſeu vt verum ſit differentiale ex differentiatione cuiuſpiam functionis binarum variabilum x et y natum. Quodſi enim haec functio ponatur $= S$ vt ſit $dS = VPdx + VQdy$, quia eſt $Pdx + Qdy = 0$ erit etiam $dS = 0$, ideoque $S = Conſt.$ quae ergo aequatio erit integrale idque completum aequationis differentialis $Pdx + Qdy = 0$. Totum ergo negotium ad inuentionem illius multiplicatoris V redit.

CAPVT II.

CAPVT II.

DE INTEGRATIONE AEQVATIONVM OPE MVLTIPLICATORVM.

Problema 58.

443.

Propositam aequationem differentialem examinare vtrum per se sit integrabilis nec ne?

Solutio.

Dispositis omnibus aequationis terminis ad eandem partem signi aequalitatis, vt huiusmodi habeatur forma $Pdx + Qdy = 0$, aequatio per se erit integrabilis, si formula $Pdx + Qdy$ fuerit verum differentiale functionis cuiuspiam binarum variabilium x et y. Hoc autem euenit, vti in calculo differentiali ostendimus, si differentiale ipsius P sumta sola y variabili ad dy eandem habeat rationem, ac differentiale ipsius Q, sumta sola x variabili ad dx, seu adhibito signandi modo, quo in Calculo differentiali sumus vsi, si fuerit $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$. Nam si Z sit ea functio, cuius differentiale est $Pdx + Qdy$, erit hoc signandi modo $P = \left(\frac{dZ}{dx}\right)$ et $Q = \left(\frac{dZ}{dy}\right)$: hinc

 ergo

ergo sequitur $\left(\frac{dP}{dy}\right)=\left(\frac{ddZ}{dx\,dy}\right)$ et $\left(\frac{dQ}{dx}\right)=\left(\frac{ddZ}{dy\,dx}\right)$. At est $\left(\frac{ddZ}{dx\,dy}\right)=\left(\frac{ddZ}{dy\,dx}\right)$, vnde colligitur $\left(\frac{dP}{dy}\right)=\left(\frac{dQ}{dx}\right)$. Quare proposita aequatione differentiali $Pdx+Qdy=0$, vtrum ea per se sit integrabilis nec ne? hoc modo dignoscetur: Quaerantur per differentiationem valores $\left(\frac{dP}{dy}\right)$ et $\left(\frac{dQ}{dx}\right)$ qui si fuerint inter se aequales, aequatio per se erit integrabilis; sin autem hi valores sint inaequales, aequatio non erit per se integrabilis.

Coroll. 1.

444. Omnes ergo aequationes differentiales, in quibus variabiles sunt a se inuicem separatae, per se sunt integrabiles: habebunt enim huiusmodi formam $Xdx+Ydy=0$, vt X sit functio solius x et Y solius y, eritque propterea $\left(\frac{dX}{dy}\right)=0$ et $\left(\frac{dY}{dx}\right)=0$.

Coroll. 2.

445. Viciſſim igitur si proposita aequatione differentiali $Pdx+Qdy=0$, fuerit $\left(\frac{dP}{dy}\right)=0$ et $\left(\frac{dQ}{dx}\right)=0$, variabiles in ea erunt separatae; littera enim P erit functio tantum ipsius x et Q tantum ipsius y. Vnde aequationes separatae quasi primum genus aequationum per se integrabilium constituunt.

Coroll. 3.

446. Euidens autem est, fieri posse, vt sit $\left(\frac{dP}{dy}\right)=\left(\frac{dQ}{dx}\right)$, etiamsi neuter horum valorum sit nihilo

hilo aequalis. Dantur ergo aequationes per se integrabiles, licet variabiles in iis non sint separatae.

Scholion.

447. Criterium hoc, quo aequationes per se integrabiles agnoscimus, maximi est momenti in hac, quam tradere suscipimus, methodo integrandi. Quodsi enim aequatio deprehendatur per se integrabilis, eius integrale per praecepta iam exposita inveniri potest; sin autem aequatio non fuerit per se integrabilis, semper dabitur quantitas, per quam si ea multiplicetur, fiat per se integrabilis; vnde totum negotium eo reuocabitur, vt proposita aequatione quacunque per se non integrabili inueniatur multiplicator idoneus, qui eam reddat per se integrabilem; qui si semper inueniri posset, nihil amplius in hac methodo integrandi esset desiderandum. Verum haec inuestigatio rarissime succedit, ac vix adhuc latius patet, quam ad eas aequationes, quas ope separationis variabilium iam tractare docuimus; interim tamen non dubito hanc methodum praecedenti longe praeferre, cum ad naturam aequationum magis videatur accommodata, atque etiam ad aequationes differentiales altiorum graduum pateat, in quibus separatio variabilium nullius est vsus.

Problema 59.

448. Aequationis differentialis, quam per se integrabilem esse constat, integrale inuenire.

Solutio.

Sit aequatio differentialis $Pdx + Qdy = o$, in qua cum sit $(\frac{dP}{dy}) = (\frac{dQ}{dx})$, erit $Pdx + Qdy$ differentiale cuiuspiam functionis binarum variabilium x et y, quae sit Z, vt sit $dZ = Pdx + Qdy$. Cum ergo habeamus hanc aequationem $dZ = o$, erit integrale quaesitum $Z = C$. Totum negotium ergo huc redit, vt ista functio Z eruatur, quod cum sciamus esse $dZ = Pdx + Qdy$ haud difficulter praestabitur. Nam quia sumta tantum x variabili, et altera y vt constante spectata, est $dZ = Pdx$, habemus hic formulam differentialem simplicem vnicam variabilem x inuoluentem, quae per praecepta superioris sectionis integrata dabit $Z = \int Pdx +$ Const vbi autem notandum est, in hac constante quantitatem hic pro constanti habitam y vtcunque inesse posse, vnde eius loco scribatur Y vt sit $Z = \int Pdx + Y$; Deinde simili modo x pro constante habeatur, spectata sola y vt variabili, et cum sit $dZ = Qdy$, erit quoque $Z = \int Qdy +$ Const. quae constans autem quantitatem x inuoluet, ita vt sit functio ipsius x, qua posita X erit $Z = \int Qdy + X$. Quanquam autem neque hic functio X neque ibi functio Y determinatur,

natur, tamen quia esse debet $\int Pdx + Y = \int Qdy + X$, hinc vtraque determinabitur. Cum enim sit

$$\int Pdx - \int Qdy = X - Y$$

haec quantitas $\int Pdx - \int Qdy$ semper in eiusmodi binas partes distinguetur, quarum altera est functio ipsius x tantum, et altera ipsius y tantum, vnde valores X et Y sponte cognoscuntur.

Coroll. 1.

449. Cum sit $Q = (\frac{dz}{dy})$, duplici integratione ne opus quidem est. Inuento enim integrali $\int Pdx$, id iterum differentietur, sumta sola y variabili, prodeatque Vdy vnde necesse est fiat $Vdy + dY = Qdy$, ideoque $dY = Qdy - Vdy = (Q - V)dy$.

Coroll. 2.

450. Aequationum ergo per se integrabilium $Pdx + Qdy = 0$ integratio ita perficietur. Quaeratur integrale $\int Pdx$ spectata y constante, idque rursus differentietur spectata sola y variabili, vnde prodeat Vdy: tum $Q - V$ erit functio ipsius y tantum; vnde quaeratur $Y = \int (Q - V)dy$, eritque aequatio integralis $\int Pdx + Y = Const.$

Coroll. 3.

451. Vel quaeratur $\int Qdy$ spectata x constante, quod integrale rursus differentietur sumta x variabili,

riabili, y autem constante, vnde prodeat Udx, tum certe erit $P-U$ functio ipsius x tantum, vnde quaeratur $X = \int(P-U)dx$, eritque aequatio integralis quaesita $\int Qdy + X = Const.$

Coroll. 4.

452. Ex rei natura patet, perinde esse vtra via procedatur, necesse enim est ad eandem aequationem integralem perueniri, si quidem aequatio differentialis proposita per se fuerit integrabilis. Tum autem certe eueniet, vt priori casu $Q-V$ sit functio solius y, posteriori autem $P-U$ functio solius x.

Scholion.

453. Haec methodus integrandi etiam tentari posset, antequam exploratum esset, num aequatio integrabilis existat; si enim vel in modo Coroll. 2. eueniret, vt $Q-V$ esset functio ipsius y tantum, vel in modo Coroll 3. vt $P-U$ esset functio ipsius x tantum, hoc ipsum indicio foret, aequationem esse per se integrabilem. Verum tamen praestat ante omnia scrutari, an aequatio integrabilis sit per se nec ne; seu an sit $(\frac{dP}{dy}) = (\frac{dQ}{dx})$? quoniam hoc examen sola differentiatione absoluitur. Exempla igitur aliquot aequationum per se integrabilium afferamus, quo non solum methodus integrandi, sed etiam insignes illae proprietates, quas commemorauimus, clarius intelligantur.

Exem-

Exemplum 1.

454. *Aequationem per se integrabilem*

$$dx(\alpha x+\beta y+\gamma)+dy(\beta x+\delta y+\varepsilon)=0$$

integrare.

Cum hic sit $P=\alpha x+\beta y+\gamma$ et $Q=\beta x+\delta y+\varepsilon$ erit $(\frac{dP}{dy})=\beta$ et $(\frac{dQ}{dx})=\beta$, qua aequalitate integrabilitas per se confirmatur. Quaeratur ergo per Coroll. 2, spectata y vt constante, $\int Pdx=\frac{1}{2}\alpha xx+\beta yx+\gamma x$, erit $Vdy=\beta xdy$, et $(Q-V)dy=dy(\delta y+\varepsilon)=dY$, ideoque $Y=\frac{1}{2}\delta yy+\varepsilon y$, vnde integrale erit $\frac{1}{2}\alpha xx+\beta yx+\gamma x+\frac{1}{2}\delta yy+\varepsilon y=C$.

Modo autem Coroll. 3. spectata x constante, erit $\int Qdy=\beta xy+\frac{1}{2}\delta yy+\varepsilon y$, quae, spectata y constante, praebet $Udx=\beta ydx$, hincque $(P-U)dx=(\alpha x+\gamma)dx$ et $X=\frac{1}{2}\alpha xx+\gamma x$, vnde $\int Qdy+X=C$ integrale dat vt ante. Hinc simul etiam intelligitur esse $\int Pdx-\int Qdy=\frac{1}{2}\alpha xx+\gamma x-\frac{1}{2}\delta yy-\varepsilon y$, quae in duas functiones $X-Y$ sponte dispescitur.

Exemplum 2.

455. *Aequationem per se integrabilem*

$$\frac{dy}{y}=\frac{xdy-ydx}{y\sqrt{(xx+yy)}} \text{ seu } \frac{dx}{\sqrt{(xx+yy)}}+\frac{dy}{y}\left(1-\frac{x}{\sqrt{(xx+yy)}}\right)=0$$

integrare.

Cum hic sit $P=\frac{1}{\sqrt{(xx+yy)}}$ et $Q=\frac{1}{y}-\frac{x}{y\sqrt{(xx+yy)}}$ pro charactere integrabilitatis per se cognoscendo est $(\frac{dP}{dy})=\frac{-y}{(xx+yy)^{\frac{3}{2}}}$ et $(\frac{dQ}{dx})=\frac{-y}{(xx+yy)^{\frac{3}{2}}}$, qui bini

valores vtique sunt aequales. Iam pro integrali inveniendo vtamur regula Coroll. 2. et habebimus

$$\int P dx = l(x+\sqrt{(xx+yy)}) \text{ et } V dy = \frac{y dy}{(x+\sqrt{(xx+yy)})\sqrt{(xx+yy)}}$$

seu supra et infra per $\sqrt{(xx+yy)}-x$ multiplicando

$$V = \frac{\sqrt{(xx+yy)}-x}{y\sqrt{(xx+yy)}} = \frac{1}{y} - \frac{x}{y\sqrt{(xx+yy)}}$$

vnde $Q-V=0$, et $Y=\int(Q-V)dy=0$,

sicque integrale quaesitum $l(x+\sqrt{(xx+yy)})=\text{Const.}$ Per regulam Coroll. 3. habemus

$$\int Q dy = ly - x\int\frac{dy}{y\sqrt{(xx+yy)}}$$

at posito $y=\frac{1}{z}$, est

$$\int\frac{dy}{y\sqrt{(xx+yy)}} = -\int\frac{dz}{\sqrt{(xxzz+1)}} = -\frac{1}{x}l(xz+\sqrt{(xxzz+1)})$$

ergo

$$\int Q dy = ly + l\frac{x+\sqrt{(xx+yy)}}{y} = l(x+\sqrt{(xx+yy)})$$

vnde $U dx = \frac{dx}{\sqrt{(xx+yy)}}$; hinc $(P-U)dx=0$.

Exemplum 3.

456. *Aequationem per se integrabilem*

$$(xx+yy-aa)dy+(aa+2xy+xx)dx=0$$

integrare.

Hic ergo est

$$P=aa+2xy+xx, \text{ et } Q=xx+yy-aa$$

vnde $\left(\frac{dP}{dy}\right)=2x$ et $\left(\frac{dQ}{dx}\right)=2x$

quae

quae aequalitas integrabilitatem per se innuit. Tum vero est

$$\int P dx = aax + xxy + \tfrac{1}{3}x^3 \text{ et } V dy = xx dy$$

vnde $(Q-V)dy = (yy-aa)dy$ et $Y = \tfrac{1}{3}y^3 - aay$.

Ergo integrale

$$aax + xxy + \tfrac{1}{3}x^3 + \tfrac{1}{3}y^3 - aay = \text{Const.}$$

Altero modo est

$$\int Q dy = xxy + \tfrac{1}{3}y^3 - aay,$$

hincque $U dx = 2xy\,dx$,

ergo $(P-U)dx = (aa+xx)dx$ et $X = aax + \tfrac{1}{3}x^3$

vnde integrale oritur vt ante.

Scholion.

457. In his exemplis licuit, integrale $\int P dx$ actu exhibere, indeque eius differentiale $V dy$ sumta sola y variabili assignare. Quodsi autem hoc integrale $\int P dx$ euolui nequeat, haud liquet quomodo inde differentiale $V dy$ elici possit, quandoquidem formula $\int P dx$ in se spectata constantem quamcunque, quae etiam y in se implicet, complectitur. Tum igitur quomodo procedendum sit, videamus. Ponamus $Z = \int P dx + Y$, et cum quaeratur $\left(\frac{d.\int P dx}{dy}\right) = V$, ob $\int P dx = Z - Y$ erit $V = \left(\frac{dZ}{dy}\right) - \frac{dY}{dy}$. At est $\left(\frac{dZ}{dx}\right) = P$, ergo $\left(\frac{ddZ}{dx\,dy}\right) = \left(\frac{dP}{dy}\right) = \left(\frac{dV}{dx}\right)$, ob $\left(\frac{dZ}{dy}\right) = V + \frac{dY}{dy}$. Hinc erit $V = \int dx \left(\frac{dP}{dy}\right)$, quare quantitas V inuenitur per inte-

integrationem huius formulae $\int dx(\frac{dP}{dy})$, in qua y vt conſtans ſpectatur, poſt quam in valore $(\frac{dP}{dy})$ inueniendo ſola y variabilis eſſet aſſumta. Verum cum hic denuo conſtans cum y implicetur, hinc illa functio Y quam quaerimus non determinatur. Ratio huius incommodi manifeſto in ambiguitate integralium $\int Pdx$ et $\int dx(\frac{dP}{dy})$ eſt ſita, cum vtraque functiones arbitrarias ipſius y recipit. Remedium ergo afferetur, ſi vtrumque integrale certa quadam conditione determinetur. Ita quando integrale $\int Pdx$ ita accipi ponimus, vt euaneſcat poſito $x=f$, vbi quidem conſtantem f pro lubitu accipere licet, tum eadem lege alterum integrale $\int dx(\frac{dP}{dy})$ capiatur. Quo facto erit $Q-\int dx(\frac{dP}{dy})$ functio ipſius y tantum et aequationis $Pdx+Qdy=0$ integrale erit $\int Pdx+\int dy(Q-\int dx(\frac{dP}{dy}))=$ Conſt. dummodo ambo integralia $\int Pdx$ et $\int dx(\frac{dP}{dy})$, in quibus y vt conſtans tractatur, ita determinentur, vt euaneſcant, dum in vtraque ipſi x idem valor f tribuitur. Quare hinc iſtam colligimus regulam

Regula pro integratione aequationis per ſe integrabilis

$Pdx+Qdy=0$, in qua $(\frac{dP}{dy})=(\frac{dQ}{dx})$.

458. Quaerantur integralia $\int Pdx$ et $\int dx(\frac{dP}{dy})$, ſpectando y vt conſtantem ita, vt ambo euaneſcant, dum ipſi x certus quidam valor, puta $x=f$, tribuitur. Tum erit $Q-\int dx(\frac{dP}{dy})$ functio ipſius y tantum

ſum, quae ſit $=Y$ et integrale quaeſitum erit $\int Pdx+\int Ydy=$Conſt. Vel quod eodem redit, quaerantur integralia $\int Qdy$ et $\int dy(\frac{dQ}{dx})$ ſpectando x vt conſtantem, ita vt ambo euaneſcant, dum ipſi y certus quidem valor, puta $y=g$, tribuitur: tum $P=\int dy(\frac{dQ}{dx})$ erit functio ipſius x tantum, qua poſita $=X$ erit integrale quaeſitum $\int Qdy+\int Xdx=$Conſt.

Demonſtratio.

Veritatem huius regulae ex praecedentibus perſpicere licet, ſi cui forte precario aſſumſiſſe videamur, ambas formulas $\int Pdx$ et $\int dx(\frac{dP}{dy})$ eadem lege determinari debere, vt dum ipſi x certus quidam valor puta $x=f$ tribuitur, ambae euaneſcant. Sed ne forte quis putet, alteram integrationem pari iure ſecundum aliam legem determinari poſſe, hanc demonſtrationem addo. Prima quidem integratio ab arbitrio noſtro pendet, quam ergo ita determinari aſſumamus, vt integrale $\int Pdx$ euaneſcat poſito $x=f$, quo facto dico alterum integrale $\int dx(\frac{dP}{dy})$ neceſſario per eandem conditionem determinari oportere. Sit enim $\int Pdx=Z$, eritque Z eiusmodi functio ipſarum x et y, quae euaneſcit poſito $x=f$, habebit ergo factorem $f-x$, vel eius quampiam poteſtatem poſitiuam $(f-x)^\lambda$ ita vt ſit $Z=(f-x)^\lambda T$. Nunc quia $\int dx(\frac{dP}{dy})$ exprimit valorem ipſius $(\frac{dZ}{dy})$ erit $\int dx(\frac{dP}{dy})=(f-x)^\lambda(\frac{dT}{dy})$, ex quo manifeſtum eſt hoc integrale etiam euaneſcere poſito $x=f$, ita vt

huius integralis determinatio non amplius arbitrio nostro relinquatur. Hoc posito erit vtique aequationis per se integrabilis $P dx + Q dy = 0$ integrale $\int P dx + \int Y dy = \text{Const.}$ existente $Y = Q - \int dx \left(\frac{dP}{dy}\right)$; nam posito $\int P dx = Z$, quatenus scilicet in hac integratione y pro constante habetur, vt habeatur haec aequatio $Z + \int Y dy = \text{Const.}$ quam esse integrale quaesitum vel ex ipsa differentiatione patebit. Cum enim sit $dZ = P dx + dy \left(\frac{dZ}{dy}\right) = P dx + dy \int dx \left(\frac{dP}{dy}\right)$, erit aequationis inuentae differentiale:

$$P dx + dy \int dx \left(\frac{dP}{dy}\right) + Y dy = 0$$

sed $Y = Q - \int dx \left(\frac{dP}{dy}\right)$, vnde prodit $P dx + Q dy = 0$ quae est ipsa aequatio differentialis proposita. Quod autem sit $Q - \int dx \left(\frac{dP}{dy}\right)$ functio ipsius y tantum, inde sequitur, quoniam aequatio differentialis per se est integrabilis.

Theorema.

459. Pro omni aequatione, quae per se non est integrabilis semper datur quantitas, per quam ea multiplicata redditur integrabilis.

Demonstratio.

Sit $P dx + Q dy = 0$ aequatio differentialis, et concipiamus eius integrale completum, quod erit aequatio quaedam inter x et y, in quam constans quantitas arbitraria ingrediatur. Ex hac aequatione eruatur haec ipsa constans arbitraria, vt prodeat

huius-

huiusmodi aequatio Conſt. = *functioni cuidam ipſarum* x *et* y, quae differentiata praebeat $0 = Mdx + Ndy$ quae aequatio iam a conſtante illa arbitraria per integrationem ingreſſa eſt libera, ideoque neceſſe eſt vt haec aequatio differentialis conueniat cum propoſita, alioquin integrale ſuppoſitum non eſſet verum. Oportet ergo, vt relatio inter dx et dy vtrinque prodeat eadem, vnde erit $\frac{P}{Q} = \frac{M}{N}$ ideoque $M = LP$ et $N = LQ$. Sed quia $Mdx + Ndy$ eſt verum differentiale ex differentiatione cuiuspiam functionis ipſarum x et y ortum, eſt $\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right)$. Quare pro aequatione $Pdx + Qdy = 0$ dabitur certo quidam multiplicator L, vt ſit $\left(\frac{d.LP}{dy}\right) = \left(\frac{d.LQ}{dx}\right)$, ſeu vt aequatio per L multiplicata fiat per ſe integrabilis.

Coroll. 1.

460. Pro omni ergo aequatione $Pdx + Qdy = 0$ datur eiusmodi functio L, vt ſit $\left(\frac{d.LP}{dy}\right) = \left(\frac{d.LQ}{dx}\right)$, ideoque euoluendo:

$$L\left(\frac{dP}{dy}\right) + P\left(\frac{dL}{dy}\right) = L\left(\frac{dQ}{dx}\right) + Q\left(\frac{dL}{dx}\right) \text{ ſeu}$$

$$L\left(\left(\frac{dP}{dy}\right) - \left(\frac{dQ}{dx}\right)\right) = Q\left(\frac{dL}{dx}\right) - P\left(\frac{dL}{dy}\right)$$

quae functio L ſi fuerit inuenta, aequatio differentialis $LPdx + LQdy = 0$ per ſe erit integrabilis.

Coroll. 2.

Coroll. 2.

461. In aequatione propofita loco Q tuto vnitatem fcribere licet, quia omnis aequatio hac forma $Pdx+dy=0$ repraefentari poteft. Hinc inuentio multiplicatoris L, qui eam reddat per fe integrabilem, pendet a refolutione huius aequationis:

$$L\left(\frac{dP}{dy}\right)=\left(\frac{dL}{dx}\right)-P\left(\frac{dL}{dy}\right)$$

vbi notandum eft effe

$$dL=dx\left(\frac{dL}{dx}\right)+dy\left(\frac{dL}{dy}\right).$$

Scholion.

462. Quoniam hic quaeritur functio binarum variabilium x et y, quarum relatio mutua minime fpectatur, quam inuoluit aequatio $Pdx+Qdy=0$, haec inueftigatio in noftrum librum fecundum incurrit, vbi huiusmodi functio ex data quadam differentialium relatione indagare debet. In hac enim inueftigatione non attendimus ad aequationem propofitam, qua formula $Pdx+Qdy$ nihilo aequalis reddi debet, fed abfolute quaeritur multiplicator L, per quam formula $Pdx+Qdy$ multiplicata abeat in verum differentiale cuiuspiam functionis finitae, quae fit Z, ita vt habeatur $dZ=LPdx+LQdy$. Quo multiplicatore L inuento tum demum aequalitas $Pdx+Qdy=0$ fpectatur, indeque concluditur functionem Z quantitati conftanti aequari oportere. Cum igitur minime expectari queat, vt methodum tradamus

damus huiusmodi multiplicatores pro quauis aequatione differentiali proposita inueniendi, eos casus percurramus, quibus talis multiplicator constat, vndecunque sit repertus. Interim tamen ad pleniorem vsum huius methodi notasse iuuabit, statim atque vnum multiplicatorem pro quapiam [illegible]ione differentiali cognouerimus [illegible] facile innumerabiles [illegible] posse, qui pariter aequationem propositam per se integrabilem reddant.

Problema 60.

463. Dato vno multiplicatore L qui aequationem $Pdx+Qdy=0$ per se integrabilem reddat, inuenire innumerabiles alios multiplicatores, qui idem officium praestent.

Solutio.

Cum ergo $L(Pdx+Qdy)$ sit differentiale verum cuiuspiam functionis Z, quaeratur per superiora praecepta haec functio Z, ita vt sit $L(Pdx+Qdy)=dZ$: et nunc manifestum est hanc formulam dZ integrationem etiam esse admissuram, si per functionem quamcunque ipsius Z, quam ita $\Phi:Z$ indicemus multiplicetur. Cum igitur etiam integrabilis sit haec formula $(Pdx+Qdy)L\Phi:Z$, erit quoque $L\Phi:Z$ multiplicator aequationis propositae $Pdx+Qdy=0$, qui eam reddat integrabilem. Quare inuento vno

 multi-

multiplicatore L, quaeratur per integrationem $Z = \int L(Pdx + Qdy)$, ac tum expressio $L\Phi:Z$ vbi pro $\Phi:Z$ functio quaecunque ipsius Z assumi potest, dabit infinitos alios multiplicatores idem officium praestantes.

Scholion.

464. Tametsi sufficiat pro quauis aequatione differentiali vnicum multiplicatorem cognouisse, tamen occurrunt casus, quibus perquam vtile est plures imo infinitos multiplicatores in promtu habere. Veluti si aequatio proposita in duas partes commode discerpatur, huiusmodi $(Pdx + Qdy) + (Rdx + Sdy) = 0$ atque omnes multiplicatores constent, quibus vtraque pars seorsim $Pdx + Qdy$ et $Rdx + Sdy$ reddatur integrabilis, inde interdum communis multiplicator vtramque integrabilem reddens concludi potest. Sit enim $L\Phi:Z$ expressio generalis pro omnibus multiplicatoribus formulae $Pdx + Qdy$ et $M\Phi:V$ expressio generalis pro omnibus multiplicatoribus formulae $Rdx + Sdy$, et quoniam $\Phi:Z$ et $\Phi:V$ functiones quascunque quantitatum Z et V denotant, si eas ita capere liceat, vt fiat $L\Phi:Z = M\Phi:V$ habebitur multiplicator idoneus pro aequatione $Pdx + Qdy + Rdx + Sdy = 0$. Intelligitur autem hoc iis tantum casibus praestari posse, quibus multiplicator pro tota aequatione, etiam singulas eius partes seorsim sumtas integrabiles reddat. Quare cauendum est, ne huic methodo nimium tribuatur,

et

et quando ea non fuccedit, aequatio pro irrefolubili habeatur, euenire enim vtique poteft, vt tota aequatio habeat multiplicatorem, qui fingulis eius partibus non conueniat. Ita propofita aequatione $Pdx+Qdy=0$, multiplicator partem Pdx feorfim integrabilem reddens manifefto eft $\frac{X}{P}$, denotante X functionem quamcunque ipfius x, et multiplicator partem alteram Qdy integrabilem reddens eft $\frac{Y}{Q}$: etiamfi autem neutiquam fieri poffit, vt fit $\frac{X}{P}=\frac{Y}{Q}$ feu $\frac{P}{Q}=\frac{X}{Y}$, nifi cafibus per fe obuiis, tamen tota formula $Pdx+Qdy$ certo femper habet multiplicatorem, quo ea integrabilis reddatur.

Exemplum I.

465. *Inuenire omnes multiplicatores, quibus formula* $\alpha ydx+\beta xdy$ *integrabilis redditur.*

Primus multiplicator fponte fe offert $\frac{1}{xy}$, qui praebet: $\frac{\alpha dx}{x}+\frac{\beta dy}{y}$, cuius integrale eft $\alpha lx+\beta ly=lx^{\alpha}y^{\beta}$. Huius ergo functio quaecunque $\Phi:x^{\alpha}y^{\beta}$ in $\frac{1}{xy}$ ducta dabit multiplicatorem idoneum, cuius itaque forma generalis eft $\frac{1}{xy}\Phi:x^{\alpha}y^{\beta}$. Functio enim quantitatis $x^{\alpha}y^{\beta}$ etiam eft functio logarithmi eiusdem quantitatis. Nam fi P fuerit functio ipfius p, et Π functio ipfius P, etiam Π eft functio ipfius p et viciffim.

 Coroll. 1.

Corollarium.

466. Si pro functione sumatur potestas quaecunque $x^{n\alpha}y^{n\beta}$, formula $\alpha y dx + \beta x dy$ integrabilis redditur, si multiplicetur per $x^{n\alpha-1}y^{n\beta-1}$, quo quidem casu integrale sponte patet, est enim $\frac{1}{n}x^{n\alpha}y^{n\beta}$.

Exemplum 2.

467. *Inuenire omnes multiplicatores, qui hanc formulam* $X y dx + dy$ *integrabilem reddant.*

Primus multiplicator $\frac{1}{y}$ sponte se offert, vnde cum sit $\int (X dx + \frac{dy}{y}) = \int X dx + ly$ seu $l e^{\int X dx} y$, omnes functiones huius quantitatis, seu huius $e^{\int X dx} y$ per y diuisae dabunt multiplicatores idoneos. Vnde expressio generalis pro omnibus multiplicatoribus erit $= \frac{1}{y} \Phi : e^{\int X dx} y$.

Corollarium.

468. Pro formula ergo $X y dx + dy$ multiplicator quoque est $e^{\int X dx}$ qui est functio ipsius x tantum; quo ergo cum etiam formula $\mathfrak{X} dx$ denotante $\mathfrak{X}$ functionem quamcunque ipsius x, integrabilis reddatur, ille multiplicator etiam huic formulae $dy + X y dx + \mathfrak{X} dx$ conueniet.

Problema 61.

469. Proposita aequatione $dy + X y dx = \mathfrak{X} dx$, in qua X et $\mathfrak{X}$ sint functiones quaecunque ipsius x,

inue-

inuenire multiplicatorem idoneum, eamque integrare.

Solutio.

Cum alterum membrum $\mathfrak{X}dx$ per functionem quamcunque ipsius x multiplicatum fiat integrabile, dispiciatur num etiam prius membrum $dy+Xydx$ per huiusmodi multiplicatorem integrabile reddi possit. Quod cum praestet multiplicator $e^{\int Xdx}$, hoc adhibito habebitur aequatio integralis quaesita:

$$e^{\int Xdx}y=\int e^{\int Xdx}\mathfrak{X}dx\text{, siue}$$
$$y=e^{-\int Xdx}\int e^{\int Xdx}\mathfrak{X}dx$$

vti iam supra inuenimus.

Coroll. 1.

470. Patet etiam si loco y adsit functio quaecunque ipsius y, vt habeatur haec aequatio $dY+YXdx=\mathfrak{X}dx$, eam per multiplicatorem $e^{\int Xdx}$ reddi integrabilem, et integrale fore:

$$e^{\int Xdx}Y=\int e^{\int Xdx}\mathfrak{X}dx.$$

Coroll. 2.

471. Quare etiam haec aequatio $dy+yXdx=y^n\mathfrak{X}dx$ quia per y^n diuisa abit in $\frac{dy}{y^n}+\frac{Xdx}{y^{n-1}}=\mathfrak{X}dx$, vbi posito $\frac{1}{y^{n-1}}=Y$, ob $-\frac{(n-1)dy}{y^n}=dY$ seu $\frac{dy}{y^n}=-\frac{dY}{n-1}$ prodit $-\frac{dY}{n-1}+YXdx=\mathfrak{X}dx$ seu $dY-(n-1)YXdx=-(n-1)\mathfrak{X}dx$, qui

qui per multiplicatorem $e^{-(n-1)\int X dx}$ fit integrabilis: eiusque integrale erit

$$e^{-(n-1)\int X dx} Y = -(n-1)\int e^{-(n-1)\int X dx}\mathfrak{X}\, dx \text{ siue}$$

$$\frac{1}{y^{n-1}} = -(n-1) e^{(n-1)\int X dx}\int e^{-(n-1)\int X dx}\mathfrak{X}\, dx.$$

Scholion.

472. Cum pro membro $dy + yX dx$ multiplicator generalis sit $\frac{1}{y}\Phi : e^{\int X dx} y$, sumta loco functionis potestate, multiplicator idoneus erit $e^{m\int X dx} y^{m-1}$, integrale praebens $\frac{1}{m} e^{m\int X dx} y^m$. Efficiendum ergo est, vt etiam idem multiplicator alterum membrum $y^n\mathfrak{X}dx$ reddat integrabile; quod euenit sumendo $m-1=-n$ seu $m=1-n$, ex quo huius membri integrale fit $\int e^{m\int X dx}\mathfrak{X}\, dx$, ita vt aequatio integralis quaesita obtineatur:

$$\frac{1}{1-n} e^{(1-n)\int X dx} y^{1-n} = \int e^{(1-n)\int X dx}\mathfrak{X}\, dx$$

quae cum modo inuenta prorsus congruit.

Problema 62.

473. Proposita aequatione differentiali:

$$\alpha y dx + \beta x dy = x^m y^n(\gamma y dx + \delta x dy)$$

inuenire multiplicatorem idoneum, qui eam integrabilem reddat, ipsumque integrale assignare.

Solutio.

Solutio.

Confideretur vtrumque membrum feorfim; ac pro priori vidimus $\alpha y dx + \beta x dy$ omnes multiplicatores idoneos contineri in hac forma $\frac{1}{xy}\Phi : x^\alpha y^\beta$. Pro altera parte $x^m y^n(\gamma y dx + \delta x dy)$ primus multiplicator eft $\frac{1}{x^{m+1}y^{n+1}}$, quo prodit $\frac{\gamma dx}{x} + \frac{\delta dy}{y}$, cuius multiplicatoribus eft $\frac{1}{x^{m+1}y^{n+1}}\Phi : x^\gamma y^\delta$. Quo nunc hi duo multiplicatores pares reddantur, loco functionum fumantur poteftates, fiatque:

$$x^{\mu\alpha-1}y^{\mu\beta-1} = x^{\nu\gamma-m-1}y^{\nu\delta-n-1}$$

vnde ftatui oportet $\mu\alpha = \nu\gamma - m$ et $\mu\beta = \nu\delta - n$; hincque colligitur:

$$\mu = \frac{\gamma n - \delta m}{\alpha\delta - \beta\gamma} \text{ et } \nu = \frac{\alpha n - \beta m}{\alpha\delta - \beta\gamma}$$

Quocirca multiplicator erit

$$x^{\mu\alpha-1}y^{\mu\beta-1} = x^{\nu\gamma-m-1}y^{\nu\delta-n-1}$$

vnde aequatio noftra induit hanc formam:

$$x^{\mu\alpha-1}y^{\mu\beta-1}(\alpha y dx + \beta x dy) = x^{\nu\gamma-1}y^{\nu\delta-1}(\gamma y dx + \delta x dy)$$

vbi vtrumque membrum per fe eft integrabile, ideoque integrale quaefitum:

$$\frac{1}{\mu}x^{\mu\alpha}y^{\mu\beta} = \frac{1}{\nu}x^{\nu\gamma}y^{\nu\delta} + \text{Conft.}$$

quod

quod conuenit cum eo, quod capite praecedente est inuentum.

Coroll. 1.

474. Posito ergo breuitatis gratia

$$\mu = \frac{\gamma n - \delta m}{\alpha\delta - \beta\gamma} \text{ et } \nu = \frac{\alpha n - \beta m}{\alpha\delta - \beta\gamma}$$

aequationis differentialis:

$$\alpha y dx + \beta x dy = x^m y^n (\gamma y dx + \delta x dy)$$

integrale completum est:

$$\tfrac{1}{\mu} x^{\mu\alpha} y^{\mu\beta} = \tfrac{1}{\nu} x^{\nu\gamma} y^{\nu\delta} + \text{Const.}$$

Coroll. 2.

475. Si eueniat, vt sit $\mu = 0$, seu $\gamma n = \delta m$ integrale ad logarithmos reducetur eritque

$$l x^\alpha y^\beta = \tfrac{1}{\nu} x^{\nu\gamma} y^{\nu\delta} + \text{Const.}$$

sin autem sit $\nu = 0$ seu $\alpha n = \beta m$, erit integrale

$$\tfrac{1}{\mu} x^{\mu\alpha} y^{\mu\beta} = l x^\gamma y^\delta + \text{Const.}$$

Scholion.

476. Hinc autem casus excipi videtur, quo $\alpha\delta = \beta\gamma$, quia tum ambo numeri μ et ν fiunt infiniti. Verum si $\delta = \frac{\beta\gamma}{\alpha}$ aequatio nostra hanc induit formam $\alpha y dx + \beta x dy = \frac{\gamma}{\alpha} x^m y^n (\alpha y dx + \beta x dy)$ seu $(\alpha y dx + \beta x dy)(1 - \frac{\gamma}{\alpha} x^m y^n) = 0$, quae cum habeat duos factores, duplex solutio ex vtroque seorsim

fim ad nihilum reducto deriuatur. Prior fcilicet nafcitur ex $\alpha y dx + \beta x dy = 0$ cuius integrale eft $x^{\alpha} y^{\beta} =$ Conft. alter vero factor per fe dat aequationem finitam $1 - \frac{\gamma}{\alpha} x^m y^n = 0$, quarum folutionum vtraque aeque fatisfacit. Atque hoc in genere tenendum eft de omnibus aequationibus differentialibus, quas in factores refoluere licet, vbi perinde atque in aequationibus finitis finguli factores praebent folutiones. Plerumque autem factores finiti ftatim, ante quam integratio fufcipitur, per diuifionem tolli folent, quandoquidem non ex natura rei, fed per operationes inftitutas demum acceffiffe cenfentur, ita vt perinde ac in Algebra faepe fieri folet, ad folutiones inutiles effent perducturi.

Problema 63.

477. Propofita aequatione differentiali homogenea, multiplicatorem idoneum inuenire, qui eam integrabilem reddat, indeque eius integrale eruere.

Solutio.

Sit $P dx + Q dy = 0$ aequatio propofita, in qua P et Q fint functiones homogeneae n dimenfionum ipfarum x et y, ac quaeramus multiplicatorem L, qui fit etiam functio homogenea, cuius dimenfionum numerus fit λ. Cum iam formula $L(P dx + Q dy)$ fit integrabilis, erit integrale functio $\lambda + n + 1$ dimenfionum ipfarum x et y, quae

 functio

functio si ponatur Z erit ex natura functionum homogenearum

$$LPx+LQy=(\lambda+n+1)Z.$$

Quare si λ sumatur $=-n-1$, quantitas $LPx+LQy$ erit vel $=0$ vel constans, vnde obtinemus $L=\frac{1}{Px+Qy}$, qui ergo est multiplicator idoneus pro nostra aequatione. Idem quoque ex separatione variabilium colligitur: posito enim $y=ux$, fiet $P=x^nU$ et $Q=x^nV$, existentibus U et V functionibus u ipsius tantum, et ob $dy=udx+xdu$

$$\text{erit } Pdx+Qdy=x^nUdx+x^nVudx+x^nVxdu$$

$$\text{seu } Pdx+Qdy=x^n(U+Vu)dx+x^{n+1}Vdu.$$

At haec formula per $x^{n+1}(U+Vu)$ diuisa fit integrabilis, ideoque et formula nostra $Pdx+Qdy$ diuisa per $x^{n+1}(U+Vu)=Px+Qy$, restitutis valoribus $U=\frac{P}{x^n}$, $V=\frac{Q}{x^n}$ et $u=\frac{y}{x}$, fiet integrabilis; seu multiplicator idoneus est $\frac{1}{Px+Qy}$, vnde haec aequatio $\frac{Pdx+Qdy}{Px+Qy}=0$ semper per se est integrabilis.

Iam ad integrale ipsius inueniendum integretur formula $\int\frac{Pdx}{Px+Qy}$ spectando y vt constantem, ac determinetur certa ratione vt euanescat posito $x=f$. Tum posito breuitatis causa $\frac{P}{Px+Qy}=R$, sumatur valor $(\frac{dR}{dy})$, et eadem lege quaeratur integrale $\int dx(\frac{dR}{dy})$ spectando

ſpectando iterum y vt conſtantem. Tum erit $\frac{Q}{Px+Qy} - \int dx \left(\frac{dR}{dy}\right)$ functio ipſius y tantum ſeu $\frac{Q}{Px+Qy} - \int dx \left(\frac{dR}{dy}\right) = Y$: atque hinc erit integrale quaeſitum $\int \frac{P\,dx}{Px+Qy} + \int Y\,dy = \text{Conſt.}$

Coroll. 1.

478. Cum ergo formula $\frac{Pdx+Qdy}{Px+Qy}$ ſit per ſe integrabilis, ſi breuitatis gratia ponamus $\frac{P}{Px+Qy} = R$ et $\frac{Q}{Px+Qy} = S$, neceſſe eſt ſit $\left(\frac{dR}{dy}\right) = \left(\frac{dS}{dx}\right)$. At eſt $\left(\frac{dR}{dy}\right) = \left(Qy\left(\frac{dP}{dy}\right) - Py\left(\frac{dQ}{dy}\right) - PQ\right) : (Px+Qy)^2$ et $\left(\frac{dS}{dx}\right) = \left(Px\left(\frac{dQ}{dx}\right) - Qx\left(\frac{dP}{dx}\right) - PQ\right) : (Px+Qy)^2$. Quamobrem habebitur $Qy\left(\frac{dP}{dy}\right) - Py\left(\frac{dQ}{dy}\right) = Px\left(\frac{dQ}{dx}\right) - Qx\left(\frac{dP}{dx}\right)$.

Coroll. 2.

479. Haec aequalitas etiam ex natura functionum homogenearum concluditur. Cum enim P et Q ſint functiones n dimenſionum ipſarum x et y, ob $dP = dx\left(\frac{dP}{dx}\right) + dy\left(\frac{dP}{dy}\right)$ et $dQ = dx\left(\frac{dQ}{dx}\right) + dy\left(\frac{dQ}{dy}\right)$ erit $nP = x\left(\frac{dP}{dx}\right) + y\left(\frac{dP}{dy}\right)$ et $nQ = x\left(\frac{dQ}{dx}\right) + y\left(\frac{dQ}{dy}\right)$. Aequalitas autem inuenta eſt $Q\left(x\left(\frac{dP}{dx}\right) + y\left(\frac{dP}{dy}\right)\right) = P\left(x\left(\frac{dQ}{dx}\right) + y\left(\frac{dQ}{dy}\right)\right)$, quae hinc abit in identicam $nPQ = nPQ$.

Coroll. 3.

480 Si aequatio homogenea $Pdx + Qdy = 0$ fuerit per ſe integrabilis, et P et Q ſint functio-

nes -1 dimensionis, erit $Px+Qy$ numerus constans. Veluti cum $\frac{xdx+ydy}{xx+yy}=0$ huiusmodi sit aequatio, si loco dx et dy scribantur x et y, prodit $\frac{xx+yy}{xx+yy}=1$.

Scholion.

481. In calculo differentiali ostendimus, si V fuerit functio homogenea n dimensionum ipsarum x et y, ponaturque $dV=Pdx+Qdy$, fore $Px+Qy=nV$. Quare si $Pdx+Qdy$ fuerit formula integrabilis, et P et Q functiones homogeneae $n-1$ dimensionum, integrale statim habetur, erit enim $V=\frac{1}{n}(Px+Qy)$, neque ad hoc vlla integratione est opus. Interim tamen videmus hinc excipi oportere casum quo $n=0$, vti fit in nostra aequatione per multiplicatorem integrabili reddita $\frac{Pdx+Qdy}{Px+Qy}=0$, vbi dx et dy multiplicantur per functiones -1 dimensionis, neque enim hic integrale sine integratione obtineri potest. Ratio autem huius exceptionis in hoc est sita, quod formulae integrabilis $Pdx+Qdy$, in qua P et Q sunt functiones homogeneae $n-1$ dimensionum, integrale tum tantum sit functio homogenea n dimensionum quando n non est $=0$, hoc enim solo casu fieri potest, vt integrale non sit functio nullius dimensionis, quemadmodum fit in hac formula differentiali $\frac{xdx+ydy}{xx+yy}$, quippe cuius integrale est $\frac{1}{2}l(xx+yy)$. Quocirca, quod formula $\frac{Pdx+Qdy}{Px+Qy}$ sit integrabilis, hoc peculiari modo demonstrauimus, ex ratione separabilitatis

tis deducto. Interim tamen fine vllo refpectu, vnde hoc cognouerimus, id in praefenti negotio maxime eft notatu dignum, omnes aequationes homogeneas $Pdx+Qdy=0$ per multiplicatorem $\frac{1}{Px+Qy}$ per fe reddi integrabiles. Methodus igitur defideratur, cuius beneficio hunc multiplicatorem a priori inuenire liceret; qua methodo fane maxima incrementa in Analyfin importarentur. Quamdiu autem eousque pertingere non licet, plurimum intererit huiusmodi multiplicatores pro pluribus cafibus probe notaffe; quod cum iam in duobus aequationum generibus praeftiterimus, pro reliquis aequationibus, quas fupra integrare docuimus, multiplicatores inueftigemus; ipfa autem reductio ad feparationem nobis hos multiplicatores patefaciet, vti in fequente problemate docebimus.

Problema 64.

482. Propofita aequatione differentiali, quam ad feparationem variabilium reducere liceat, inuenire multiplicatorem, per quem ea per fe integrabilis reddatur.

Solutio.

Sit $Pdx+Qdy=0$, quae certa quadam fubftitutione, dum loco x et y aliae binae variabiles t et u introducuntur, ad feparationem accommodetur, ponamus ergo facta hac fubftitutione fieri $Pdx+Qdy=Rdt+Sdu$, nunc autem hanc for-

 mulam

mulam $R\,dt + S\,du$ si per V diuidatur, separari, ita vt in hac formula $\frac{R\,dt + S\,du}{V}$ quantitas $\frac{R}{V}$ sit functio solius t, et $\frac{S}{V}$ functio solius u. Cum igitur formula $\frac{R\,dt + S\,du}{V}$ per se sit integrabilis, etiam integrabilis erit haec $\frac{P\,dx + Q\,dy}{V}$ quippe illi aequalis, siquidem in V variabiles x et y restituantur. Hinc ergo ex reductione ad separabilitatem aequationis $P\,dx + Q\,dy = 0$ discimus multiplicatorem, quo ea integrabilis reddatur esse $\frac{1}{V}$, sicque quas aequationes ad separationem variabilium perducere licet, pro iisdem multiplicatorem, qui illas integrabiles reddat, assignare possumus.

Coroll. 1.

483. Methodus ergo per multiplicatores integrandi aequationes differentiales aeque late patet ac prior methodus, ope separationis variabilium; propterea quod ipsa separatio pro quauis aequatione, vbi succedit, multiplicatorem suppeditat.

Coroll. 2.

484. Contra autem methodus per multiplicatores integrandi latius patet altera, si pro eiusmodi aequationibus multiplicatores assignare liceat, quas quomodo ad separationem perduci debeant, non constet.

Scholion.

Scholion.

485. Etsi autem ex reductione ad separationem idoneum multiplicatorem elicere licet, tamen nondum intelligitur, quomodo cognito multiplicatore separatio variabilium institui debeat, quare etiam ob hanc rationem methodus per multiplicatores integrandi alteri longe praeferenda videtur. Quamuis enim hactenus ipsa separatio nos ad inuentionem multiplicatorum perduxerit, nullum tamen est dubium quin detur via multiplicatores inueniendi, nullo respectu ad separationem habito, licet haec via etiamnum nobis sit incognita. Ea autem paullatim planior reddetur, si pro quamplurimis aequationibus multiplicatores idoneos cognouerimus, ex quo quos adhuc ex separatione eruere licet, indagemus in subiunctis exemplis.

Exemplum 1.

486. *Proposita aequatione differentiali primi ordinis* $dx(\alpha x+\beta y+\gamma)+dy(\delta x+\epsilon y+\zeta)=0$, *pro ea multiplicatorem idoneum assignare.*

Haec aequatio ad separationem praeparatur ponendo primo:

$$\alpha x+\beta y+\gamma=r \text{ et } \delta x+\epsilon y+\zeta=s$$

ideoque

$$\alpha dx+\beta dy=dr \text{ et } \delta dx+\epsilon dy=ds,$$

vnde oritur

$$dx=\frac{\epsilon dr-\beta ds}{\alpha\epsilon-\beta\delta} \text{ et } dy=\frac{\alpha ds-\delta dr}{\alpha\epsilon-\beta\delta}$$

hinc-

hincque aequatio noſtro omiſſo denominatore vtpote conſtante erit

$$\varepsilon r dr - \beta r ds + \alpha s ds - \delta s dr = 0$$

quae cum ſit homogenea per $\varepsilon rr - (\beta + \delta) rs + \alpha ss$ diuiſa, fit integrabilis. Quod idem ex ſeparatione colligitur poſito enim $r = su$ prodit

$$\varepsilon ssudu + \varepsilon suuds - \beta suds + \alpha sds - \delta ssdu - \delta suds = 0$$

ſeu $ssdu(\varepsilon u - \delta) + sds(\varepsilon uu - \beta u - \delta u + \alpha) = 0$

quae diuiſa per $ss(\varepsilon uu - \beta u - \delta u + \alpha)$ ſeparatur. Quare multiplicator noſtrae aequationis propoſitae eſt

$$\frac{1}{ss(\varepsilon uu - \beta u - \delta u + \alpha)} = \frac{1}{\varepsilon rr - \beta rs - \delta rs + \alpha ss} = \frac{1}{r(\varepsilon r - \beta s) + s(\alpha s - \delta r)}$$

qui reſtitututis valoribus fit

$$\frac{1}{(\alpha x + \beta y + \gamma)((\alpha\varepsilon - \beta\delta)x + \gamma\varepsilon - \beta\zeta) + (\delta x + \varepsilon y + \zeta)((\alpha\varepsilon - \beta\delta)y + \alpha\zeta - \gamma\delta)}$$

atque euolutione facta:

$$\frac{1}{(\alpha\varepsilon - \beta\delta)(\alpha xx + (\beta + \delta)xy + \varepsilon yy + \gamma x + \zeta y) + \alpha\zeta\zeta - (\beta + \delta)\gamma\zeta + \gamma\gamma\varepsilon + (\alpha\gamma\varepsilon - (\beta - \delta)\alpha\zeta - \gamma\delta\delta)x + (\alpha\varepsilon\zeta + (\beta - \delta)\gamma\varepsilon - \beta\beta\zeta)y}$$

Quare per ſe integrabilis erit haec aequatio

$$\frac{dx(\alpha x + \beta y + \gamma) + dy(\delta x + \varepsilon y + \zeta)}{(\alpha\varepsilon - \beta\delta)(\alpha xx + (\beta + \delta)xy + \varepsilon yy + \gamma x + \zeta y) + Ax + By + C} = 0$$

exiſtente

$$A = \alpha\gamma\varepsilon - (\beta - \delta)\alpha\zeta - \gamma\delta\delta$$

$$A = \alpha\varepsilon\zeta + (\beta - \delta)\gamma\varepsilon - \beta\beta\zeta$$

$$C = \alpha\zeta\zeta - (\beta - \delta)\gamma\zeta + \gamma\gamma\varepsilon.$$

Corolla-

Corollarium.

487. Etiamsi forte fiat $\alpha\varepsilon-\beta\delta=0$, hic multiplicator non turbatur, cum tamen separatio non succedat hac quidem operatione. Sit enim $\alpha=ma$, $\beta=mb$, $\delta=na$, $\varepsilon=nb$, vt habeatur haec aequatio

$$dx(m(ax+by)+\gamma)+dy(n(ax+by)+\zeta)=0$$

ob $A=a(na-mb)(m\zeta-n\gamma)$

$B=b(na-mb)(m\zeta-n\gamma)$ et

$C=(m\zeta-n\gamma)(a\zeta-b\gamma)$

omisso factore communi multiplicator est

$$\frac{1}{(na-mb)(ax+by)+a\zeta-b\gamma}$$

ita vt haec aequatio per se fit integrabilis:

$$\frac{(ax+by)(mdx+ndy)+\gamma dx+\zeta dy}{(na-mb)(ax+by)+a\zeta-\beta\gamma}=0.$$

Exemplum 2.

488. *Proposita aequatione differentiali*

$$ydx(c+nx)-dy(y+a+bx+nxx)=0$$

multiplicatorem idoneum inuenire.

Fiat substitutio $\frac{y(c+nx)}{y+a+bx+nxx}=u$, seu $y=\frac{u(a+bx+nxx)}{c+nx-u}$ vt contrahatur aequatio nostra in hanc formam

$$ydx(c+nx)-\frac{ydy(c+nx)}{u}=0$$

seu $\frac{y(c+nx)}{u}(udx-dy)=0$ vel $\frac{yy(c+nx)}{u}\left(\frac{dy}{y}-\frac{udx}{y}\right)=0$ probe enim cauendum est, ne hic vllus factor omittatur. At facta substitutione reperitur $\frac{dy}{y}-\frac{udx}{y}=\frac{du}{u}+\frac{dx(b+2nx)}{a+bx+nxx}+\frac{du-ndx}{c+nx-u}-\frac{dx(c+nx-u)}{a+bx+nxx}=\frac{du(c+nx)}{u(c+nx-u)}-\frac{dx(na+cc-bc+(b-2c)u+uu)}{(c+nx-u)(a+bx+nxx)}$. Vnde aequatio nostra induet hanc formam

$$\frac{yy(c+nx)^2}{u(c+nx-u)}\left(\frac{du}{u}-\frac{dx(na+cc-bc+(b-2c)u+uu)}{(a+bx+nxx)(c+nx)}\right)=0$$

quae ergo separabitur ducta in hunc multiplicatorem

$$\frac{u(c+nx-u)}{yy(c+nx)^2(na+cc-bc+(b-2c)u+uu)}$$

tum enim prodit

$$\frac{du}{u(na+cc-bc+(b-2c)u+uu)}-\frac{dx}{(a+bx+nxx)(c+nx)}=0.$$

Quo igitur multiplicatorem quaesitum consequamur, ibi loco u tantum opus est suum valorem restituere tum autem reperitur multiplicator

$$\frac{a+bx+nxx}{n(a+bx+nxx)y^4+(a+bx+nxx)(2na-bc+n(b-2c)x)yy+(na+cc-bc)(a+bx+nxx)^2}$$

qui reducitur ad hanc formam

$$\frac{1}{ny^4+(2na-bc)yy+n(b-2c)xyy+(na+cc-bc)(a+bx+nxx)}$$

Exemplum 3.

489. *Proposita aequatione differentiali*

$$\frac{ndx(1+yy)\sqrt{(1+yy)}}{\sqrt{(1+xx)}}+(x-y)dy=0$$

inuenire multiplicatorem qui eam integrabilem reddat.

Posui-

Posuimus supra (435.) $y = \frac{x-u}{1+xu}$ seu $u = \frac{x-y}{1+xy}$, vnde fit $x-y = \frac{u(1+xu)}{1+xy}$ et $1+yy = \frac{(1+xx)(1+uu)}{(1+xu)^2}$, hincque nostra aequatio hanc induit formam

$$\frac{ndx(1+xx)(1+uu)^{\frac{3}{2}}}{(1+xu)^3} + \frac{udx(1+xx)(1+uu) - udu(1+xx)^2}{(1+xu)^3} = 0$$

quae primo multiplicata per $(1+xu)^3$ tum diuisa per $(1+xx)^2(1+uu)(u+n\sqrt{(1+uu)})$ separatur. Quare aequationis nostrae multiplicator erit

$$\frac{(1+xu)^3}{(1+xx)^2(1+uu)(u+n\sqrt{(1+uu)})}$$

qui primo ob $1+uu = \frac{(1+yy)(1+xu)^2}{1+xx}$ abit in

$$\frac{1+xu}{(1+xx)(1+yy)(u+n\sqrt{(1+uu)})}.$$

Nunc ob $u = \frac{x-y}{1+xy}$; est $\sqrt{(1+uu)} = \frac{\sqrt{(1+xx)(1+yy)}}{1+xy}$ et $1+xu = \frac{1+xx}{1+xy}$ ideoque noster multiplicator colligitur:

$$\frac{1}{(1+yy)(x-y+n\sqrt{(1+xx)(1+yy)})}$$

ita vt per se sit integrabilis haec aequatio

$$\frac{ndx(1+yy)\sqrt{(1+yy)} + (x-y)dy\sqrt{(1+xx)}}{(1+yy)(x-y+n\sqrt{(1+xx)(1+yy)})\sqrt{(1+xx)}} = 0$$

cuius integrationi non immoror, cum iam supra integrale exhibuerim.

Exemplum 4.

490. *Aliud exemplum memoratu dignum suppeditat haec aequatio:*

$$ydx - xdy + ax^n ydy(x^n+b)^{\frac{1}{n}} = 0$$

 quae

quae si hac forma rapraesentetur:

$$xdy - ydx + \tfrac{1}{b}x^{n+1}dy = \tfrac{1}{b}x^{n+1}dy + ax^n ydy(x^n+b)^{\frac{1}{n}}$$

euenit, vt vtrumque integrabile existat, si ducatur in hunc multiplicatorem:

$$\frac{y^{n-1}}{x^{n+1} + abx^n y(x^n + b)^{\frac{1}{n}}}$$

ad quem inueniendum ex separatione variabilium, adhibeatur haec substitutio non adeo obuia $\frac{x}{(x^n+b)^{\frac{1}{n}}} = vy$, vnde fit $x^n = \frac{bv^n y^n}{1-v^n y^n}$, et hinc aequatio $\frac{ydx - xdy}{(x^n+b)^{\frac{1}{n}}}$ $+ ax^n ydy = 0$, abit in hanc $\frac{yydv + v^{n+1}y^{n+1}dy + abv^n y^{n+1}dy}{1-v^n y^n} = 0$, quae multiplicata per $\frac{1-v^n y^n}{yyv^n(ab+v)}$ separatur $\frac{dv}{v^n(ab+v)} + y^{n-1}dy = 0$, vnde idem ille multiplicatur colligitur.

Exemplum 5.

491. *Proposita aequatione differentiali:*

$$dx + yydx - \frac{adx}{x^4} = 0$$

inuenire multiplicatorem, quo ea integrabilis reddatur.

Secun-

Secundum §. 440. ponatur $x=\frac{1}{t}$ et ob $dx=-\frac{dt}{tt}$; nostra formula erit $dy-\frac{yydt}{tt}+attdt$, in qua porro statuatur $y=t-ttz$, et prodibit $-tt(dz+zzdt-adt)$, quae per $tt(zz-a)$ diuisa separatur, ergo et nostra aequatio diuisa per $tt(zz-a)=\frac{(t-y)^2-at^4}{tt}$ $=(1-xy)^2-\frac{a}{xx}$ fiet integrabilis, ex quo multiplicator erit $=\frac{xx}{xx(1-xy)^2-a}$ et aequatio per se integrabilis $\frac{x^4dy+x^4yydx-adx}{x^4(1-xy)^2-axx}=0$. Spectetur iam x vt constans eritque ex dy natum integrale:

$$\frac{1}{2\sqrt{a}}l\frac{x(1-xy)+\sqrt{a}}{\sqrt{a}-x(1-xy)}+X,$$

pro quo vt valor ipsius X obtineatur, differentietur denuo, ac prodibit

$$\frac{2xydx-dx}{xx(1-xy)^2-a}+dX=\frac{x^4yydx-adx}{x^4(1-xy)^2-axx}$$

vnde

$$dX=\frac{x^4yydx-adx-2x^3ydx+xxdx}{x^4(1-xy)^2-axx}=\frac{dx}{xx}, \text{ et } X=-\frac{1}{x}+C$$

quare aequatio integralis completa erit

$$l\frac{\sqrt{a}+x(1-xy)}{\sqrt{a}-x(1-xy)}=\frac{2\sqrt{a}}{x}+C.$$

Scholion.

492. En ergo plures casus aequationum differentialium pro quibus multiplicatores nouimus, ex quorum contemplatione haec insignis inuestigatio non parum adiuuari videtur. Quanquam autem adhuc longe absumus a certa methodo pro quouis casu multiplicatores idoneos inueniendi; hinc tamen for-

mas aequationum colligere poterimus, vt per datos multiplicatores integrabiles reddantur; quod negotium cum in hac ardua doctrina maximam vtilitatem allaturum videatur, in sequente capite aequationes inuestigabimus, quibus dati multiplicatores conueniant? exempla scilicet hic euoluta idoneas multiplicatorum formas nobis suppeditant, quibus nostram inuestigationem superstruere licebit.

CAPVT III.

CAPVT III.

DE INVESTIGATIONE AEQVATIONVM DIFFERENTIALIVM QVAE PER MVLTIPLICATORES DATAE FORMAE INTEGRABILES REDDANTVR.

Problema 65.

493.

Definire functiones P et Q ipſius x, vt aequatio differentialis $Pydx + (y+Q)dy = 0$, per multiplicatorem $\frac{1}{y^3 + Myy + Ny}$, vbi M et N ſunt functiones ipſius x, fiat integrabilis.

Solutio.

Neceſſe igitur eſt, vt factoris ipſius dx, qui eſt $\frac{Py}{y^3 + Myy + Ny}$, differentiale ex variabilitate ipſius y natum; aequale ſit differentiali factoris ipſius dy, qui eſt $\frac{y+Q}{y^3 + Myy + Ny}$, dum ſola x variabilis ſumitur. Horum valorum aequalium neglecto denominatore communi aequalitas dat:

$$-2Py^3 - PMy^2 = (y^3 + Myy + Ny)\frac{dQ}{dx} - (y+Q)\frac{(yydM + ydN)}{dx}$$

quae

quae secundum potestates ipsius y ordinata praebet:

$$\begin{aligned}0 = &\, 2Py^3dx + PMy^2dx \\ &+ y^3dQ \;\; + My^2dQ + NydQ \\ &- y^2dM \;\; - y^2dN \\ &\qquad\quad - Qy^2dM - QydN\end{aligned}$$

vnde singulis potestatibus seorsim ad nihilum perductis, nanciscimur primo $NdQ - QdN = 0$, seu $\frac{dN}{N} = \frac{dQ}{Q}$, ex cuius integratione sequitur $N = \alpha Q$. Tum binae reliquae conditiones sunt

$$\text{I.}\;\; 2Pdx + dQ - dM = 0 \;\text{ et}$$

$$\text{II.}\;\; PMdx + MdQ - \alpha dQ - QdM = 0$$

vnde I. M − II. 2 suppeditat:

$$-MdQ - MdM + 2\alpha dQ + 2QdM = 0 \;\text{ seu}$$

$$dQ + \frac{2QdM}{2\alpha - M} = \frac{MdM}{2\alpha - M}$$

quae per $(2\alpha - M)^2$ diuisa et integrata dat:

$$\frac{Q}{(2\alpha - M)^2} = \int \frac{MdM}{(2\alpha - M)^3} = -\int \frac{dM}{(2\alpha - M)^2} + 2\alpha \int \frac{dM}{(2\alpha - M)^3}$$

seu

$$\frac{Q}{(2\alpha - M)^2} = \frac{-1}{2\alpha - M} + \frac{\alpha}{(2\alpha - M)^2} + \beta = \frac{M - \alpha}{(2\alpha - M)^2} + \beta$$

Erit ergo

$$Q = M - \alpha + \beta(2\alpha - M)^2$$

hincque

$$2Pdx = dM - dQ = +2\beta dM(\alpha - M)$$

sicque

ſicque pro M functionem quamcunque ipſius x ſumere licet. Capiatur ergo $M = 2\alpha - X$, erit $P dx = -\beta X dX$ et $Q = \alpha - X + \beta XX$ atque $N = \alpha\alpha - \alpha X + \alpha\beta XX$. Quocirca pro hac aequatione

$$-\beta y X dX + dy(\alpha - X + \beta XX + y) = 0$$

habemus hunc multiplicatorem

$$\frac{1}{y^3 + (2\alpha - X)yy + \alpha(\alpha - X + \beta XX)y}$$

quo ea integrabilis redditur.

Coroll. 1.

494. Tribuatur aequationi haec forma:

$$dy(y + A + BV + CVV) - CyVdV = 0$$

eritque $\alpha = A$; $X = -BV$; $\beta XX = \beta BBVV = CVV$ ergo $\beta = \frac{C}{BB}$, vnde multiplicator fiet

$$\frac{1}{y^3 + (2A + BV)yy + A(A + BV + CVV)y}$$

Coroll. 2.

495. Si hic ſumatur $V = a + x$, obtinebitur aequatio ſimilis illi, quam ſupra §. 488. integrauimus, et multiplicator quoque cum eo, quem ibi dedimus, conuenit. Hic autem multiplicator commodius hac forma exhibetur:

$$\frac{1}{y(y+A)^2 + BVy(y+A) + ACVVy}.$$

Coroll. 3.

496. Si ponamus $y+A=z$, nostra aequatio erit

$$dz(z+BV+CVV)-C(z-A)VdV=0$$

cui conuenit multiplicator $\frac{1}{(z-A)(zz+BVz+ACVV)}$, ita vt per se integrabilis sit haec aequatio:

$$\frac{dz(z+BV+CVV)-C(z-A)VdV}{(z-A)(zz+BVz+ACVV)}=0.$$

Scholion.

497. Quemadmodum hic aequationis $Pydx+(y+Q)dy=0$ multiplicatorem assumsimus $=\frac{y^{-1}}{yy+My+N}$ ita generalius eius loco sumere poterimus $\frac{y^{n-1}}{yy+My+N}$, vt haec aequatio $\frac{Py^n dx+(y^n+Qy^{n-1})dy}{yy+My+N}=0$, per se debeat esse integrabilis, qua comparata cum forma $Rdx+Sdy=0$, vt sit $\left(\frac{dR}{dy}\right)=\left(\frac{dS}{dx}\right)$, habebimus:

$$(n-2)Py^{n+1}+(n-1)PMy^n+nPNy^{n-1}=(yy+My+N)y^{n-1}\frac{dQ}{dx}$$
$$-(y^n+Qy^{n-1})\left(\frac{ydM}{dx}+\frac{dN}{dx}\right)$$

siue ordinata aequatione:

$$\left.\begin{matrix}(n-2)Py^{n+1}dx & +(n-1)PMy^n dx & +nPNy^{n-1}dx \\ -y^{n+1}dQ & -My^n dQ & -Ny^{n-1}dQ \\ +y^{n+1}aM & +y^n dN & +y^{n-1}QdN \\ & +y^n QdM & \end{matrix}\right\}=0$$

vnde

vnde fingulis membris ad nihilum reductis fit

$$\text{I. } (n-2)Pdx = dQ - dM$$

$$\text{II. } (n-1)MPdx = MdQ - QdM - dN$$

$$\text{III. } nNPdx = NdQ - QdN.$$

Sit $Pdx = dV$, eritque ex prima $Q = A + M + (n-2)V$, quo valore in fecunda fubftituto prodit

$$MdV + (n-2)VdM + AdM + dN = 0$$

et tertia fit

$$2NdV + (n-2)VdN + MdN - NdM + AdN = 0$$

vnde eliminando dV reperitur:

$$(n-2)V + A = \frac{MMdN - MNdM - 2NdN}{2NdM - MdN}.$$

Verum fi hinc vellemus V elidere, in aequationem differentio-differentialem illaberemur. Cafus tamen quo $n = 2$ expediri poteft.

Exemplum.

498. *Sit in euolutione huius cafus* $n = 2$, *vt per fe integrabilis effe debeat haec aequatio*

$$\frac{y(Pydx + (y+Q)dy)}{yy + My + N} = 0.$$

Ac primo effe oportet $Q = A + M$ tum vero

$$2ANdM - AMdN = M(MdN - NdM) - 2NdN$$

quam ergo aequationem integrare debemus, quae cum in nulla iam tractatarum contineatur, viden-

 dum

dum eſt, quomodo tractabilior reddi queat. Ponatur ergo $M = Nu$, vt fiat

$MdN - NdM = -NNdu$ et $2NdM - MdN = 2NNdu + NudN$

hinc

$$2ANNdu + ANudN + N^3udu + NdN = 0$$

ſiue

$$\frac{2dN}{NN} + \frac{AudN}{NN} + \frac{2Adu}{N} + udu = 0$$

ſtatuatur porro $\frac{1}{N} = v$ ſeu $N = \frac{1}{v}$ habebitur:

$$-2dv - Audv + 2Avdu + udu = 0 \text{ ſeu}$$

$$dv - \frac{2Avdu}{2+Au} = \frac{udu}{2+Au}$$

vbi variabilis v vnicam habet dimenſionem, et hanc ob rem patet hanc aequationem integrabilem reddi, ſi diuidatur per $(2+Au)^2$ prodibitque:

$$\frac{v}{(2+Au)^2} = \int \frac{udu}{(2+Au)^3} = \frac{C}{AA} - \frac{1-Au}{AA(2+Au)^2}$$

ideoque $v = \frac{C(2+Au)^2 - 1 - Au}{AA}$. Sumto ergo pro u functione quacunque ipſius x erit

$$N = \frac{AA}{C(2+Au)^2 - 1 - Au} \text{ et } M = \frac{AAu}{C(2+Au)^2 - 1 - Au}$$

atque $Q = \frac{AC(2+Au)^2 - A}{C(2+Au)^2 - 1 - Au}$. Iam ex tertia aequatione adipiſcimur $2NPdx = NdQ - QdN$, ſeu $2Pdx = Nd.\frac{Q}{N}$, at $\frac{Q}{N} = \frac{C(2+Au)^2 - 1}{A}$, vnde $d.\frac{Q}{N} = 2Cdu(2+Au)$, ideoque

$$Pdx = \frac{AACdu(2+Au)}{C(2+Au)^2 - 1 - Au}.$$

Quocirca aequatio noſtra per ſe integrabilis eſt

$$\frac{AACyydu(2+Au) + ydy(C(2+Au)^2y - (1+Au)y + AC(2+Au)^2 - A)}{C(2+Au)^2yy - (1+Au)yy + AAuy + AA} = 0$$

quae

quae posito $Au+2=t$ induet hanc formam:

$$y.\frac{ACytdt+ydy(Ctt-t+1)+Ady(Ctt-1)}{Cttyy-(t-2)yy+A(t-2)y+AA}=0.$$

Hinc autem posito $A=\alpha$; $C=\frac{\alpha\gamma}{\beta\beta}$ et $t=-\frac{\beta x}{\alpha}$ invenimus

$$y.\frac{\alpha\gamma xydx+ydy(\alpha+\beta x+\gamma xx)-\alpha dy(\alpha-\gamma xx)}{(\alpha+\beta x+\gamma xx)yy-\alpha(2\alpha+\beta x)y+\alpha^2}=0.$$

Coroll. 1.

499. Hoc igitur modo integrari potest haec aequatio

$$\alpha\gamma xydx+ydy(\alpha+\beta x+\gamma xx)-\alpha dy(\alpha-\gamma xx)=0$$

quae quomodo ad separationem reduci debeat, non statim patet. Est autem multiplicator idoneus:

$$\frac{y}{(\alpha+\beta x+\gamma xx)yy-\alpha(2\alpha+\beta x)y+\alpha^2}.$$

Coroll. 2.

500. Hic multiplicator etiam hoc modo exprimi potest, vt eius denominator in factores resolvatur:

$$\frac{(\alpha+\beta x+\gamma xx)y}{((\alpha+\beta x+\gamma xx)y-\alpha(\alpha+\tfrac{1}{2}\beta x)+\alpha x\sqrt{(\tfrac{1}{4}\beta\beta-\alpha\gamma)})((\alpha+\beta x+\gamma xx)y-\alpha(\alpha+\tfrac{1}{2}\beta x)-\alpha x\sqrt{(\tfrac{1}{4}\beta\beta-\alpha\gamma)})}.$$

 Coroll. 3.

Coroll. 3.

501. Si ergo ponamus $(\alpha+\beta x+\gamma xx)y-\alpha(\alpha+\frac{1}{2}\beta x)=\alpha z$, erit multiplicator $\frac{\alpha+\frac{1}{2}\beta x+z}{(z+x\sqrt{(\frac{1}{4}\beta\beta-\alpha\gamma)})(z-x\sqrt{(\frac{1}{4}\beta\beta-\alpha\gamma)})}$. At ob $y=\frac{\alpha\alpha+\frac{1}{2}\alpha\beta x+\alpha z}{\alpha+\beta x+\gamma xx}$, aequatio nostra erit

$$\gamma xy\,dx+dy(z+\tfrac{1}{2}\beta x+\gamma xx)=0.$$

At est

$$dy=\frac{-\frac{1}{2}\alpha(\alpha\beta+4\alpha\gamma x+\beta\gamma xx)dx-\alpha z dx(\beta+2\gamma x)+\alpha dz(\alpha+\beta x+\gamma xx)}{(\alpha-\beta x+\gamma xx)^2}$$

hoc autem valore substituto prodit aequatio nimis complicata.

Problema 66.

502. Inuenire aequationem differentialem huius formae:

$$yP\,dx+(Qy+R)dy=0$$

in qua P, Q et R sint functiones ipsius x, vt ea integrabilis euadat per hunc multiplicatorem $\frac{y^m}{(1+Sy)^n}$ vbi S est etiam functio ipsius x.

Solutio.

Quia dx per $\frac{y^{m+1}P}{(1+Sy)^n}$ et dy per $\frac{Qy^{m+1}+Ry^m}{(1+Sy)^n}$ multiplicatur, oportet sit:

$$(m+1)Py^m(1+Sy)-nPSy^{m+1}=\frac{(1+Sy)(y^{m+1}dQ+y^m dR)-nydS(Qy^{m+1}+Ry^m)}{dx}$$

qua

qua euoluta aequatione erit

$$\left.\begin{array}{llll}(m+1)Py^m dx+(m+1-n)PSy^{m+1}dx-y^{m+1}SdQ \\ -y^m dR \quad -y^{m+1}dQ \quad +ny^{m+1}QdS \\ \quad -y^{m+1}SdR \\ \quad +ny^{m+1}RdS\end{array}\right\}=0$$

hinc fit $Pdx=\frac{dR}{m+1}$, et $SdQ=nQdS$, ideoque $Q=AS^n$ et $dQ=nAS^{n-1}dS$, quibus in membro medio substitutis fit

$$\frac{m+1-n}{m+1}SdR-nAS^{n-1}dS-SdR+nRdS=0$$

seu $-\frac{SdR}{m+1}-AS^{n-1}dS+RdS=0$, ideoque

$$dR-\frac{(m+1)RdS}{S}=-(m+1)AS^{n-1}dS$$

quae per S^{m+1} diuisa et integrata praebet

$$\frac{R}{S^{m+1}}=B-\frac{(m+1)AS^{n-m-1}}{n-m-1}.$$

Ponamus $A=(m+2-n)C$ vt sit $Q=(m+2-n)CS^n$ et $R=BS^{m+1}+(m+1)CS^{n-1}$, ideoque $Pdx=BS^m dS+(n-1)CS^{n-2}dS$. Quocirca habebimus hanc aequationem

$$ydS(BS^m+(n-1)CS^{n-2})+dy((m+2-n)CS^n y+BS^{m+1}+(m+1)CS^{n-1})=0$$

quae multiplicata per $\frac{y^m}{(1+Sy)^n}$ fit integrabilis, vbi pro S functionem quamcunque ipsius x capere licet.

Coroll. 1.

Coroll. 1.

503. Integrari ergo poterit haec aequatio

$$ByS^m dS + BS^{m+1}dy + (n-1)CyS^{n-2}dS + (m+1)CS^{n-1}dy$$
$$+ (m+2-n)CS^n y dy = 0$$

quae ſponte reſoluitur in has duas partes:

$$BS^m(ydS + Sdy) +$$
$$CS^{n-2}((n-1)ydS + (m+1)Sdy + (m+2-n)S^2 y dy) = 0$$

quarum vtraque ſeorſim per $\frac{y^m}{(1+Sy)^n}$ multiplicata fit integrabilis.

Coroll. 2.

504. Prior pars $BS^m(ydS + Sdy)$ integrabilis redditur per hunc multiplicatorem $\frac{1}{S^m}\Phi.Sy$; eſt enim haec formula $B(ydS + Sdy)\Phi.Sy$ per ſe integrabilis. Vnde pro hac parte multiplicator erit $S^{\lambda-m}y^\lambda(1+Sy)^\mu$ qui vtique continet aſſumtum $\frac{y^m}{(1+Sy)^n}$, ſi quidem capiatur $\lambda = m$ et $\mu = -n$. Eſt vero $\int\frac{y^m}{(1+Sy)^n}.BS^m(ydS + Sdy) = B\int\frac{v^m dv}{(1+v)^n}$ poſito $Sy = v$.

Coroll. 3.

Coroll. 3.

505. Pro altera parte, quae posito $S = \frac{1}{v}$ abit in $\frac{C}{v^n}(-(n-1)y\,dv + (m+1)v\,dy + (m+2-n)y\,dy)$, habebimus

$$-\frac{(n-1)Cy}{v^n}\left(dv - \frac{(m+1)v\,dy}{(n-1)y} - \frac{(m+2-n)dy}{(n-1)}\right) =$$

$$-\frac{(n-1)Cy^{\frac{m+n}{n-1}}}{v^n}\left(y^{\frac{-m-1}{n-1}}dv - \frac{m+1}{n-1}y^{\frac{-m-n}{n-1}}v\,dy - \frac{m+2-n}{n-1}y^{\frac{-m-1}{n-1}}dy\right)$$

$$= -\frac{(n-1)Cy^{\frac{m+n}{n-1}}}{v^n}d.\left(y^{\frac{-m-n}{n-1}}v + y^{\frac{n-m-1}{n-1}}\right).$$

Ideoque haec altera pars ita repraesentabitur:

$$-(n-1)CS^n y^{\frac{m+n}{n-1}}d.\frac{1+Sy}{y^{\frac{m+1}{n-1}}S}.$$

Multiplicator ergo hanc partem integrabilem reddens erit in genere

$$\frac{1}{S^n y^{\frac{m+n}{n-1}}}\Phi.\frac{1+Sy}{Sy^{\frac{m+1}{n-1}}}.$$

Coroll. 4.

506. Pro altera ergo parte multiplicator erit $\frac{(1+Sy)^\mu}{S^{n+\mu}y^{\frac{m+n+\mu(m+1)}{n-1}}}$, quo haec pars fit

$$-(n-1)C.\frac{(1+Sy)^\mu}{S^\mu y^{\frac{\mu(m+1)}{n-1}}}d.\frac{1+Sy}{y^{\frac{m+1}{n-1}}S},$$

 cuius

cuius integrale est

$$-\frac{(n-1)CZ^{\mu+1}}{\mu+1} \text{ posito } Z = \frac{1+Sy}{y^{\frac{m+1}{n-1}}S}.$$

Coroll. 5.

507. Iam multiplicator pro prima parte $S^{\lambda-m}y^{\lambda}(1+Sy)^{\mu}$ congruens reddetur cum multiplicatore alterius partis modo exhibito, si sumatur $\lambda = m$ et $\mu = -n$, vnde resultat multiplicator communis $\frac{y^m}{(1+Sy)^n}$, hincque posito $Sy = v$ et $\frac{1+Sy}{y^{\frac{m+1}{n-1}}S} = z$, nostrae aequationis integrale erit

$$B\int\frac{v^m dv}{(1+v)^n} + Cz^{1-n} = D \text{ siue}$$

$$B\int\frac{v^m dv}{(1+v)^n} + \frac{CS^{n-1}y^{m+1}}{(1+Sy)^{n-1}} = D.$$

Scholion.

508. Aequatio ergo, quam hoc problemate integrare didicimus, per principia iam supra stabilita tractari potest, dum pro binis eius partibus seorsim multiplicatores quaeruntur, iique inter se congruentes redduntur, cuius methodi hic insignem vsum declarauimus. Possemus etiam multiplicatori hanc formam dare $\frac{y^m}{(1+Sy+Tyy)^n}$, ita vt haec

aequa-

aequatio:

$$\frac{y^m(y P dx + (Q y + R) dy)}{(1 + S y + T y y)^n} = 0$$

per se debeat esse integrabilis, et calculo vt ante instituto inueniemus:

$$y^m \left\{ \begin{matrix} (m+1) P dx \\ - dR \end{matrix} \right\} + y^{m+1} \left\{ \begin{matrix} +(m+1-n) P S dx \\ - dQ \\ - S dR \\ + n R dS \end{matrix} \right\} + y^{m+2} \left\{ \begin{matrix} +(m+1-2n) P T dx \\ - S dQ \\ - T dR \\ + n Q dS \\ + n R dT \end{matrix} \right\} + y^{m+3} \left\{ \begin{matrix} - T dQ \\ + n Q dT \end{matrix} \right\} = 0$$

vnde ex vltimo membro $-T dQ + n Q dT = 0$ concludimus $Q = A T^n$ et ex primo $P dx = \frac{dR}{m+1}$, qui valores in binis mediis substituti praebent:

$$R dS - \frac{S dR}{m+1} - A T^{n-1} dT = 0 \text{ et}$$

$$R dT - \frac{2 T dR}{m+1} + A T^n dS - A S T^{n-1} dT = 0$$

quarum illa fit integrabilis per se si $m = -2$, haec vero integrari potest si $m = 2n - 1$, fit enim

$$R dT - \frac{T dR}{n} + A T^{n-1}(T dS - S dT) = 0 \text{ seu}$$

$$\frac{n R dT - T dR}{n T^{n+1}} + \frac{A (T dS - S dT)}{TT} = 0$$

cuius integrale est $\frac{-R}{n T^n} + \frac{AS}{T} = \frac{-B}{n}$ hincque $R = B T^n + n A T^{n-1} S$. Praeterea vero notari meretur casus $m = -1$, quem cum illis in subiunctis exemplis euoluamus.

Exemplum I.

509 *Definire hanc aequationem*

$$yPdx+(Qy+R)dy=0$$

vt multiplicata per $\frac{1}{y(1+Sy+Tyy)^n}$ *fiat per se integrabilis.*

Ob $m=-1$, habemus statim $dR=0$, ideoque $R=C$ tum est vt ante $Q=AT^n$ et $dQ=nAT^{n-1}dT$ vnde binae reliquae determinationes erunt:

$$-PSdx-AT^{n-1}dT+CdS=0$$
$$-2PTdx-AST^{n-1}dT+AT^n dS+CdT=0$$

hinc eliminando Pdx prodit:

$$ASST^{n-1}dT-2AT^n dT-AT^n SdS+2CTdS-CSdT=0.$$

Statuatur hic $SS=Tv$, vt fiat $2TdS-SdT=TS(\frac{2dS}{S}-\frac{dT}{T})=\frac{TSdv}{v}=\frac{Tdv\sqrt{T}}{\sqrt{v}}$, eritque

$$\tfrac{1}{2}AT^n v dT-2AT^n dT-\tfrac{1}{2}AT^{n+1}dv+\frac{CTdv\sqrt{T}}{\sqrt{v}}=0$$

seu hoc modo:

$$-\tfrac{1}{2}AT^{n+2}d.\frac{v-4}{T}+\frac{CTdv\sqrt{T}}{\sqrt{v}}=0$$

cuius prior pars integrabilis redditur per multiplicatorem $\frac{1}{T^{n+2}}\Phi.\frac{v-4}{T}$, posterior vero per $\frac{1}{T\sqrt{T}}\Phi.v$ vnde communis multiplicator erit $\frac{1}{T(v-4)^{n+\frac{1}{2}}\sqrt{T}}$ hincque

hincque aequatio elicitur integralis haec:

$$\frac{AT^{n-\frac{1}{2}}}{(2n-1)(v-4)^{n-\frac{1}{2}}}+C\int\frac{dv}{(v-4)^{n+\frac{1}{2}}\sqrt{v}}=D$$

vnde T definitur per v, tum vero est $S=\sqrt{Tv}$; $R=C$, $Q=AT^n$ et $Pdx=\frac{CdS-AT^{n-1}dT}{S}$.

Coroll. 1.

510. Casu quo est $n=\frac{1}{2}$, ob $\frac{1}{0}z^0=lz$ habetur

$\frac{1}{2}Al\frac{T}{v-4}+C\int\frac{dv}{(v-4)\sqrt{v}}=\frac{1}{2}D$ seu

$\frac{1}{2}Al\frac{T}{v-4}-\frac{1}{2}Cl\frac{\sqrt{v}+2}{\sqrt{v}-2}=\frac{1}{2}D$

vnde posito $v=4uu$ et $C=\lambda A$ erit

$l\frac{T}{1-uu}-\lambda l\frac{1+u}{1-u}=$ Const. seu $T=E(1-uu)\left(\frac{1+u}{1-u}\right)^{\lambda}$.

Hinc porro $S=2u\sqrt{T}=2u\left(\frac{1+u}{1-u}\right)^{\frac{\lambda}{2}}\sqrt{E(1-uu)}$, et $R=C=\lambda A$; tum $Q=A\left(\frac{1+u}{1-u}\right)^{\frac{\lambda}{2}}\sqrt{E(1-uu)}$, atque $Pdx=\frac{\lambda A\,du}{u}+\frac{\lambda A\,dT}{2T}-\frac{A\,dT}{2Tu}$. At est $\frac{dT}{T}=\frac{-2udu+2\lambda du}{1-uu}$. Ergo $Pdx=\frac{A\,du(1+\lambda\lambda-2\lambda u)}{1-uu}$. Quocirca pro hac aequatione

$$\frac{Ay\,du(1+\lambda\lambda-2\lambda u)}{1-uu}+A\,dy\left(\lambda+y\left(\frac{1+u}{1-u}\right)^{\frac{\lambda}{2}}\sqrt{E(1-uu)}\right)=0$$

multiplicator erit

$$\frac{1}{y\sqrt{\left(1+2uy\left(\frac{1+u}{1-u}\right)^{\frac{\lambda}{2}}\sqrt{E(1-uu)}+Eyy(1-uu)\left(\frac{1+u}{1-u}\right)^{\lambda}\right)}}.$$

 Coroll. 2.

Coroll. 2.

511. Casu quo $n = -\frac{1}{2}$ habemus:

$$-\frac{A(v-4)}{2T} + 2C\sqrt{v} = -2D \text{ seu } T = \frac{A(v-4)}{4D+4C\sqrt{v}}.$$

Ponamus $v = 4uu$, vt sit $T = \frac{A(uu-1)}{D+2Cu}$, tum fit $S = 2u\sqrt{T} = 2u\sqrt{\frac{A(uu-1)}{D+2Cu}}$; $R = C$; $Q = \sqrt{\frac{A(D+2Cu)}{uu-1}}$ et $Pdx = \frac{Cdu}{u} + \frac{CdT}{2T} - \frac{AdT}{2TTu} = \frac{du(C+Du+Cuu)(Cu^3-3Cu-D)}{u(uu-1)^2(D+2Cu)}$ vnde tam aequatio quam multiplicator definitur.

Exemplum 2.

512. *Definire aequationem* $yPdx + (Qy+R)dy = 0$, *vt multiplicata per* $\frac{1}{y^2(1+Sy+Tyy)^n}$ *fiat per se integrabilis.*

Ob $m = -2$, ex superioribus habemus:

$$RS = \frac{A}{n}T^n + B \text{ seu } R = \frac{AT^n}{nS} + \frac{B}{S}$$

qui valor in altera aequatione substitutus praebet:

$$\frac{(2n+1)AT^n dT}{nS} - \frac{2AT^{n+1}dS}{nSS} + AT^n dS - AST^{n-1}dT + \frac{BdT}{S} - \frac{2BTdS}{SS} = 0$$

quae in has tres partes distinguatur:

$$\frac{AS}{nT^n}\left(\frac{(2n+1)T^{2n}dT}{S^2} - \frac{2T^{2n+1}dS}{S^3}\right) + AT^{n+1}\left(\frac{dS}{T} - \frac{SdT}{TT}\right) + BS\left(\frac{dT}{SS} - \frac{2BTdS}{S^3}\right) = 0$$

seu

seu

$$\frac{AS}{nT^n}d.\frac{T^{2n+1}}{SS}+AT^{n+1}d.\frac{S}{T}+BSd.\frac{T}{SS}=0.$$

Statuamus ad abreuiandum:

$$\frac{T^{2n+1}}{SS}=p;\ \frac{S}{T}=q \text{ et } \frac{T}{SS}=r$$

fiet $S=\frac{1}{qr}$; $T=\frac{1}{qqr}$, hinc $p=\frac{1}{q^{4n}r^{2n-1}}$; nostraque aequatio ita se habebit:

$$\frac{A}{nq\sqrt{pr}}dp+\frac{A\sqrt{p}}{qqr\sqrt{r}}dq+\frac{B}{qr}dr=0$$

seu

$$\frac{A\sqrt{r}}{n\sqrt{p}}dp+\frac{A\sqrt{p}}{q\sqrt{r}}dq+Bdr=0.$$

Quas tres partes seorsim consideremus, ac prima fit integrabilis multiplicata per $\frac{\sqrt{p}}{\sqrt{r}}\Phi:p$, secunda vero per $\frac{q\sqrt{r}}{\sqrt{p}}\Phi:q$, tertia vero per $\Phi:r$. Vt bini primi conueniant, ponatur $\frac{\sqrt{p}}{\sqrt{r}}.p^\lambda=\frac{q\sqrt{r}}{\sqrt{p}}.q^\mu$ seu $p^{\lambda+1}=q^{\mu+1}r$, hinc $p=q^{\frac{\mu+1}{\lambda+1}}r^{\frac{1}{\lambda+1}}=q^{-4n}r^{-2n+1}$. Fit ergo $\lambda+1=-\frac{1}{2n-1}$ et $\mu+1=-4n(\lambda+1)=\frac{4n}{2n-1}$, sicque $\mu=\frac{2n+1}{2n-1}$ et $\lambda=-\frac{2n}{2n+1}$. Multiplicetur ergo aequatio per $\frac{q^{\frac{4n}{2n-1}}\sqrt{r}}{\sqrt{p}}=q^{2n+\frac{4n}{2n-1}}r^{n+1}$ ac prodibit

$$\frac{A}{n}p^\lambda dp+Aq^\mu dq+Bq^{2n+\frac{4n}{2n+1}}r^{n+1}dr=0$$

seu

seu $A\,d.\left(\frac{p^{\lambda+1}}{n(\lambda+1)}+\frac{q^{\mu+1}}{\mu+1}\right)+Bq^{\frac{4nn+6n}{2n+1}}r^{n+1}dr=0$

vel $\frac{(2n+1)A}{4n}\,d.q^{\frac{4n}{2n+1}}(1-4r)+Bq^{\frac{4nn+6n}{2n+1}}r^{n+1}dr=0.$

Multiplicetur per $q^{\frac{4\nu n}{2n+1}}(1-4r)^{\nu}$ vt prodeat

$$\frac{(2n+1)A}{4n}.q^{\frac{4\nu n}{2n+1}}(1-4r)^{\nu}d.q^{\frac{4n}{2n+1}}(1-4r)+Bq^{\frac{4nn+6n+4\nu n}{2n+1}}r^{n+1}dr(1-4r)^{\nu}=0.$$

Fiat ergo $4\nu+4n+6=0$ seu $\nu=-n-\frac{3}{2}$, et ambo membra integrari poterunt, eritque

$$\frac{(2n+1)A}{4n(\nu+1)}q^{\frac{4n(\nu+1)}{2n+1}}(1-4r)^{\nu+1}+B\int r^{n+1}dr(1-4r)^{\nu}=\text{Const.}$$

at est $\nu+1=-n-\frac{1}{2}=-\frac{2n-1}{2}$ sicque habebitur:

$$-\frac{A}{2n}q^{-2n}(1-4r)^{\frac{-2n-1}{2}}+B\int\frac{r^{n+1}dr}{(1-4r)^{\frac{2n+3}{2}}}=\text{Const.}$$

Dabitur ergo q per r, eritque $S=\frac{1}{qr}$; $T=\frac{S}{q}$, tum $R=\frac{AT^n}{nS}+\frac{B}{S}$, $Q=AT^n$ et $Pdx=-dR$

Coroll. 1.

513. Si sit $n=-\frac{1}{2}$ erit $Aq+\frac{2Br\surd r}{3}=\frac{C}{3}$ seu $q=\frac{C-2Br\surd r}{3A}$; hincque $S=\frac{3A}{Cr-2Br^2\surd r}$; $T=\frac{9AA}{r(C-2Br\surd r)^2}$; $Q=\frac{C\surd r-2Brr}{3}$, et $R=\frac{Q+nB}{nS}=\frac{B-2Q}{S}=\frac{r(C-2Br\surd r)(3B-2C\surd r+4Brr)}{9A}$ seu $R=\frac{3BCr-2CCr\surd r-6BBrr\surd r+8BCr^3-8BBr^4\surd r}{9A}.$

Coroll. 2.

Coroll. 2.

514. Ponamus eodem casu $r = uu$ erit:

$$S = \frac{3A}{Cuu - 2Bu^5}; \quad T = \frac{9AA}{uu(C - 2Bu^3)^2}; \quad Q = \frac{u(C - 2Bu^3)}{3}; \text{ et}$$

$$R = \frac{3BCu^2 - 2CCu^5 - 6BBu^5 + 8BCu^6 - 8BBu^9}{9A} \text{ hincque}$$

$$Pdx = \frac{-6BCu + 6CCuu + 30BBu^4 - 48BCu^5 + 72BBu^8}{9A}\,du$$

eritque aequatio $yPdx + (Qy + R)dy = 0$ integrabilis si multiplicetur per

$$\frac{\sqrt{(1 + Sy + Tyy)}}{yy} = \frac{1}{yy}\sqrt{\left(1 + \frac{3Ay}{uu(C - 2Bu^3)} + \frac{9AAyy}{uu(C - 2Bu^3)^2}\right)}.$$

Exemplum 3.

515. *Definire aequationem* $yPdx + (Qy + R)dy = 0$ *quae multiplicata per* $\frac{y^{2n-1}}{(1 + Sy + Tyy)^n}$ *fiat per se integrabilis.*

Hic est $m = 2n - 1$, $Q = AT^n$, et $Pdx = \frac{dR}{2n}$, tum vero ex superioribus $R = nAT^{n-1}S + BT^n$; ac superest aequatio $RdS - \frac{SdR}{2n} - AT^{n-1}dT = 0$, quae loco R substituto valore inuento abit in

$$(2n-1)AT^{n-1}SdS - (n-1)AT^{n-2}SSdT - 2AT^{n-1}dT$$
$$+ 2BT^n dS - BT^{n-1}SdT = 0 \text{ seu}$$

$$(2n-1)ATSdS - (n-1)ASSdT - 2ATdT$$
$$+ 2BTTdS - BTSdT = 0.$$

Prius membrum posito $SS=u$ abit in

$$(n-\tfrac{1}{2})AT\,du-(n-1)Au\,dT-2AT\,dT, \text{ seu}$$

$$(n-\tfrac{1}{2})AT\left(du-\frac{(n-1)u\,dT}{(n-\frac{1}{2})T}-\frac{2\,dT}{n-\frac{1}{2}}\right) \text{ siue}$$

$$\tfrac{1}{2}(2n-1)AT^{\frac{4n-3}{2n-1}}\left(\frac{du}{T^{\frac{2n-2}{2n-1}}}-\frac{2(n-1)u\,dT}{(2n-1)T^{\frac{4n-3}{2n-1}}}-\frac{4\,dT}{(2n-1)T^{\frac{2n-2}{2n-1}}}\right)=$$

$$\tfrac{1}{2}(2n-1)AT^{\frac{4n-3}{2n-1}}d.\left(\frac{u}{T^{\frac{2n-2}{2n-1}}}-4T^{\frac{1}{2n-1}}\right) \text{ vel}$$

$$\tfrac{1}{2}(2n-1)AT^{\frac{4n-3}{2n-1}}d.T^{\frac{1}{2n-1}}\left(\frac{SS}{T}-4\right)+\frac{BT^{3}}{S}d.\frac{SS}{T}=0, \text{ seu}$$

$$(2n-1)AT^{\frac{-1}{2n-1}}d.T^{\frac{1}{2n-1}}\left(\frac{SS}{T}-4\right)+\frac{2BT}{S}d.\frac{SS}{T}=0.$$

Ponatur $\frac{SS}{T}=p$ et $T^{\frac{1}{2n-1}}(\frac{SS}{T}-4)=q=T^{\frac{1}{2n-1}}(p-4)$, vt fit $T^{\frac{1}{2n-1}}=\frac{q}{p-4}$, vnde $T=\frac{q^{2n-1}}{(p-4)^{2n-1}}$ et $S=\sqrt{\frac{pq^{2n-1}}{(p-4)^{2n-1}}}$.

Ergo

$$\frac{(2n-1)A(p-4)dq}{q}+\frac{2B\sqrt{q^{2n-1}}}{\sqrt{p(p-4)^{2n-1}}}dp=0$$

siue

$$\frac{(2n-1)A\,dq}{q^{n+\frac{1}{2}}}+\frac{2B\,dp:\sqrt{p}}{(p-4)^{n+\frac{1}{2}}}=0$$

quae

quae integrata praebet

$$\frac{-2A}{q^{n-\frac{1}{2}}} + 2B\int \frac{dp:\sqrt{p}}{(p-4)^{n+\frac{1}{2}}} = 2C$$

et facto $\frac{p}{p-4} = vv$ seu $p = \frac{4vv}{vv-1}$ fiet

$$\frac{+A}{q^{n-\frac{1}{2}}} - \frac{B}{4^{n-1}}\int dv(vv-1)^{n-1} = C$$

Scholion.

516. Haec fusius non prosequor, quia ista exempla eum in finem potissimum attuli, vt methodus supra tradita aequationes differentiales tractandi exerceretur; in his enim exemplis casus non parum difficiles se obtulerunt, quos ita per partes resoluere licuit, vt pro singulis multiplicatores idonei quaererentur, ex iisque multiplicator communis definiretur; nunc igitur alia aequationum genera, quae per multiplicatores integrabiles reddi queant, inuestigemus.

Problema 67.

517. Ipsius x functiones P, Q, R, S definire, vt haec aequatio $(Py+Q)dx+ydy=0$, per hunc multiplicatorem $(yy+Ry+S)^n$ integrabilis reddatur.

Solutio.

Necesse igitur est, sit

$$\left(\frac{d.(Py+Q)(yy+Ry+S)^n}{dy}\right)=\left(\frac{d.y(yy+Ry+S)^n}{dx}\right)$$

vnde coligitur per $(yy+Ry+S)^{n-1}$ diuidendo

$$P(yy+Ry+S)+n(Py+Q)(2y+R)=\frac{ny(ydR+dS)}{dx}$$

seu

$$\left.\begin{array}{lll}(2n+1)Pyydx+(n+1)PRydx+RSdx \\ -nyydR \quad +2nQydx \quad +nQRdx \\ \qquad -nydS.\end{array}\right\}=0$$

Hinc ergo concluditur $Pdx=\frac{ndR}{2n+1}$; et $\frac{(n+1)RdR}{2n+1}+2Qdx-dS=0$; $\frac{SdR}{2n+1}+QRdx=0$ porroque

$$Qdx=\frac{-SdR}{(2n+1)R}=\frac{-(n+1)RdR}{2(2n+1)}+\frac{dS}{2} \text{ ergo}$$

$$dS+\frac{2SdR}{(2n+1)R}=\frac{(n+1)RdR}{2n+1},$$

quae per $R^{\frac{2}{2n+1}}$ multiplicata et integrata dat

$$R^{\frac{2}{2n+1}}S=C+\tfrac{1}{4}R^{\frac{4n+4}{2n+1}} \text{ hincque}$$

$$S=\tfrac{1}{4}RR+CR^{\frac{-2}{2n+1}} \text{ atque}$$

$$Qdx=\frac{-RdR}{4(2n+1)}-\frac{C}{2n+1}R^{\frac{-2n-3}{2n+1}}dR \text{ et } Pdx=\frac{ndR}{2n+1},$$

vnde aequationem obtinemus

$$(ny-\tfrac{1}{4}R-CR^{\frac{-2n-3}{2n+1}})dR+(2n+1)ydy=0$$

quae

quae integrabilis redditur per hunc multiplicatorem

$$(yy+Ry+\tfrac{1}{4}RR+CR^{\frac{-2}{2n+1}})^n.$$

Coroll. 1.

518. Casu quo $n=-\frac{1}{2}$, fit $dR=0$ et $R=A$, et reliquae aequationes sunt:

$$(n+1)APdx+2nQdx-ndS=0 \quad \text{et} \quad PSdx+nAQdx=0.$$

Ergo $Pdx=\frac{AQdx}{2S}=\frac{2Qdx-dS}{A}$, ideoque $(AA-4S)Qdx=-2SdS$, seu $Qdx=\frac{-2SdS}{AA-4S}$ et $Pdx=\frac{-AdS}{AA-4S}$ sicque haec aequatio $\frac{(Ay+2S)dS}{4S-AA}+ydy=0$ integrabilis redditur per hunc multiplicatorem $\frac{1}{\sqrt{(yy+Ay+S)}}$.

Coroll. 2.

519. Si hic ponamus $A=2a$ et $S=x$, haec aequatio $\frac{(ay+x)dx+2ydy(x-aa)}{(x-aa)\sqrt{(yy+2ay+x)}}=0$ per se est integrabilis, vnde integrale inueniri potest huius aequationis

$$xdx+aydx+2xydy-2aaydy=0$$

quae diuisa per $(x-aa)\sqrt{(yy+2ay+x)}$ fit integrabilis.

Coroll. 3.

520. Ad integrale inueniendum, sumatur primo x constans et partis $\frac{2ydy}{\sqrt{(yy+2ay+x)}}$ integrale est

$$2\sqrt{(yy+2ay+x)}+2al(a+y-\sqrt{(yy+2ay+x)})+X$$

 cuius

cuius differentiale sumto y constante

$$\frac{dx}{\sqrt{(yy+2ay+x)}} - \frac{adx:\sqrt{(yy+2ay+x)}}{a+y-\sqrt{(yy+2ay+x)}} + dX$$

si alteri aequationis parti $\frac{(ay+x)dx}{(x-aa)\sqrt{(yy+2ay+x)}}$ aequetur, reperitur $dX = \frac{adx}{aa-x}$ et $X = -al(aa-x)$. Ex quo integrale completum erit

$$\sqrt{(yy+2ay+x)} + al\frac{a+y-\sqrt{(yy+2ay+x)}}{\sqrt{(aa-x)}} = C$$

Coroll. 4.

521. Memoratu dignus est etiam casus $n=-1$, qui scripto a loco $C+\frac{1}{4}$ praebet hanc aequationem:

$$(y+aR)dR+ydy=0$$

quae diuisa per $yy+Ry+aRR$ fit integrabilis, haec autem aequatio est homogenea.

Scholion.

522. Potest etiam aequationis

$$(Py+Q)dx+ydy=0$$

multiplicator statui $(y+R)^m(y+S)^n$, fierique debet

$$\left(\frac{d.(Py+Q)(y+R)^m(y+S)^n}{dy}\right) = \left(\frac{d.y(y+R)^m(y+S)^n}{dx}\right)$$

vnde reperitur

$$Pdx(y+R)(y+S)+mdx(Py+Q)(y+S)+ndx(Py+Q)(y+R)$$
$$=my(y+S)dR+ny(y+R)dS$$

quae

quae euoluitur in

$$\left.\begin{array}{lll}(m+n+1)Pyydx & +(n+1)PRydx & +PRSdx \\ \quad -myydR & +(m+1)PSydx & +mQSdx \\ \quad -nyydS & +(m+n)Qydx & +nQRdx \\ & -mSydR & \\ & -nRydS & \end{array}\right\} = 0$$

vnde colligitur

$$Pdx = \frac{mdR+ndS}{m+n+1} \text{ et } Qdx = \frac{-PRSdx}{mS+nR} = \frac{-RS(mdR+ndS)}{(m+n+1)(mS+nR)};$$

hincque

$$\frac{(mdR+ndS)((n+1)R+(m+1)S)}{m+n+1} - \frac{(m+n)RS(mdR+ndS)}{(m+n+1)(mS+nR)} - mSdR - nRdS = 0$$

seu

$$\begin{array}{l} +m(n+1)RdR - mnRdS - \frac{m(m+n)RSdR - n(m+n)RSdS}{mS+nR} = 0 \\ +n(m+1)SdS - mnSdR \end{array}$$

quae reducitur ad hanc formam:

$$\begin{array}{l} +(n+1)RRdR + (m-n-1)RSdR - SSdR \\ +(m+1)SSdS + (n-m-1)RSdS - RRdS \end{array} = 0$$

quae cum sit homogenea, diuidatur per

$$(n+1)R^3 + (m-2n-1)R^2S + (n-2m-1)RSS + (m+1)S^3$$

seu per $(R-S)^2((n+1)R+(m+1)S)$ vt fiat integrabilis.

At ipsa illa aequatio per $R-S$ diuisa erit

$$(n+1)RdR + mSdR - nRdS - (m+1)SdS = 0.$$

Diuida-

Diuidatur per $(R-S)((n+1)R+(m+1)S)$ et resoluatur in fructiones partiales:

$$\frac{dR}{m+n+2}\left(\frac{m+n+1}{R-S}+\frac{n+1}{(n+1)R+(m+1)S}\right)+\frac{dS}{m+n+2}\left(\frac{m+n+1}{S-R}+\frac{m+1}{(n+1)R+(m+1)S}\right)=0$$

seu $\frac{(m+n+1)(dR-dS)}{R-S}+\frac{(n+1)dR+(m+1)dS}{(n+1)R+(m+1)S}=0$

vnde integrando obtinemus:

$$(R-S)^{m+n+1}((n+1)R+(m+1)S)=C.$$

Sit $R-S=u$ erit $(n+1)R+(m+1)S=\frac{C}{u^{m+n+1}}$ hincque

$$R=\frac{(m+1)u}{m+n+2}+\frac{a}{u^{m+n+1}} \text{ et } S=\frac{(n+1)u}{m+n+2}+\frac{a}{u^{m+n+1}}$$

tum vero

$$Pdx=\frac{(m-n)du}{m+n+2}-\frac{(m+n)adu}{u^{m+n+2}} \text{ et}$$

$$Qdx=+\frac{du}{u}\left(\frac{a}{u^{m+n+1}}+\frac{(m+1)u}{m+n+2}\right)\left(\frac{a}{u^{m+n+1}}-\frac{(n+1)u}{m+n+2}\right).$$

Coroll. 1.

523. Hinc ergo integrari potest ista aequatio

$$ydy+ydu\left(\frac{m-n}{m+n+2}-\frac{(m+n)a}{u^{m+n+2}}\right)+\frac{du}{u}\left(\frac{aa}{u^{2m+2n+2}}+\frac{(m-n)a}{(m+n+2)u^{m+n}}-\frac{(m+1)(n+1)uu}{(m+n+2)^2}\right)=0$$

quippe quae per se fit integrabilis, si multiplicetur per

$$\left(y+\frac{a}{u^{m+n+1}}+\frac{(m+1)u}{m+n+2}\right)^m\left(y+\frac{a}{u^{m+n+1}}-\frac{(n+1)u}{m+n+2}\right)^n.$$

Coroll. 2.

Coroll. 2.

524. Sit $m=n$, et aequatio nostra erit

$$ydy - \frac{2naydu}{u^{2n+2}} + \frac{aadu}{u^{4n+2}} - \tfrac{1}{4}udu = 0$$

cuius multiplicator est $((y+\frac{a}{u^{2n+1}})^2 - \frac{1}{4}uu)^n$. Quare si ponamus $y = z - \frac{a}{u^{2n+1}}$, aequatio prodit

$$zdz - \frac{adz}{u^{2n+1}} + \frac{azdu}{u^{2n+2}} - \tfrac{1}{4}udu = 0$$

quae integrabilis fit multiplicata per $(zz - \frac{1}{4}uu)^n$. Vel ponatur $z = \frac{1}{2}y$ et $a = \frac{1}{2}b$ erit

$$ydy - udu - \frac{bdy}{u^{2n+1}} + \frac{bydu}{u^{2n+2}} = 0$$

et multiplicator $(yy-uu)^n$.

Coroll. 3.

525. Si $m=-n$ prodit haec aequatio:

$$ydy - nydu + \tfrac{aadu}{u^3} + \tfrac{1}{4}(nn-1)udu - \tfrac{nadu}{u} = 0$$

quae integrabilis redditur multiplicata per

$$(y + \tfrac{a}{u} - \tfrac{1}{2}(n+1)u)^n (y + \tfrac{a}{u} - \tfrac{1}{2}(n-1)u)^{-n}.$$

Posito autem $y + \frac{a}{u} = z$ prodit haec aequatio

$$zdz - nzdu + \tfrac{1}{4}(nn-1)udu - \tfrac{adz}{u} + \tfrac{azdu}{uu} = 0$$

 quam

quam integrabilem reddit hic multiplicator:

$$(z-\tfrac{1}{2}(n+1)u)^n(z-\tfrac{1}{2}(n-1)u)^{-n}.$$

Coroll. 4.

526. Ponamus hic $z=uv$, et habebitur ista aequatio

$$uuvdv+udu(vv-nv+\tfrac{1}{4}(nn-1))=adv$$

quae si multiplicetur per $\left(\frac{v-\frac{1}{2}(n+1)}{v-\frac{1}{2}(n-1)}\right)^n$ vtrumque membrum fiet integrabile. Posito enim $\frac{v-\frac{1}{2}(n+1)}{v-\frac{1}{2}(n-1)}=s$, seu $v=\frac{n+1-(n-1)s}{2(1-s)}$ oritur $\frac{s^{n+1}udu}{(1-s)^2}+\frac{n+1-(n-1)s}{2(1-s)^3}uus^n ds$ $=\frac{as^n ds}{(1-s)^2}$; cuius integrale est $\frac{s^{n+1}uu}{2(1-s)^2}=a\int\frac{s^n ds}{(1-s)^2}$.

Scholion.

527. Quo nostram aequationem in genere concinniorem reddamus, ponamus $m=-\lambda-1+\mu$ et $n=-\lambda-1-\mu$ vt fit $m+n+2=-2\lambda$, fietque aequatio:

$$ydy-ydu\left(\tfrac{\mu}{\lambda}-2(\lambda+1)au^{2\lambda}\right)+udu\left(\tfrac{\mu\mu-\lambda\lambda}{4\lambda\lambda}-\tfrac{\mu}{\lambda}au^{2\lambda}+aau^{4\lambda}\right)=0$$

quae per hunc multiplicatorem integrabilis redditur

$$\left(y+au^{2\lambda+1}-\tfrac{(\mu-\lambda)u}{2\lambda}\right)^{\mu-\lambda-1}\left(y+au^{2\lambda+1}-\tfrac{(\mu+\lambda)u}{2\lambda}\right)^{-\mu-\lambda-1}.$$

Ponatur

Ponatur $y+au^{2\lambda+1}=uz$ et orietur haec aequatio

$$uzdz-au^{2\lambda+1}dz+du\left(zz-\frac{\mu}{\lambda}z+\frac{\mu\mu-\lambda\lambda}{4\lambda\lambda}\right)=0$$

cui respondet multiplicator:

$$u^{-2\lambda-1}\left(z+\frac{\lambda-\mu}{2\lambda}\right)^{\mu+\lambda-1}\left(z-\frac{\lambda-\mu}{2\mu}\right)^{-\mu-\lambda-1}.$$

Reperitur autem integrale

$$C=a\int dz\left(z+\frac{\lambda-\mu}{2\lambda}\right)^{\mu-\lambda-1}\left(z-\frac{\lambda-\mu}{2\lambda}\right)^{-\mu-\lambda-1}+\frac{1}{2\lambda u^{2\lambda}}\left(z+\frac{\lambda-\mu}{2\lambda}\right)^{\mu-\lambda}\left(z-\frac{\lambda-\mu}{2\lambda}\right)^{-\mu-\lambda}$$

quod ergo conuenit huic aequationi differentiali

$$zdz+\frac{du}{u}\left(z+\frac{\lambda-\mu}{2\lambda}\right)\left(z-\frac{\lambda-\mu}{2\lambda}\right)=au^{2\lambda}dz.$$

Problema 68.

528. Ipsius x functiones P, Q, R et X definire, vt haec aequatio $dy+yydx+Xdx=0$ integrabilis reddatur per hunc multiplicatorem $\frac{1}{Pyy+Qy+R}$.

Solutio.

Debet ergo esse

$$\frac{1}{dy}d.\frac{yy+X}{Pyy+Qy+R}=\frac{1}{dx}d.\frac{1}{Pyy+Qy+R}$$

hincque

$$2y(Pyy+Qy+R)-(yy+X)(2Py+Q)=-\frac{yydP-ydQ-dR}{dx}$$

ergo fieri debet

$$\left.\begin{matrix}+Qyydx+2Rydx-QXdx\\+yydP-2PXydx+dR\\+ydQ\end{matrix}\right\}=0$$

 Quare

Quare habetur $Q = -\frac{dP}{dx} = \frac{dR}{Xdx}$, et $X = -\frac{dR}{dP}$. Sumto ergo dx conſtante eſt $dQ = -\frac{ddP}{dx}$, vnde fieri oportet

$$2Rdx + \frac{2PdRdx}{dP} - \frac{ddP}{dx} = 0 \text{ ſeu}$$

$$RdP + PdR = \frac{dPddP}{2dx^2}, \text{ cuius integratio praebet}$$

$$PR = \frac{dP^2}{4dx^2} + C, \text{ hinc } R = \frac{dP^2}{4Pdx^2} + \frac{C}{P}$$

tum

$$Q = -\frac{dP}{dx} \text{ et } X = \frac{C}{PP} + \frac{dP^2}{4PPdx^2} - \frac{ddP}{2Pdx^2}.$$

Ponamus $P = SS$, vt S ſit functio quaecunque ipſius x, obtinebimusque:

$$P = SS;\ Q = -\frac{2SdS}{dx};\ R = \frac{C}{SS} + \frac{dS^2}{dx^2} \text{ et } X = \frac{C}{S^4} - \frac{ddS}{Sdx^2},$$

quibus ſumtis valoribus per ſe integrabilis erit haec aequatio $\frac{dy + yydx + Xdx}{Pyy + Qy + R} = 0$.

Scholion.

529. Haec ſolutio commodius inſtitui poterit ſi multiplicatori tribuatur haec forma $\frac{P}{yy + 2Qy + R}$, vt fieri debeat

$$\frac{1}{dy} d.\frac{P(yy + X)}{yy + 2Qy + R} = \frac{1}{dx} d.\frac{P}{yy + 2Qy + R},$$

vnde oritur:

$$\left.\begin{array}{lll} 2PQyydx & +2PRydx & -2PQXdx \\ -yydP & -2PXydx & -RdP \\ & -2QydP & +PdR \\ & +2QydQ & \end{array}\right\} = 0$$

vbi ex ſingulis commode definitur $\frac{dP}{P}$ ſcilicet

$$\frac{dP}{P} = 2Q\,dx = \frac{R\,dx - X\,dx + dQ}{Q} = \frac{dR - 2QX\,dx}{R}$$

Hinc colligitur $2Q(R+X)dx = dR$, vnde nunc ipſum elementum dx definiamus, $dx = \frac{dR}{2Q(R+X)}$, quo valore ſubſtituto adipiſcimur:

$$\frac{Q\,dR}{R+X} = \frac{(R-X)\,dR}{2Q(R+X)} + dQ \text{ ſeu}$$

$$2QQ\,dR = R\,dR - X\,dR + 2QR\,dQ + 2QX\,dQ$$

vnde colligimus $X = \frac{2QQdR - 2QRdQ - RdR}{2QdQ - dR}$ et $R+X = \frac{2(QQ-R)dR}{2QdQ - dR}$, hinc $dx = \frac{2QdQ - dR}{4Q(QQ-R)}$ atque $\frac{dP}{P} = \frac{2QdQ - dR}{2(QQ-R)}$ ideoque $P = A\sqrt{(QQ-R)}$.

Fiat $QQ - R = S$ ac reperietur:

$$dx = \frac{dS}{4QS};\quad X = \frac{4QS\,dQ}{dS} - QQ - S;\quad R = QQ - S$$

atque $P = A\sqrt{S}$. Quocirca habebimus hanc aequationem:

$$dy + \frac{yy\,dS}{4QS} + dQ - \frac{(QQ+S)\,dS}{4QS} = 0$$

quae integrabilis redditur, per hunc multiplicatorem

$$\frac{\sqrt{S}}{yy + 2Qy + QQ - S} = \frac{\sqrt{S}}{(y+Q)^2 - S}.$$

Ad eius integrale inueniendum, ſumantur Q et S conſtantes, prodibitque

$$\int \frac{dy\sqrt{S}}{(y+Q)^2 - S} = \frac{1}{2} l \frac{y+Q-\sqrt{S}}{y+Q+\sqrt{S}} + V$$

existente V certa functione ipsius S vel Q. Iam differentietur haec forma sumta y constante, proditque

$$\frac{dQ\sqrt{S}-\frac{(Q+y)dS}{2\sqrt{S}}}{(y+Q)^2-S}+dV=\frac{yydS+4QSdQ-QQdS-SdS}{4Q((y+Q)^2-S)\sqrt{S}}$$

ideoque

$$dV=\frac{yydS+2QydS+QQdS-SdS}{4Q((y+Q)^2-S)\sqrt{S}}=\frac{dS}{4Q\sqrt{S}}.$$

Ex quo aequationis nostrae integrale est

$$\tfrac{1}{2}l\frac{y+Q-\sqrt{S}}{y+Q+\sqrt{S}}+\tfrac{1}{4}\int\frac{dS}{Q\sqrt{S}}=C.$$

Coroll. 1.

530. Singularis est casus, quo $R=QQ$, fit enim

$$\frac{dP}{P}=2Qdx=\frac{QQdx-Xdx+dQ}{Q}=\frac{2dQ-2Xdx}{Q}$$

vnde has duas aequationes elicimus:

$$QQdx+Xdx-dQ=0 \text{ et } QQdx+Xdx-dQ=0$$

quae cum inter se conueniant, erit

$$Xdx=dQ-QQdx \text{ et } lP=2\int Qdx.$$

Coroll. 2.

531. Sumto ergo Q negatiuo, vt habeamus hanc aequationem

$$dy+yydx-dQ-QQdx=0$$

haec

haec integrabilis redditur, per hunc multiplicatorem $\frac{e^{-2\int Q\,dx}}{(y-Q)^2}$. Et integrale erit

$$\frac{-1}{y-Q}e^{-2\int Q\,dx}+V=\text{Const.}$$

vbi V est functio ipsius x, ad quam definiendam, differentietur sumta y constante:

$$\frac{-dQ}{(y-Q)^2}e^{-2\int Q\,dx}+\frac{2Q\,dx}{y-Q}e^{-2\int Q\,dx}+dV=\frac{yydx-dQ-QQdx}{(y-Q)^2}e^{-2\int Q\,dx}$$

vnde fit $V=\int e^{-2\int Q\,dx}dx$, ita vt integrale sit

$$\int e^{-2\int Q\,dx}dx-\frac{e^{-2\int Q\,dx}}{y-Q}=C.$$

Coroll. 3.

532. Proposita ergo aequatione

$$dy+yydx+Xdx=0$$

si eius integrale particulare quoddam constet, $y=Q$ vt sit

$$dQ+QQdx+Xdx=0$$

ideoque

$$dy+yydx-dQ-QQdx=0$$

multiplicator pro ea erit $\frac{1}{(y-Q)^2}e^{-2\int Q\,dx}$ et integrale completum

$$Ce^{2\int Q\,dx}+\frac{1}{y-Q}=e^{2\int Q\,dx}\int e^{-2\int Q\,dx}dx$$

Scholion.

Scholion.

533. Aequatio autem in praecedente scholio inuenta

$$dy + \frac{yvdS}{4QS} + dQ - \frac{(QQ+S)dS}{4QS} = 0$$

non multum habet in recessu, posito enim $y + Q = z$ prodit

$$dz - \frac{zdS}{2S} + \frac{dS(zz-S)}{4QS} = 0$$

in qua vt bini priores termini in vnum contrahantur, ponatur $z = v\sqrt{S}$, reperieturque

$$dv\sqrt{S} + \frac{vvdS}{4Q} - \frac{dS}{4Q} = 0 \text{ seu } \frac{dv}{vv-1} + \frac{dS}{4Q\sqrt{S}} = 0$$

quae cum sit separata integrale erit $\frac{1}{2}l\frac{1+v}{1-v} = \frac{1}{4}\int\frac{dS}{Q\sqrt{S}}$ vbi est $v = \frac{y+Q}{\sqrt{S}}$.

Aequatio autem in ipsa solutione inuenta

$$dy + yydx + \frac{Cdx}{S^4} - \frac{ddS}{Sdx} = 0$$

vbi S est functio quaecunque ipsius x, et $\frac{ddS}{dx} = d.\frac{dS}{dx}$, magis ardua videtur, dum per se fit integrabilis si diuidatur per

$$SSyy - \frac{2SydS}{dx} + \frac{dS^2}{dx^2} + \frac{C}{SS} = (Sy - \frac{dS}{dx})^2 + \frac{C}{SS}.$$

At sumto x constante integrale reperitur

$$\frac{1}{\sqrt{C}}\text{Arc. tang.}\frac{SSydx - SdS}{dx\sqrt{C}} + V = \text{Const.}$$

nunc ergo ad functionem V inueniendam sumatur differentiale posita y constante, quod est

$$\frac{2SydS - \frac{SddS}{dx} - \frac{dS^2}{dx}}{SS(Sy - \frac{dS}{dx})^2 + C} + dV$$

et

et aequari debet alteri parti

$$\frac{\frac{Cdx}{S^4} - \frac{ddS}{Sdx} + yydx}{(Sy - \frac{dS}{dx})^2 + \frac{C}{SS}} = \frac{\frac{Cdx}{SS} - \frac{SddS}{dx} + SSyydx}{SS(Sy - \frac{dS}{dx})^2 + C}.$$

Ergo

$$dV = \frac{SS\,yydx - 2S\,ydS + \frac{dS^2}{dx} + \frac{Cdx}{SS}}{SS(Sy - \frac{dS}{dx})^2 + C} = \frac{dx}{SS}.$$

Quocirca integrale completum est

$$\frac{1}{\sqrt{C}}\text{ Arc. tang. } \frac{SSydx - SdS}{dx\sqrt{C}} + \int \frac{dx}{SS} = D.$$

Quod si sumamus $S = x$ huius aequationis

$$dy + yydx + \frac{Cdx}{x^4} = 0$$

integrale completum est

$$\frac{1}{\sqrt{C}}\text{ Arc. tang. } \frac{xxy - x}{\sqrt{C}} - \frac{1}{x} = D.$$

Sin autem fit $S = x^n$, ob $\frac{dS}{dx} = nx^{n-1}$ et $d.\frac{dS}{dx} = n(n-1)x^{n-2}dx$ integrari poterit haec aequatio

$$dy + yydx + \frac{Cdx}{x^{4n}} - \frac{n(n-1)dx}{xx} = 0$$

integrale enim erit

$$\frac{1}{\sqrt{C}}\text{ Arc. tang. } \frac{x^{2n}y - nx^{2n-1}}{\sqrt{C}} - \frac{1}{(2n-1)x^{2n-1}} = D.$$

Supra autem inuenimus hanc aequationem

$$dy + yydx + Cx^m dx = 0$$

ad separationem reduci posse, quoties fuerit $m = \frac{-4i}{2i \pm 1}$, iisdem ergo casibus functionem S assignare licebit,

vt fiat $\frac{C}{s^4} - \frac{dds}{sdx^2} = Cx^m$; quod cum ad aequationes differentiales secundi gradus pertineat, hic non attingemus.

Problema 69.

534. Definire functiones P et Q ambarum variabilium x et y, vt aequatio differentialis $Pdx+Qdy=0$, diuisa per $Px+Qy$ fiat per se integrabilis.

Solutio.

Cum formula $\frac{Pdx+Qdy}{Px+Qy}$ debeat esse integrabilis, statuamus $Q=PR$, vt habeamus $\frac{dx+Rdy}{x+Ry}$, sitque $dR=Mdx+Ndy$. Quare fieri oportet $\frac{1}{dy}d.\frac{1}{x+Ry} = \frac{1}{dx}d.\frac{R}{x+Ry}$, vnde nanciscimur $\frac{-R-Ny}{(x+Ry)^2} = \frac{Mx-R}{(x+Ry)^2}$ seu $N=-\frac{Mx}{y}$; hinc fit $dR=Mdx-\frac{Mxdy}{y}=My.\frac{ydx-xdy}{yy}$, quae formula cum debeat esse integrabilis, necesse est sit My functio ipsius $\frac{x}{y}$, quia $\frac{ydx-xdy}{yy}=d.\frac{x}{y}$; atque ex hac integratione prodit $R=$ funct. $\frac{x}{y}$; seu quod eodem redit, R erit functio nullius dimensionis ipsarum x et y. Quocirca cum $\frac{Q}{P}=R$, manifestum est huic conditioni satisfieri, si P et Q fuerint functiones homogeneae eiusdem dimensionum numeri ipsarum x et y; hoc ergo modo eandem integrationem aequationum homogenearum sumus assecuti, quam in capite superiori docuimus.

Coroll. 1.

535. Cum igitur $\frac{dt+Rdu}{t+Ru}$ sit integrabile si, fuerit $R=f:\frac{t}{u}$, seu $R=\frac{t}{u}f:\frac{t}{u}$, erit etiam haec for-

formula $\frac{\frac{dt}{t}+\frac{du}{u}f:\frac{t}{u}}{1+f:\frac{t}{u}}$ integrabilis, quae ita repraesentari potest $\frac{\frac{dt}{t}+\frac{du}{u}f:(\int\frac{dt}{t}-\int\frac{du}{u})}{1+f:(\int\frac{dt}{t}-\int\frac{du}{u})}$, vbi littera f denotat functionem quamcunque quantitatis suffixae.

Coroll. 2.

536. Ponatur $\frac{dt}{t}=\frac{dx}{X}$ et $\frac{du}{u}=\frac{dy}{Y}$, atque haec formula:

$$\frac{\frac{dx}{X}+\frac{dy}{Y}f:(\int\frac{dx}{X}-\int\frac{dy}{Y})}{1+f:(\int\frac{dx}{X}-\int\frac{dy}{Y})}=\frac{dx+\frac{X\,dy}{Y}f:(\int\frac{dx}{X}-\int\frac{dy}{Y})}{X+Xf:(\int\frac{dx}{X}-\int\frac{dy}{Y})}$$

erit per se integrabilis. Quare posito $R=\frac{X}{Y}f:(\int\frac{dx}{X}-\int\frac{dy}{Y})$ haec formula $\frac{dx+R\,dy}{X+RY}$ erit per se integrabilis, quaecunque functio sit X ipsius x, et Y ipsius y.

Coroll. 3.

537. Quare si quaerantur functiones P et Q, vt haec aequatio $Pdx+Qdy=0$ fiat integrabilis, si diuidatur per $PX+QY$ existente X functione quacunque ipsius x, et Y ipsius y, debet esse

$$\frac{Q}{P}=\frac{X}{Y}\text{ funct. }(\int\frac{dx}{X}-\int\frac{dy}{Y}).$$

Coroll. 4.

538. Quare si signa Φ et ψ functiones quascunque indicent, fueritque

$$P=\frac{V}{X}\Phi(\int\frac{dx}{X}-\int\frac{dy}{Y})\text{ et }Q=\frac{V}{Y}\psi(\int\frac{dx}{X}-\int\frac{dy}{Y})$$

 haec

haec aequatio $Pdx+Qdy=0$ integrabilis reddetur, si diuidatur per $PX+QY$.

Scholion.

539. Hinc ergo innumerabiles aequationes proferri possunt, quas integrare licebit, etiamsi alioquin difficillime pateat, quomodo eae ad separationem variabilium reduci queant. Verum haec inuestigatio proprie ad librum secundum Calculi Integralis est referenda, cuius iam egregia specimina hic habentur; definiuimus enim functionem R binarum variabilium x et y ex certa conditione inter M et N proposita scilicet $Mx+Ny=0$ seu $x(\frac{dR}{dx})+y(\frac{dR}{dy})=0$, hoc est ex certa differentialium conditione.

CAPVT IV.

CAPVT IV.

DE INTEGRATIONE PARTICVLARI AEQVATIONVM DIFFERENTIALIVM.

Definitio.

540.

Integrale particulare aequationis differentialis est relatio variabilium aequationi satisfaciens, quae nullam nouam quantitatem constantem in se complectitur. Opponitur ergo integrali completo, quod constantem in differentiali non contentam inuoluit, in quo tamen contineatur necesse est.

Coroll. 1.

541. Cognito ergo integrali completo, ex eo innumerabilia integralia particularia exhiberi possunt, prout constanti illi arbitrariae alii atque alii valores determinati tribuuntur.

Coroll. 2.

542. Proposita ergo aequatione differentiali inter variabiles x et y, omnes functiones ipsius x, quae loco y substitutae aequationi satisfaciunt, dabunt integralia particularia, nisi forte sint completa.

Coroll. 3.

543. Cum omnis aequatio differentialis ad hanc formam $\frac{dy}{dx} = V$ reuocetur, exiſtente V functione quacunque ipſarum x et y, ſi eiusmodi conſtet relatio inter x et y, vnde pro $\frac{dy}{dx}$ et V reſultent valores aequales, ea pro integrali particulari erit habenda.

Scholion 1.

544. Interdum facile eſt integrale particulare quaſi diuinatione colligere; veluti ſi propoſita ſit haec aequatio

$$aady + yydx = aadx + xydx.$$

Statim liquet ei ſatisfieri ponendo $y = x$, quae relatio cum non ſolum nullam nouam conſtantem, ſed ne eam quidem a, quae in ipſa aequatione differentiali continetur, implicet, vtique eſt integrale particulare: vnde nihil pro integrali compl[e]to colligere licet. Saepe numero quidem cognitio integralis particularis ad inuentionem completi viam patefacit, quemadmodum in hoc ipſo exemplo vſu venit, in quo ſi ſtatuamus $y = x + z$ fit

$$aadx + aadz + xxdx + 2xzdx + zzdx = aadx + xxdx + xzdx$$

ſeu $aadz + xzdx + zzdx = 0$

quae aequatio poſito $z = \frac{aa}{v}$ abit in hanc

$$dv - \frac{xvdx}{aa} = dx$$

quae

quae per $e^{-\int\frac{xdx}{aa}}=e^{\frac{-xx}{2aa}}$ multiplicata fit integrabilis, et dat:

$$e^{\frac{-xx}{2aa}}v=\int e^{\frac{-xx}{2aa}}dx \text{ seu } v=e^{\frac{xx}{2aa}}\int e^{\frac{-xx}{2aa}}dx$$

quod ergo eſt maxime tranſcendens, cum tamen ſimpliciſſimum illud particulare inuoluat: ſcilicet ſi conſtans integratione $\int e^{\frac{-xx}{2aa}}dx$ inuecta ſumatur infinita, fit $v=\infty$ et $z=0$ vnde $y=x$. Interdum autem integrale particulare parum iuuat ad completum inueſtigandum, veluti ſi habeatur haec aequatio

$$a^3dy+y^3dx=a^3dx+x^3dx$$

cui manifeſto ſatisfacit $y=x$, poſito autem $y=x+z$ prodit

$$a^3dz+3xxzdx+3xzzdx+z^3dx=0$$

cuius reſolutio haud facilior videtur, quam illius.

Scholion 2.

545. In his exemplis integrale particulare ſtatim in oculos incurrit, dantur autem caſus quibus difficilius perſpicitur; et quanquam raro inde via pateat ad integrale completum perueniendi, tamen ſaepe numero plurimum intereſt integrale particulare noſſe, cum eo nonnunquam totum negotium confici poſſit Iam enim animaduertimus in omnibus problematibus, quorum ſolutio ad aequationem differentialem perducitur, conſtantem arbitrariam per

per integrationem inuectam ex ipsis conditionibus, cuique problemati adiunctis, determinari, ita vt semper integrali tantum particulari sit opus; quare si eueniat, vt hoc ipsum integrale particulare cognosci possit, sine subsidio completi, solutio problematis exhiberi poterit, etiamsi integratio aequationis differentialis non sit in potestate. Quibus ergo casibus sine integratione vera solutio inueniri est censenda; propterea quod proprie loquendo nulla aequatio differentialis integrari existimatur, nisi eius integrale completum assignetur. Quocirca vtile erit eos casus perpendere, quibus integrale particulare exhibere licet.

Scholion 3.

546. Maximi autem est momenti hic animaduertisse, non omnes valores aequationi cuipiam differentiali satisfacientes pro eius integrali particulari haberi posse. Veluti si habeatur haec aequatio $dy = \frac{dx}{\sqrt{(a-x)}}$, seu $\frac{dx}{dy} = \sqrt{(a-x)}$ posito $x = a$ fit tam $\sqrt{(a-x)} = 0$ quam $\frac{dx}{dy} = 0$, ita vt aequatio $x = a$ illi differentiali satisfaciat, cum tamen nequaquam eius sit integrale particulare. Integrale namque completum est $y = C - 2\sqrt{(a-x)}$ seu $a - x = \frac{1}{4}(C-y)^2$, vnde quicunque valor constanti C tribuatur, nunquam sequitur $a - x = 0$. Simili modo huic aequationi $dy = \frac{x\,dx + y\,dy}{\sqrt{(xx+yy-aa)}}$ satisfacit haec aequatio finita $xx + yy = aa$, quae tamen inter integralia particularia admitti nequit, propterea quod

in

in integrali completo $y = C + \sqrt{(xx + yy - aa)}$ neutiquam continetur. Quare ad integrale particulare non sufficit, vt eo aequationi differentiali satisfiat, sed insuper hanc conditionem adiungi oportet, vt in integrali completo contineatur; ex quo inuestigatio integralium particuliarium maxime est lubrica, nisi simul integrale completum innotescat; hoc autem cognito superuacuum esset methodo peculiari in integralia particularia inquirere. Tum enim potissimum iuuat ad inuestigationem integralium particularium confugere, quando integrale completum elicere non licet. Quo igitur hinc fructum percipere queamus, criteria tradi conueniet, ex quibus valores, qui aequationi cuipiam differentiali satisfaciunt, diiudicare liceat, vtrum sint integralia particularia, nec ne? Etiamsi scilicet omnia integralia sint eiusmodi valores, qui aequationi differentiali satisfaciant, tamen non vicissim omnes valores, qui satisfaciunt, sunt integralia. Quod cum parum adhuc sit animaduersum, operam dabo, vt hoc argumentum dilucide euoluam.

Problema 70.

547. Si in aequatione differentiali $dy = \frac{dx}{Q}$, functio Q euanescat posito $x = a$, determinare quibus casibus haec aequatio $x = a$ sit integrale particulare aequationis differentialis propositae.

Solutio.

Cum sit $Q=\frac{dx}{dy}$, posito $x=a$ fit tam $Q=0$ quam $\frac{dx}{dy}=0$, vnde hic valor $x=a$ aequationi differentiali propositae $dy=\frac{dx}{Q}$ vtique satisfacit, neque tamen hinc sequitur eum esse integrale. Hoc solum scilicet non sufficit, sed insuper requiritur, vt aequatio $x=a$ in integrali completo contineatur, si quidem constanti per integrationem inuectae certus quidam valor tribuatur. Ponamus ergo P esse integrale formulae $\frac{dx}{Q}$, vt integrale completum sit $y=C+P$; cui aequationi ponendo $x=a$ satisfieri nequit, nisi posito $x=a$ fiat $P=\infty$, tum enim sumta constante C pariter infinita positione $x=a$ quantitas y manet indeterminata, ideoque si posito $x=a$ fiat $P=\infty$, tum demum aequatio $x=a$ pro integrali particulari erit habenda. En ergo criterium, ex quo dignoscere licet, vtrum valor $x=a$ aequationi differentiali $dy=\frac{dx}{Q}$ satisfaciens simul sit eius integrale particulare nec ne? scilicet tum demum erit integrale, si posito $x=a$ non solum fiat $Q=0$, sed etiam integrale $P=\int\frac{dx}{Q}$ abeat in infinitum. Quod quo clarius exponamus, quoniam posito $x=a$ fit $Q=0$, ponamus $Q=(a-x)^n R$, denotante n numerum quemcunque positiuum, et cum aequatio

$$dy=\frac{dx}{Q}=\frac{dx}{(a-x)^n R}$$

induere

induere queat hanc formam

$$dy = \frac{\alpha dx}{(a-x)^n} + \frac{\beta dx}{(a-x)^{n-1}} + \frac{\gamma dx}{(a-x)^{n-2}} + \ldots\ldots + \frac{Sdx}{R}$$

ratio illius infiniti P pendebit a termino $\int \frac{\alpha dx}{(a-x)^n}$, qui si posito $x = a$ euadat infinitus, etiam integrale $P = \int \frac{dx}{Q}$ erit infinitum, vtcunque se habeant reliqua membra. At est $\int \frac{\alpha dx}{(a-x)^n} = \frac{\alpha}{(n-1)(a-x)^{n-1}}$ quae expressio fit infinita posito $x = a$, dummodo $n-1$ sit numerus positiuus, vel etiam $n = 1$. Quare dummodo exponens n non sit vnitate minor, posito $Q = (a-x)^n R$ aequatio $x = a$ pro integrali particulari erit habenda.

Coroll. 1.

548. Quoties ergo posito $Q = (a-x)^n R$ exponens n est vnitate minor, aequationi $dy = \frac{dx}{Q}$ non conuenit integrale particulare $x = a$, etiamsi hoc modo aequationi differentiali satisfiat.

Coroll. 2.

549. Si exponens n est vnitate minor, formula $\frac{dQ}{dx}$ fit infinita posito $x = a$; vnde nouum criterium adipiscimur: Scilicet proposita aequatione $dy = \frac{dx}{Q}$, si posito $x = a$ fiat quidem $Q = 0$, at $\frac{dQ}{dx} = \infty$, tum valor $x = a$ non est integrale particulare illius aequationis.

Coroll. 3.

550. His igitur casibus exclusis aequationis $dy=\frac{dx}{Q}$ vbi posito $x=a$ fit $Q=0$, integrale particulare semper erit $x=a$, nisi eodem casu $x=a$ fiat $\frac{dQ}{dx}=\infty$; hoc est quoties valor formulae $\frac{dQ}{dx}$ fuerit vel finitus vel euanescat.

Scholion 1.

551. Haec conclusio inuersioni propositionum hypotheticarum innixa licet videri queat suspecta ac regulis Logicae aduersa, verum totum ratiocinium regulis apprime est consentaneum, cum a sublatione consequentis ad sublationem antecedentis concludat. Quoties enim posito $Q=(a-x)^n R$ exponens n est vnitate minor, toties $\frac{dQ}{dx}$ fit $=\infty$ posito $x=a$. Quare si posito $x=a$ non fiat $\frac{dQ}{dx}=\infty$, ideoque eius valor vel finitus, vel euanescat, tum certe exponens n non est vnitate minor, erit ergo vel maior vnitate vel ipsi aequalis, vtroque autem casu integrale $P=\int\frac{dx}{Q}$ posito $x=a$ fit infinitum, ideoque aequatio $x=a$ est integrale particulare. Quare si in aequatione differentiali $dy=\frac{dx}{Q}$ posito $x=a$ fiat $Q=0$, examinatur valor $\frac{dQ}{dx}$ pro casu $x=a$, qui si fuerit vel finitus vel euanescat, aequatio $x=a$ est integrale particulare; sin autem is sit infinitus, ea inter integralia locum non habet, etiamsi aequationi differentiali satisfiat. Eadem regula quoque locum habet,

habet, si aequatio differentialis fuerit huiusmodi $dy = \frac{P\,dx}{Q}$ seu $\frac{dy}{dx} = \frac{P}{Q}$ ac posito $x = a$ fiat $Q = 0$, quaecunque fuerit P functio ipsarum x et y; quin etiam necesse non est, vt Q sit functio solius variabilis x, sed simul alteram y vtcunque implicare potest.

Scholion 2.

552. Demonstratio quidem inde est petita, quod quantitas Q, quae posito $x = a$ euanescit, factorem implicet potestatem quampiam ipsius $a - x$, quod in functionibus algebraicis est manifestum. Verum in functionibus transcendentibus eadem regula locum habet, cum potestate talibus dignitatibus aequiualeant. Veluti si sit $dy = \frac{dx}{lx - la}$, vbi $Q = lx - la = l\frac{x}{a}$, fitque $Q = 0$ posito $x = a$, quaeratur $\frac{dQ}{dx} = \frac{1}{x}$, quae formula cum non fiat infinita posito $x = a$, integrale particulare erit $x = a$. Quod etiam valet pro aequatione $dy = \frac{P\,dx}{lx - la}$, dummodo P non fiat $= 0$ posito $x = a$. Sit enim $P = \frac{1}{x}$, erit integrando $y = C + l(lx - la)$ et $l\frac{x}{a} = e^{y - C}$. Sumta iam constante $C = \infty$ fit $l\frac{x}{a} = 0$ ideoque $x = a$, quod ergo est integrale particulare. Simili modo si fit $dy = P\,dx : (e^{\frac{x}{a}} - e)$, vbi $Q = e^{\frac{x}{a}} - e$ ideoque posito $x = a$ fit $Q = 0$; quia $\frac{dQ}{dx} = \frac{1}{a}e^{\frac{x}{a}}$, hincque posito $x = a$ fit $\frac{dQ}{dx} = \frac{e}{a}$, erit $x = a$ etiam integrale particulare. Sumatur $P = e^{\frac{x}{a}}$ vt integratio succedat, et quia

 $y = C$

$y = C + al(e^{\frac{x}{a}} - e)$, hincque $e^{\frac{x}{a}} = e + e^{\frac{y-C}{a}}$ statuatur $C = \infty$, erit $e^{\frac{x}{a}} = e$, ideoque $x = a$, quod ergo manifesto est integrale particulare.

Exemplum 1.

553. *Proposita aequatione differentiali* $dy = \frac{P\,dx}{\sqrt{S}}$, *in qua* S *euanescat posito* $x = a$, *definire casus, quibus aequatio* $x = a$ *est eius integrale particulare.*

Cum hic sit $\sqrt{S} = Q$, erit $dQ = \frac{dS}{2\sqrt{S}}$: ergo vt integrale particulare sit $x = a$, necesse est, vt posito $x = a$ fiat $\frac{dQ}{dx} = \frac{dS}{2\,dx\sqrt{S}}$ quantitas finita. Hinc eodem casu quantitas $\frac{dS^2}{S\,dx^2}$ fieri debet finita, vnde cum S euanescat, etiam $\frac{dS^2}{dx^2}$ ac proinde $\frac{dS}{dx}$ euanescere debet: Tum autem posito $x = a$ illius fractionis valor est $\frac{2\,dS\,ddS}{dS\,dx^2} = \frac{2\,ddS}{dx^2}$, quem ergo finitum esse oportet, vel $= 0$. Quare vt aequatio $x = a$ sit integrale particulare aequationis propositae, hae conditiones requiruntur, primo vt posito $x = a$ fiat $S = 0$. Secundo vt fiat $\frac{dS}{dx} = 0$, ac tertio vt huius formulae $\frac{ddS}{dx^2}$ valor prodeat vel finitus, vel $= 0$, dummodo ne fiat infinite magnus. Si S sit functio rationalis haec eo redeunt, vt S factorem habeat $(a - x)^2$ vel potestatem altiorem.

Scholion.

Scholion.

554. Haec resolutio vsum habet in motu corporis ad centrum virium attracti dignoscendo, num in circulo fiat. Si enim distantia corporis a centro ponatur $= x$, et vis centripeta huic distantiae conueniens $= X$ pro tempore t talis reperitur aequatio $dt = \frac{x\,dx}{\sqrt{(Exx - c^4 - 2\alpha xx \int X\,dx)}}$, vbi E est constans per praecedentem integrationem ingressa, cuius valor quaeritur, vt hinc aequationi satisfaciat valor $x = a$, quo casu corpus in circulo reuoluetur. Hic ergo est $S = Exx - c^4 - 2\alpha xx \int X dx$, vel sumi potest $S = E - \frac{c^4}{xx} - 2\alpha \int X\,dx$. Non solum ergo haec quantitas, sed etiam eius differentiale $\frac{dS}{dx} = \frac{2c^4}{x^3} - 2\alpha X$ euanescere debet posito $x = a$, neque tamen differentio-differentiale $\frac{ddS}{dx^2} = -\frac{6c^4}{x^4} - \frac{2\alpha\,d\,X}{dx}$ in infinitum abire debet. Inde ergo constans a erit valor ipsius x, ex hac aequatione $\alpha x^3 X = c^4$ resultans, qui est radius circuli, in quo corpus reuolui poterit, dummodo constans E, a qua celeritas pendet, ita fuerit comparata, vt posito $x = a$ fiat $E = \frac{c^4}{aa} + 2\alpha \int X dx$; nisi forte eodem casu expressio $\frac{6c^4}{x^4} + \frac{2\alpha\,dX}{dx}$ seu saltem haec $\frac{dX}{dx}$ fiat infinita. Hoc enim si eueniret motus in circulo tolleretur; ad quod ostendendum ponamus $X = b + \sqrt{(a-x)}$, vt $\frac{dX}{dx} = -\frac{1}{2\sqrt{(a-x)}}$ fiat infinitum posito $x = a$, et aequatio $\alpha x^3 X = c^4$ dabit $\alpha a^3 b = c^4$. Tum vero ob

$\int X dx = bx - \frac{2}{3}(a-x)^{\frac{3}{2}}$ erit $E = \alpha ab + 2\alpha ab = 3\alpha ab$

nostra-

noſtraque aequatio fit

$$dt = \frac{x\,dx}{\sqrt{(3\alpha abxx - \alpha a^3 b - 2\alpha bx^3 + \frac{4}{3}\alpha xx(a-x)^{\frac{3}{2}})}}$$

cui valor $x = a$ certe non conuenit tanquam integrale. Fit enim

$$S = \alpha(a-x)(-aab - abx + 2bxx + \tfrac{4}{3}xx\sqrt{(a-x)})$$

cuius factor cum non ſit $(a-x)$ ſed tantum $(a-x)^{\frac{3}{2}}$, integrale particulare $x = a$ locum habere nequit.

Exemplum 2.

555. *Propoſita aequatione differentiali* $dy = \frac{P\,dx}{\sqrt[n]{S^m}}$, *in qua* S *euaneſcat poſito* $x = a$, *inuenire caſus quibus integrale particulare eſt* $x = a$.

Cum fiat $S = 0$ poſito $x = a$, concipere licet $S = (a-x)^\lambda R$, eritque denominator $\sqrt[n]{S^m} = (a-x)^{\frac{\lambda m}{n}} R^{\frac{m}{n}}$, vnde patet aequationem $x = a$ fore integrale particulare aequationis propoſitae, ſi fuerit $\frac{\lambda m}{n}$ numerus poſitiuus vnitate maior, ſeu ſaltem vnitati aequalis, hoc eſt, ſi ſit vel $\lambda = \frac{n}{m}$ vel $\lambda > \frac{n}{m}$, quae diiudicatio ſi S ſit functio algebraica, facillime inſtituitur. Sin autem ſit tranſcendens, vt exponens λ in numeris exhiberi nequeat, vti licebit altera regula: ſcilicet, cum ſit $\sqrt[n]{S^m} = Q$, erit $\frac{dQ}{dx} = \frac{mS^{\frac{m-n}{n}}\,dS}{n\,dx}$ cuius

valor

valor debet esse finitus vel nullus posito $x=a$, siquidem integrale sit $x=a$. Sit igitur quoque necesse est hoc casu quantitas $\frac{S^{m-n}dS^n}{dx^n}$ finita. Quaeratur ergo huius formulae valor casu $x=a$, qui si prodeat infinite magnus, aequatio $x=a$, non erit integrale, sin autem sit vel finitus vel nullus, erit ea certe integrale particulare aequationis propositae. Hic duo constituendi sunt casus, prout fuerit vel $m>n$ vel $m<n$.

I. Si $m>n$, quia posito $x=a$ fit $S^{m-n}=c$, nisi eodem casu fiat $\frac{dS}{dx}=\infty$ certe erit $x=a$ integrale. Sin autem fiat $\frac{dS}{dx}=\infty$, vtrumque euenire potest, vt sit integrale et vt non sit. Ad quod dignoscendum ponatur $\frac{dx}{dS}=T$, vt nostra formula euadat $\frac{S^{m-n}}{T^n}$, cuius tam numerator, quam denominator euanescit posito $x=a$, ex quo eius valor reducitur ad

$$\frac{(m-n)S^{m-n-1}dS}{nT^{n-1}dT}=\frac{-(m-n)S^{m-n-1}dS^{n+1}}{ndx^n ddS}$$

qui si sit vel finitus vel nullus, integrale erit $x=a$. Simili modo vlterius progredi licet distinguendo casus $m>n+1$ et $m<n+1$.

II. Si $m<n$, formula nostra erit $\frac{dS^n}{S^{n-m}dx^n}$, cuius valor vt fiat finitus, necesse est vt sit $\frac{dS}{dx}=0$, ac praeterea, quia numerator ac denominator posito

$x = a$ euanescit, formulae nostrae valor erit

$$= \frac{ndS^{n-1}ddS}{(n-m)S^{n-m-1}dSdx^n} = \frac{ndS^{n-1}ddS}{(n-m)S^{n-m-1}dx^n},$$

quem finitum esse oportet.

Facillime autem iudicium absoluetur ponendo statim $x = a + \omega$, cum enim posito $x = a$ fiat $S = 0$, hac substitutione quantitas S semper resolui poterit in huiusmodi formam

$$P\omega^\alpha + Q\omega^\beta + R\omega^\gamma + \text{etc.}$$

cuius tantum vnus terminus $P\omega^\alpha$ infimam potestatem ipsius ω complectens spectetur; ac si fuerit vel $\alpha = \frac{n}{m}$ vel $\alpha > \frac{n}{m}$, aequatio $x = a$ certe erit integrale particulare.

Scholion.

556. Haec vltima methodus est tutissima, ac semper etiam in formulis transcendentibus optimo successu adhiberi potest. Scilicet proposita aequatione $dy = \frac{Pdx}{Q}$, in qua posito $x = a$ fiat $Q = 0$, neque vero etiam numerator P euanescat: statuatur $x = a + \omega$, et quantitas ω spectetur vt infinite parua; vt omnes eius potestates prae infima euanescant, atque quantitas Q huiusmodi formam $R\omega^\lambda$ accipiet, ex qua patebit nisi exponens λ vnitate fuerit minor, aequationem $x = a$ certe fore integrale particulare aequationis propositae. Veluti si habeamus $dy = \frac{dx}{\sqrt{(1 + \operatorname{cos}. \frac{\pi x}{a})}}$, cuius denominator euanescit

sumto

ſumto $x = a$ ob coſ. $\pi = -1$, ponamus $x = a - \omega$, erit coſ. $\frac{\pi x}{a} =$ coſ. $(\pi - \frac{\pi\omega}{a}) = -1 + \frac{\pi\pi\omega\omega}{2aa}$ ob ω infinite paruum, hinc noſtrae aequationis denominator fiet $= \frac{\pi\omega}{a\sqrt{2}}$, vnde concludimus integrale particulare vtique eſſe $x = a$. Non autem foret integrale huius aequationis $dy = \frac{dx}{\sqrt[3]{(1 + \text{coſ.} \frac{\pi x}{a})}}$.

Problema 71.

557. Propoſita aequatione differentiali, in qua variabiles ſunt a ſe inuicem ſeparatae, inueſtigare eius integralia particularia.

Solutio.

Sit propoſita haec aequatio $\frac{dx}{X} = \frac{dy}{Y}$, in qua X ſit functio ipſius x, et Y ipſius y tantum. Ac primo ponatur $X = 0$ indeque quaerantur valores ipſius x, quorum quisque ſit $x = a$, ita vt poſito $x = a$, fiat $X = 0$; tum examinetur valor formulae $\frac{dX}{dx}$ poſito $x = a$, qui niſi fiat infinitus, aequationis propoſitae integrale particulare certe erit $x = a$. Vel ponatur $x = a + \omega$, ſpectando ω vt quantitatem infinite paruam, ac ſi prodeat $X = P\omega^\lambda$, exponens λ, niſi ſit vnitate minor, indicabit integrale $x = a$; ſin autem ſit vnitate minor, aequatio $x = a$ pro integrali non erit habenda.

Simili modo examinetur alterius partis denominator Y qui ſi euaneſcat poſito $y = b$, hocque

casu formula $\frac{dY}{dy}$ non fiat infinita, aequatio $y=b$ erit integrale particulare; quod ergo etiam euenit, si posito $y=b\pm\omega$, prodeat $Y=Q\omega^{\lambda}$, vbi exponens λ vnitate non sit minor.

Coroll. 1.

558. Nisi ergo membra aequationis separatae fuerint fractiones, quarum denominatores certis casibus euanescant, huiusmodi integralia particularia non dantur; nisi forte in tali aequatione $Pdx=Qdy$, factores P et Q certis casibus fiant infiniti, qui autem casus ad praecedentem facile reducitur.

Coroll. 2.

559. Veluti si habeatur $dx\,\mathrm{tang.}\frac{\pi x}{2a}=\frac{dy}{b-y}$, primo quidem integrale particulare est $y=b$, tum vero quia posito $x=a$ fit $\mathrm{tang.}\frac{\pi x}{2a}=\infty$, prius membrum ita exhibeatur $\frac{dx}{\mathrm{cot.}\frac{\pi x}{2a}}$, cuius denominator posito $x=a-\omega$ fit $\mathrm{cot.}(\frac{\pi}{2}-\frac{\pi\omega}{2a})=\mathrm{tang.}\frac{\pi\omega}{2a}=\frac{\pi\omega}{2a}$, vbi cum exponens ipsius ω vnitate non sit minor, aequatio $x=a$ erit quoque integrale particulare.

Coroll. 3.

560. Hinc ergo interdum pro eadem aequatione duo pluraue integralia particularia assignari possunt. Veluti pro hac aequatione $\frac{m\,dx}{a-x}=\frac{n\,dy}{b-y}$ integralia particularia sunt $a-x=0$ et $b-y=0$, quae

quae etiam ex integrali completo $(a-x)^m = C(b-y)^n$ consequuntur, illud sumendo $C = 0$, hoc vero sumendo $C = \infty$.

Coroll. 4.

561. Simili modo huius aequationis $\frac{madx}{aa-xx} = \frac{nbdy}{bb-yy}$ quatuor dantur integralia particularia $a+x=0$; $a-x=0$; $b+y=0$; $b-y=0$. Integrale completum vero est $\frac{m}{2} l \frac{a+x}{a-x} = \frac{1}{2} C + \frac{n}{2} l \frac{b+y}{b-y}$, seu $\left(\frac{a+x}{a-x}\right)^m = C\left(\frac{b+y}{b-y}\right)^n$, vel $(a+x)^m (b-y)^n = C(a-x)^m (b+y)^n$, vnde illa sponte fluunt.

Coroll. 5.

562. Hinc patet si fuerit $dy = \frac{Pdx}{(a+x)^\alpha (b+x)^\beta (c+x)^\gamma}$ integralia particularia fore $a+x=0$, $b+x=0$, $c+x=0$, si modo exponentes α, β, γ etc. non fuerint vnitate minores. Quare si Q sit functio rationalis ipsius x, proposita aequatione $dy = \frac{Pdx}{Q}$ omnes factores ipsius Q nihilo aequales positi praebent integralia particularia.

Scholion 1.

563. Hoc etiam pro factoribus imiginariis valet, etiamsi inde parum lucri nanciscamur. Si enim proposita sit aequatio $dy = \frac{adx}{aa+xx}$, ex denominatore $aa+xx$ oriuntur integralia particularia $x = a\sqrt{-1}$ et $x = -a\sqrt{-1}$, quae ex integrali completo, quod est $y = C + \text{Ang. tang.} \frac{x}{a}$ minus sequi videntur. Verum

 posito

posito $x = a\sqrt{-1}$ notandum est, esse Ang. tang. $\sqrt{-1} = \infty\sqrt{-1}$, vnde si constanti C similis forma signo contrario affecta tribuatur, altera quantitas y manet indeterminata, etiamsi ponatur $x = a\sqrt{-1}$, quae positio propterea pro integrali particulari est habenda. Est enim in genere Ang. tang $u\sqrt{-1} = \int\frac{du\sqrt{-1}}{1-uu} = \frac{\sqrt{-1}}{2}l\frac{1+u}{1-u}$, vnde posito $u = +1$ vel $u = -1$, prodit $\infty\sqrt{-1}$, quod infinitum in causa est, vt integralia assignata locum habeant. Quocirca in genere affirmare licet, si fuerit $dy = \frac{Pdx}{Q}$, denominatorque Q factorem habeat $(a+x)^\lambda$, cuius exponens λ vnitate non sit minor, semper aequationem $a+x=0$ fore integrale particulare. Sin autem λ sit vnitate minor etsi positiuus, non erit $a+x=0$ integrale particulare, etiamsi posito $x=-a$ aequationi differentiali satisfaciat.

Scholion 2.

564. Insigne hoc est paradoxon a nemine adhuc, quantum mihi quidem constat, obseruatum, quod aequationi differentiali eiusmodi valor satisfacere queat, qui tamen eius non sit integrale; atque adeo vix patet, quomodo haec cum solita integralium idea conciliari possint. Quoties enim proposita aequatione differentiali eiusmodi relationem variabilium exhibere licet, quae ibi substituta satisfaciat, seu aequationem identicam producat, vix cuiquam in mentem venit dubitare, an illa relatio pro integrali saltem particulari sit habenda, cum tamen hinc procliue sit in errorem

errorem delabi. Veluti etiamsi huic aequationi $dy\sqrt{(aa-xx-yy)}=xdx+ydy$ satisfaciat haec aequatio finita $xx+yy=aa$, tamen enormem errorem committeremus, si eam pro integrali particulari habere vellemus, propterea quod ea in integrali completo $y=C-\sqrt{(aa-xx-yy)}$ neutiquam continetur. Quamobrem etsi omne integrale aequationi differentiali satisfacere debet, tamen non vicissim concludere licet, omnem aequationem finitam, quae satisfaciat, eius esse integrale; verum praeterea requiritur, vt ea certa quadam proprietate sit praedita, cuiusmodi hic exposuimus, et qua demum efficitur, vt in integrali completo contineatur. Hoc autem minime aduersatur verae integralium notioni, quam hic stabiliuimus, neque huiusmodi dubium vnquam in integralia per certas regulas inuenta cadere potest; sed tantum in eiusmodi integralibus, quae diuinando quasi sumus assecuti, locum habet. Saepe numero autem, quando integratio non succedit, diuinationi plurimum tribui solet, tum igitur maxime cauendum est, ne relationem quampiam satisfacientem temere pro integrali particulari proferamus. Quod cum iam in aequationibus separatis simus assecuti, quomodo in omnibus aequationibus differentialibus huiusmodi errores vitari oporteat, sedulo inuestigemus.

Proble-

Problema 72.

565. Si quaepiam relatio inter binas variabiles satisfaciat aequationi differentiali, definire vtrum ea sit integrale particulare nec ne?

Solutio.

Sit $Pdx = Qdy$ aequatio differentialis proposita, vbi P et Q sint functiones quaecunque ipsarum x et y, cui satisfaciat relatio quaepiam inter x et y, ex qua fiat $y = X$, functioni scilicet cuidam ipsius x, ita vt si loco y vbique scribatur X reuera prodeat $Pdx = Qdy$ seu $\frac{dy}{dx} = \frac{P}{Q}$. Quaeritur ergo vtrum hic valor $y = X$ pro integrali aequationis propositae haberi possit nec ne? Ad hoc diiudicandum ponatur $y = X + \omega$, fietque $\frac{dX}{dx} + \frac{d\omega}{dx} = \frac{P}{Q}$, vbi notetur si esset $\omega = 0$, fore $\frac{dX}{dx} = \frac{P}{Q}$. Quare ob ω expressio $\frac{P}{Q}$ hac substitutione reducetur ad $\frac{dX}{dx}$ vna cum quantitate ita per ω affecta, vt euanescat posito $\omega = 0$. In hoc negotio sufficit ω vt particulam infinite paruam spectasse, cuius ergo potestates altiores prae infima negligere liceat. Ponamus igitur hinc fieri $\frac{P}{Q} = \frac{dX}{dx} + S\omega^\lambda$, habebiturque $\frac{d\omega}{dx} = S\omega^\lambda$ seu $\frac{d\omega}{\omega^\lambda} = S\,dx$. Ex superioribus iam perspicuum est tum demum fore $y = X$ integrale particulare, seu $\omega = 0$ cum exponens λ fuerit vnitati aequalis vel maior: similis enim hic est ratio ac supra, qua requi-

requiritur, vt integrale $\int S\,dx = \int \frac{d\omega}{\omega^\lambda}$ fiat infinitum casu proposito, quo $\omega = 0$, hoc autem non euenit, nisi λ sit vnitati aequalis, vel > 1. Quodsi ergo aequationi $P\,dx = Q\,dy$ seu $\frac{dy}{dx} = \frac{P}{Q}$ satisfaciat valor $y = X$, statuatur $y = X + \omega$, spectata particula ω infinite parua, et inuestigetur hinc forma $\frac{Q}{P} = \frac{dX}{dx} + S\omega^\lambda$, ex qua nisi sit $\lambda < 1$ concludetur, illum valorem $y = X$ esse integrale particulare aequationis propositae.

Scholion.

566. Cum ω tractetur vt quantitas infinite parua, valor ipsius $\frac{P}{Q}$ posito $y = X + \omega$ per differentiationem commodissime inueniri posse videtur. Cum enim $\frac{P}{Q}$ sit functio ipsarum x et y, statuamus $d.\frac{P}{Q} = M\,dx + N\,dy$, et quia posito $y = X$ fractio $\frac{P}{Q}$ abit in $\frac{dX}{dx}$ per hypothesin, si loco y scribatur $X + \omega$, ea in $\frac{dX}{dx} + N\omega$ transibit, vnde ob exponentem ipsius ω vnitatem sequeretur, aequationem $y = X$ semper esse integrale particulare, quod tamen secus euenire potest. Ex quo patet differentiationem loco substitutionis adhiberi non posse; quod quo clarius ostendatur, ponamus esse $\frac{P}{Q} = \sqrt{(y - X)} + \frac{dX}{dx}$ vnde posito $y = X + \omega$ manifesto oritur $\frac{P}{Q} = \frac{dX}{dx} + \sqrt{\omega}$. At differentiatione vtentes ponendo $d.\frac{P}{Q} = M\,dx + N\,dy$ fiet $N = \frac{1}{2\sqrt{(y - X)}}$, hincque $\frac{P}{Q} = \frac{dX}{dx} + N\omega$, quae expressio ab illa discrepat. Illa scilicet aequationem

 $y = X$

$y = X$ ex integralium numero remouet, haec vero admittere videtur. Verum et hic notandum est quantitatem N ipſam poteſtatem ipſius ω negatiue inuoluere; vnde poteſtas ω deprimatur. Quare ne hanc rationem ſpectare opus ſit, ſemper praeſtat vera ſubſtitutione vti, differentiatione ſepoſita. Hoc obſeruato haud difficile erit omnes valores, qui aequationi cuipiam differentiali ſatisfaciunt, diiudicare, vtrum ſint vera integralia nec ne?

Exemplum 1.

567. *Cum huic aequationi* $dx(1-y^m)^n = dy(1-x^m)^n$ *manifeſto ſatisfaciat* $y = x$, *vtrum ſit eius integrale particulare nec ne? definire.*

Ponatur $y = x + \omega$, et ſpectato ω vt quantitate minima, eſt $y^m = x^m + mx^{m-1}\omega$, et $(1-y^m)^n = (1-x^m-mx^{m-1}\omega)^n = (1-x^m)^n - mnx^{m-1}\omega(1-x^m)^{n-1}$, vnde aequatio $\frac{dy}{dx} = \frac{(1-y^m)^n}{(1-x^m)^n}$ abit in $1 + \frac{d\omega}{dx} = 1 - \frac{mnx^{m-1}\omega}{1-x^m}$, ſeu $\frac{d\omega}{\omega} = -\frac{mnx^{m-1}dx}{1-x^m}$ vbi cum ω habeat dimenſionem integram, aequatio $y = x$ certe eſt integrale particulare aequationis differentialis propoſitae.

Exemplum 2.

568. *Cum huic aequationi* $ady - adx = dx\sqrt{(yy-xx)}$ *ſatisfaciat valor* $y = x$ *inueſtigare, vtrum is ſit eius integrale particulare nec ne?*

Ponatur

Ponatur $y=x+\omega$, et sumta ω quantitate infinite parua, cum sit $\sqrt{(yy-xx)}=\sqrt{2x\omega}$ erit $ad\omega=dx\sqrt{2x\omega}$ seu $\frac{ad\omega}{\sqrt{\omega}}=dx\sqrt{2x}$. Quoniam igitur hic $d\omega$ diuiditur per potestatem ipsius ω cuius exponens est vnitate minor, sequitur valorem $y=x$ non esse integrale particulare aequationis propositae, etiamsi ei satisfaciat. Scilicet si eius integrale completum exhibere liceret, pateret, quomodocunque constans arbitraria per integrationem ingressa definiretur, in ea aequationem $y=x$ non contentum iri.

Scholion.

569. Hinc noua ratio intelligitur, cur diiudicatio integralis ab exponente ipsius ω pendeat. Cum enim in exemplo proposito facto $y=x+\omega$ prodeat $\frac{ad\omega}{\sqrt{\omega}}=dx\sqrt{2x}$, erit integrando $2a\sqrt{\omega}=C+\frac{2}{3}x\sqrt{2x}$. Verum per hypothesin ω est quantitas infinite parua, hinc autem vtcunque definiatur constans C, quantitas ω obtinet valorem finitum, qui adeo quantumvis magnus euadere potest, quod cum hypothesi aduersetur, necessario sequitur aequationem $y=x$ integrale esse non posse; hocque semper euenire debere, quoties $d\omega$ prodit diuisum per potestatem ipsius ω, cuius exponens vnitate est minor. Contra vero patet, si facta substitutione exposita prodeat $\frac{d\omega}{\omega}=Rdx$, vt posito $\int Rdx=S$ fiat $l\omega=lC+lS$, seu $\omega=CS$, sumta constante C euanescente vtique ipsam quantitatem ω euanescere; quod idem euenit si

ſi prodeat $\frac{d\omega}{\omega^\lambda} = R\,dx$, exiſtente $\lambda > 1$. Erit enim

$\frac{1}{(\lambda-1)\omega^{\lambda-1}} = C - S$ ſeu $(\lambda-1)\omega^{\lambda-1} = \frac{1}{C-S}$, vnde ſumto $C = \infty$, quantitas ω reuera ſit euaneſcens, vt hypotheſis exigit.

Caeterum aequatio huius exempli poſito $x = pp - qq$ et $y = pp + qq$ ab irrationalitate liberatur, fitque $4aqdq = 4pq(pdp - qdq)$ ſiue $adq = ppdp - pqdq$, quae nullo modo tractari poſſe videtur; neque ergo eius integrale completum exhiberi poteſt. Cui aequationi cum non amplius ſatisfacit $x = y$ ſeu $q = 0$, hinc quoque concludendum eſt valorem $y = x$ non eſſe integrale particulare.

Exemplum 3.

570. *Cum huic aequationi* $aady - aadx = dx(yy - xx)$ *ſatisfaciat valor* $y = x$, *inueſtigare, vtrum is ſit eius integrale particulare nec ne?*

Ponatur $y = x + \omega$ ſpectata ω vt quantitate infinite parua, et ob $yy - xx = 2x\omega$ aequatio noſtra hanc induet formam $aad\omega = 2x\omega dx$ ſeu $\frac{aad\omega}{\omega} = 2xdx$. Quia igitur hic $d\omega$ diuiditur per poteſtatem primam ipſius ω, aequatio $y = x$ vtique erit integrale particulare aequationis propoſitae, atque adeo etiam in integrali completo continetur. Hoc enim inuenitur ponendo $y = x - \frac{aa}{u}$ quo fit:

$$\frac{a^4\,du}{uu} = dx\left(\frac{a^4}{uu} - \frac{2aax}{u}\right) \text{ ſeu } du + \frac{2uxdx}{aa} = dx.$$

Multi-

Multiplicetur per $e^{\frac{xx}{aa}}$ et integrale prodit

$$e^{\frac{xx}{aa}}u = C + \int e^{\frac{xx}{aa}}dx$$

hincque

$$y = x - aae^{\frac{xx}{aa}} : (C + \int e^{\frac{xx}{aa}}dx).$$

Quodsi ergo constans C capiatur infinita, fit $y = x$.

Scholion.

571. Si in hac aequatione vt supra ponatur $x = pp - qq$ et $y = pp + qq$ oritur $aadq = ppq(pdp - qdq)$, cui satisfacit $q = 0$, vnde casus $y = x$ nascitur. At facta hac transformatione difficulter patet, quomodo eius integrale inueniri oporteat. Si quidem superiorem reductionem perpendamus, intelligemus hanc aequationem integrabilem reddi si multiplicetur per $e^{(pp-qq)^2:aa}:q^3$; quod cum per se haud facile pateat, consultum erit hac substitutione vti $pp - qq = rr$, qua fit $pp = qq + rr$ et $pdp - qdq = rdr$, vnde aequatio abit in $aadq = qrdr(qq + rr)$, seu $\frac{aadq}{q^3} = rdr + \frac{r^3dr}{qq}$, quae posito $\frac{r}{qq} = s$ facile integratur. Quoties ergo licet eiusmodi relationem inter variabiles colligere, quae aequationi differentiali satisfaciat, hoc modo iudicari poterit, vtrum ea relatio pro integrali particulari sit habenda nec ne? Pro inuentione autem huiusmodi integralium particularium regulae vix tradi possunt; quae enim habentur regulae aeque ad integralia completa inuenienda patent. Ita quae

ſupra circa aequationes ſeparatas obſeruauimus, ob id ipſum quod ſunt ſeparatae, via ſimul ad integrale completum eſt patefacta. Simili modo ſi altera methodus per factores ſuccedat, plerumque ex ipſis factoribus, quibus aequatio integrabilis redditur, integralia particularia concludi poſſunt; quemadmodum in ſequentibus propoſitionibus declarabimus.

Theorema.

572. Si aequatio differentialis $Pdx + Qdy = 0$ per functionem M multiplicata reddatur integrabilis, integrale particulare erit $M = 0$, niſi eodem caſu P vel Q abeat in infinitum.

Demonſtratio.

Ponamus u eſſe factorem ipſius M, et oſtendendum eſt aequationem $u = 0$ eſſe integrale particulare aequationis propoſitae. Cum u aequetur certae functioni ipſarum x et y, definiatur inde altera variabilis y, vt aequatio prodeat inter binas variabiles x et u quae ſit $Rdx + Sdu = 0$, vnde poſito multiplicatore $M = Nu$, integrabilis erit haec forma:

$$NRudx + NSudu = 0.$$

Quodſi iam neque R neque S per u diuidatur, quo caſu poſito $u = 0$ neque P neque Q abit in infinitum, integrale vtique per u erit diuiſibile. Nam ſiue id colligatur ex termino $NRudx$ ſpectata u vt conſtante,

stante, siue ex termino $NSudu$ spectata x constante integrale prodit factorem u implicans, si quidem in integratione constans omittatur. Vnde concludimus integrale completum huiusmodi formam esse habiturum $Vu = C$. Quare si haec constans C nihilo aequalis capiatur, integrale particulare erit $u = 0$, iis scilicet casibus exceptis, quibus functiones R et S iam ipsae per u essent diuisae, ideoque ratiocinium nostrum vim suam amitteret. His ergo casibus exclusis, quoties aequatio $Pdx + Qdy = 0$ per functionem M multiplicata fit per se integrabilis, eaque functio M factorem habeat u, integrale particulare erit $u = 0$, quod similiter de singulis factoribus functionis M valet.

Scholion.

573. Limitatio adiecta absolute est necessaria, cum ea neglecta vniuersum ratiocinium claudicet. Quod quo facilius intelligatur, consideremus hanc aequationem

$$\frac{a\,dx}{y-x} + dy - dx = 0$$

quae per $y-x$ multiplicata manifesto fit integrabilis: ponamus ergo hunc multiplicatorem $y-x=u$, seu $y=x+u$ vnde nostra aequatio erit $\frac{adx}{u} + du = 0$, quae per u multiplicata, abit in $adx + udu = 0$: vbi cum pars adx non per u sit multiplicata, neutiquam concludere licet integrale per u fore diuisibile, quippe quod est $ax + \frac{1}{2}uu$. Hinc patet, si modo

modo pars dx per u esset multiplicata, etiamsi altera pars du factore u careret, tamen integrale per u diuisibile fore, veluti euenit in $udx+xdu$, cuius integrale xu vtique factorem habet u. Ex quo intelligitur si formula $Pudx+Qdu$ fuerit per se integrabilis, dummodo Q non diuidatur per u vel per potestatem eius prima altiorem, etiam integrale omnia scilicet constante fore per u diuisibile.

Theorema.

574. Si aequatio differentialis $Pdx+Qdy=0$ per functionem M diuisa euadat per se integrabilis, integrale particulare erit $M=0$, nisi posito $M=0$ vel P vel Q euanescat.

Demonstratio.

Habeat diuisor M factorem u, vt sit $M=Nu$, et ostendi oportet, integrale particulare futurum $u=0$, id quod de singulis factoribus diuisoris M, si quidem plures habeat est tenendum. Cum igitur u sit functio ipsarum x et y, definiatur inde altera y per x et u, vt prodeat huiusmodi aequatio $Rdx+Sdu=0$, quae ergo per Nu diuisa per se erit integrabilis. Quaeri igitur oportet integrale formulae $\frac{Rdx}{Nu}+\frac{Sdu}{Nu}$, vbi assumimus neque R neque S per u multiplicari, neque hoc modo factorem u ex denominatore tolli. Quod si iam hoc integrale ex solo membro $\frac{Rdx}{Nu}$ colli-

colligatur, fpectando u vt conftantem, prodit id $\int \frac{R\,dx}{N} + f:u$, fin autem ex altero membro $\frac{S\,du}{Nu}$, fu[illegible]ta x conftante colligatur, quia S non factorem habet u, id femper ita erit comparatum, vt pofito $u=0$, fiat infinitum. Ex quo integrale, quod fit V, ita erit comparatum, vt fiat $=\infty$ pofito $u=0$; quare cum integrale completum futurum fit $V=C$, huic aequationi fumta conftante C infinita fatisfit ponendo $u=0$. Concludimus itaque, fi diuifor $M=Nu$ reddat aequationem differentialem $Pdx+Qdy=0$ per fe integrabilem, ex quolibet diuiforis M factore u obtineri integrale particulare $u=0$, nifi forte pofito $u=0$, quantitates P et Q, vel R et S euanefcant.

Coroll. 1.

575. Si aequatio $Pdx+Qdy=0$ fuerit homogenea, ea vt fupra vidimus integrabilis redditur, fi diuidatur per $Px+Qy$, quare integrale eius particulare erit $Px+Qy=0$. Quae aequatio cum etiam fit homogenea, factores habebit formae $\alpha x+\beta y$ quorum quisque nihilo aequatus dabit integrale particulare.

Coroll. 2.

576. Pro hac aequatione

$$ydx(c+nx)-dy(y+a+bx+nxx)=0$$

diuiforem, quo integrabilis redditur, fupra §. 489.

 cuius

exhibuimus vnde integrale particulare concluditur $1°\; y = 0$, tum vero:

$$nyy + (2na - bc)y + n(b - 2c)xy + (na + cc - bc)(a + bx + nxx) = 0$$

cuius radices sunt:

$$ny = \tfrac{1}{2}bc - na + n(c - \tfrac{1}{2}b)x \pm (c + nx)\sqrt{(\tfrac{1}{4}bb - na)}.$$

Coroll. 3.

577. Pro hac aequatione differentiali:

$$\frac{n\,dx(1+yy)\sqrt{(1+yy)}}{\sqrt{(1+xx)}} + (x-y)dy = 0$$

diuisorem, quo integrabilis redditur, supra §. 490. dedimus vnde integrale particulare concludimus

$$x - y + n\sqrt{(1+xx)(1+yy)} = 0$$

seu $yy - 2xy + xx = nn + nnxx + nnyy + nnxxyy$, ex quo porro fit $y = \frac{x \pm n(1+xx)\sqrt{(1-nn)}}{1 - nn(1+xx)}$.

Coroll. 4.

578. Pro hac aequatione differentiali

$$dy + yy\,dx - \frac{a\,dx}{x^4} = 0$$

multiplicatorem supra §. 491. inuenimus $\frac{xx}{xx(1-xy)^2 - a}$, vnde integrale particulare concludimus $xx(1-xy)^2 - a = 0$ hincque $x(1-xy) = \pm\sqrt{a}$ seu $y = \frac{1}{x} \pm \frac{\sqrt{a}}{xx}$, ita vt bina habeamus integralia particularia, quae autem imaginaria euadunt, si a fuerit quantitas negatiua.

Scholion.

Scholion.

579. Haec fere sunt, quae circa tractationem aequationum differentialium adhuc sunt explorata, nonnulla tamen subsidia euolutio aequationum differentialium secundi gradus infra suppeditabit. Huc autem commode referri possunt, quae circa comparationem certarum formularum transcendentium haud ita pridem sunt inuestigata. Quemadmodum enim logarithmi et arcus circulares, etsi sunt quantitates transcendentes, inter se comparari atque adeo aeque ac quantitates algebraicae in calculo tractari possunt, ita similem comparationem inter certas quantitates transcendentes altioris generis instituere licet, quae scilicet continentur in formula hac: $\int \frac{dx}{\surd(A+Bx+Cx^2+Dx^3+Ex^4)}$, vbi etiam numerator rationalis veluti $\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2+$ etc. addi potest. Quod argumentum cum sit maxime arduum, atque adeo vires Analyseos superare videatur, nisi certa ratione expediatur, in Analysin inde haud spernenda incrementa redundant; imprimis autem resolutio aequationum differentialium non mediocriter perfici videtur. Cum enim proposita fuerit huiusmodi aequatio

$$\frac{dx}{\surd(A+Bx+Cx^2+Dx^3+Ex^4)}=\frac{dy}{\surd(A+By+Cy^2+Dy^3+Ey^4)}$$

statim quidem patet eius integrale particulare $x=y$, verum integrale completum maxime transcendens fore videtur, cum vtraque formula per se neque ad logarithmos, neque ad arcus circulares reduci

 queat.

queat. Quare eo magis erit mirandum, quod integrale completum per aequationem adeo algebraicam inter x et y exhiberi possit. Quo autem methodus ad haec sublimia ducens clarius perspiciatur, eam primo ad quantitates transcendentes notas hac formula $\int \frac{dx}{\sqrt{(A+Bx+Cxx)}}$ contentas applicemus, deinceps eius usum in formulis illis magis complexis ostensuri.

CAPVT V.

CAPVT V.

DE COMPARATIONE QVANTITATVM TRANSCENDENTIVM IN FORMA $\int\frac{Pdx}{\sqrt{(A+2Bx+Cxx)}}$ CONTENTARVM.

Problema 73.

580.

Proposita inter x et y hac aequatione algebraica

$$\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy=0$$

inuenire formulas integrales formae praescriptae, quae inter se comparari queant.

Solutio.

Differentietur aequatio proposita, et ex eius differentiali

$$2\beta dx+2\beta dy+2\gamma xdx+2\gamma ydy+2\delta xdy+2\delta ydx=0$$

colligetur haec aequatio:

$$dx(\beta+\gamma x+\delta y)+dy(\beta+\gamma y+\delta x)=0.$$

 Statua-

Statuatur $\beta+\gamma x+\delta y=p$ et $\beta+\gamma y+\delta x=q$, atque ex priori erit

$$pp=\beta\beta+2\beta\gamma x+2\beta\delta y+\gamma\gamma xx+2\gamma\delta xy+\delta\delta yy$$

a qua subtrahatur aequatio proposita per γ multiplicata

$$0=\alpha\gamma+2\beta\gamma x+2\gamma\beta y+\gamma\gamma xx+\gamma\gamma yy+2\gamma\delta xy$$

fietque

$$pp=\beta\beta-\alpha\gamma+2\beta(\delta-\gamma)y+(\delta\delta-\gamma\gamma)yy$$

similique modo reperietur

$$qq=\beta\beta-\alpha\gamma+2\beta(\delta-\gamma)x+(\delta\delta-\gamma\gamma)xx$$

vnde erit $pdx+qdy=0$. Cum iam sit p functio ipsius y, et q similis functio ipsius x, ponatur

$$\beta\beta-\alpha\gamma=A;\ \beta(\delta-\gamma)=B \text{ et } \delta\delta-\gamma\gamma=C$$

vnde colligitur

$$\delta-\gamma=\frac{B}{\beta} \text{ et } \delta+\gamma=\frac{C}{\delta-\gamma}=\frac{\beta C}{B}$$

hincque

$$\delta=\frac{BB+\beta\beta C}{2B\beta} \text{ et } \gamma=\frac{\beta\beta C-BB}{2B\beta}$$

prima vero dat

$$\alpha=\frac{\beta\beta-A}{\gamma}=\frac{2B\beta(\beta\beta-A)}{\beta\beta C-BB}.$$

Quibus valoribus pro α, γ, δ assumtis aequatio $\frac{dx}{q}+\frac{dy}{p}=0$ abit in hanc

$$\frac{dx}{\sqrt{(A+2Bx+Cxx)}}+\frac{dy}{\sqrt{(A+2By+Cyy)}}=0$$

cui

cui ergo aequationi differentiali satisfacit aequatio:

$$\frac{2B\beta(\beta\beta-A)}{\beta\beta C-BB}+2\beta(x+y)+\frac{\beta\beta C-BB}{2B\beta}(xx+yy)+\frac{BB+\beta\beta C}{B\beta}xy=0$$

quae cum contineat constantem nouam β, erit adeo integrale completum aequationis differentialis inuentae.

Neque vero opus est, vt formulae illae ipsis litteris A, B, C aequentur; sed sufficit vt ipsis sint proportionales, vnde fit

$$\frac{\beta\beta-\alpha\gamma}{\beta(\delta-\gamma)}=\frac{A}{B} \text{ et } \frac{\delta+\gamma}{\beta}=\frac{C}{B}.$$

Ergo

$$\delta=\frac{\beta C}{B}-\gamma \text{ et } \alpha=\frac{\beta\beta}{\gamma}-\frac{\beta A}{\gamma B}(\delta-\gamma)$$

seu

$$\alpha=\frac{\beta\beta}{\gamma}-\frac{\beta\beta AC}{\gamma BB}+\frac{2\beta A}{B}.$$

Quare aequationis differentialis

$$\frac{dx}{\sqrt{(A+2Bx+Cxx)}}+\frac{dy}{\sqrt{(A+2By+Cyy)}}=0$$

integrale completum est

$$\beta\beta(BB-AC)+2\beta\gamma AB+2\beta\gamma BB(x+y)+\gamma\gamma BB(xx+yy)$$
$$+2\gamma B(\beta C-\gamma B)xy=0$$

vbi ratio $\frac{\beta}{\gamma}$ constantem arbitrariam exhibet.

Coroll. 1.

581. Ex aequatatione proposita radicem extrahendo fit:

$$y=\frac{-\beta-\delta x+\sqrt{(\beta\beta+2\beta\delta x+\delta\delta xx-\alpha\gamma-2\beta\gamma x-\gamma\gamma xx)}}{\gamma}$$

seu

seu loc α et δ substitutis valoribus:

$$y = -\frac{\beta}{\gamma} - \frac{(\beta C - \gamma B)}{\gamma B}x + \sqrt{\left(\frac{\beta\beta C - 2\beta\gamma B}{\gamma\gamma BB}\right)(A + 2Bx + Cxx)}.$$

Coroll. 2.

282. Si ergo $x = 0$, fit

$$y = -\frac{\beta}{\gamma} + \sqrt{\frac{\beta\beta AC - 2\beta\gamma AB}{\gamma\gamma BB}}$$

ponatur hic valor $= a$, vt fit

$$\gamma Ba + \beta B = \sqrt{(\beta\beta AC - 2\beta\gamma AB)};$$

vnde sumtis quadratis oritur

$$\gamma\gamma BBaa + 2\beta\gamma BBa + \beta\beta BB = \beta\beta AC - 2\beta\gamma AB$$

hincque

$$\frac{\gamma}{\beta} = \frac{-A - Ba + \sqrt{A(A + 2Ba + Caa)}}{Baa}$$

seu

$$\frac{\beta}{\gamma} = \frac{B(A + Ba + \sqrt{A(A + 2Ba + Caa)})}{AC - BB}.$$

Scholion 1.

283. Vt aequatio assumta:

$$\alpha + 2\beta(x + y) + \gamma(xx + yy) + 2\delta xy = 0$$

satisfaciat aequationi differentiali

$$\frac{dx}{\sqrt{(A + 2Bx + Cxx)}} + \frac{dy}{\sqrt{(A + 2By + Cyy)}} = 0$$

necesse est vt sit:

$$\beta\beta - \alpha\gamma = mA;\ \beta(\delta - \gamma) = mB \text{ et } \delta\delta - \gamma\gamma = mC$$

vnde

vnde fit

$$\beta+\gamma y+\delta x=\sqrt{m}(A+2Bx+Cxx) \text{ et}$$
$$\beta+\gamma x+\delta y=\sqrt{m}(A+2By+Cyy).$$

At ex datis A, B, C litterarum $\alpha, \beta, \gamma, \delta$ et m tres tantum definiuntur; quare cum binae maneant indeterminatae, aequatio assumta, etiamsi per quemvis coefficientium diuidatur, vnam tamen constantem continet nouam, ex quo ea pro integrali completo erit habenda. Quare etsi aequationis differentialis neutra pars integrationem algebraice admittit, tamen integrale completum algebraice exhiberi potest. Loco constantis arbitrariae is valor ipsius y introduci potest, quem recipit posito $x=o$: cum autem euenire possit, vt hic valor fiat imaginarius, conveniet istam constantem ita definiri, vt posito $x=a$ fiat $y=b$, quo pacto ad omnes casus applicatio fieri poterit. Hinc erit

$$\frac{\beta+\gamma b+\delta a}{\beta+\gamma a+\delta b}=\sqrt{\frac{A+2Ba+Caa}{A+2Bb+Cbb}},$$

vnde colligitur

$$\beta=\frac{(\gamma a+\delta b)\sqrt{(A+2Ba+Caa)}-(\gamma b+\delta a)\sqrt{(A+2Bb+Cbb)}}{-\sqrt{(A+2Ba+Caa)}+\sqrt{(A+2Bb+Cbb)}}$$

$$\text{et } \sqrt{m}(A+2Ba+Caa)=\frac{(\delta-\gamma)(b-a)\sqrt{(A+2Ba+Caa)}}{\sqrt{(A+2Bb+Cbb)}-\sqrt{(A+2Ba+Caa)}}$$

$$\text{seu } \sqrt{m}=\frac{(\delta-\gamma)(b-a)}{\sqrt{(A+2Bb+Cbb)}-\sqrt{(A+2Ba+Caa)}}.$$

Ponatur breuitatis gratia:

$$\sqrt{(A+2Ba+Caa)}=\mathfrak{A} \text{ et } \sqrt{(A+2Bb+Cbb)}=\mathfrak{B}$$

vt fit

$$\sqrt{m} = \frac{(\delta-\gamma)(b-a)}{\mathfrak{B}-\mathfrak{A}} \text{ et}$$

$$\beta = \frac{\mathfrak{A}(\gamma a+\delta b)-\mathfrak{B}(\gamma b+\delta a)}{\mathfrak{B}-\mathfrak{A}}$$

et aequatio $\beta(\delta-\gamma)=m\mathrm{B}$ induet hanc formam:

$$\mathfrak{A}(\gamma a+\delta b)-\mathfrak{B}(\gamma b+\delta a)=\frac{\mathrm{B}(\delta-\gamma)(b-a)^2}{\mathfrak{B}-\mathfrak{A}}$$

vnde fit:

$$\left.\begin{array}{l}+\gamma\mathfrak{A}\mathfrak{B}-\gamma\mathrm{A}-\gamma\mathrm{B}(a+b)-\gamma\mathrm{C}(aa-ab+bb)\\ +\delta\mathfrak{A}\mathfrak{B}-\delta\mathrm{A}-\delta\mathrm{B}(a+b)-\delta\mathrm{C}ab\end{array}\right\}=0.$$

Statuatur ergo

$$\gamma=n\mathfrak{A}\mathfrak{B}-n\mathrm{A}-n\mathrm{B}(a+b)-n\mathrm{C}ab$$

$$\delta=n\mathrm{A}+n\mathrm{B}(a+b)+n\mathrm{C}(aa-ab+bb)-n\mathfrak{A}\mathfrak{B}$$

$$\sqrt{m}=\frac{n(b-a)(\mathfrak{B}^2+\mathfrak{A}^2-2\mathfrak{A}\mathfrak{B})}{\mathfrak{B}-\mathfrak{A}}=n(b-a)(\mathfrak{B}-\mathfrak{A})$$

$$\beta=n\mathrm{B}(b-a)^2 \text{ ergo } \delta-\gamma=\frac{m}{n(b-a)^2}$$

vnde cum fit $\delta+\gamma=n\mathrm{C}(b-a)^2$ erit vtique $\delta\delta-\gamma\gamma=m\mathrm{C}$. Supereft vt fiat $\alpha\gamma=\beta\beta-m\mathrm{A}$ hoc eft

$$\alpha\gamma=nn\mathrm{BB}(b-a)^4-nn\mathrm{A}(b-a)^2(\mathfrak{B}-\mathfrak{A})^2 \text{ feu}$$

$$\alpha\gamma=nn(b-a)^2(\mathrm{BB}(b-a)^2-\mathrm{A}(\mathfrak{B}-\mathfrak{A})^2).$$

Vel cum pofito $x=a$ fiat $y=b$ erit quoque

$$\alpha=-2\beta(a+b)-\gamma(aa+bb)-2\delta ab$$

hincque

$$\alpha=n(a-b)^2(\mathrm{A}-\mathrm{B}(a+b)-\mathrm{C}ab-\mathfrak{A}\mathfrak{B})$$

vnde

vnde aequatio nostra assumta est

$$(b-a)^2(A-B(a+b)-Cab-\mathfrak{A}\mathfrak{B})+2B(b-a)^2(x+y)$$
$$-(A+B(a+b)+Cab-\mathfrak{A}\mathfrak{B})(xx+yy)$$
$$+2(A+B(a+b)+C(aa-ab+bb)-\mathfrak{A}\mathfrak{B})xy=0.$$

Scholion 2.

584. Si ponatur $\beta=0$, vt aequatio sit

$$\alpha+\gamma(xx+yy)+2\delta xy=0 \text{ erit}$$
$$y=\frac{-\delta x+\sqrt{(-\alpha\gamma+(\delta\delta-\gamma\gamma)xx)}}{\gamma}.$$

Posito ergo $-\alpha\gamma=mA$ et $\delta\delta-\gamma\gamma=mC$, vt sit

$$\gamma y+\delta x=\sqrt{m(A+Cxx)} \text{ erit}$$
$$\frac{dx}{\sqrt{(A+Cxx)}}+\frac{dy}{\sqrt{(A+Cyy)}}=0$$

cuius aequationis integrale completum erit ipsa aequatio assumta, pro qua habebitur, $\frac{C}{A}=\frac{\gamma\gamma-\delta\delta}{\alpha\gamma}$ seu $\delta=\sqrt{(\gamma\gamma-\frac{\alpha\gamma C}{A})}$. Sin autem posito $x=0$ fieri debeat $y=b$, ob $\gamma b=\sqrt{mA}$, erit $\gamma=\frac{\sqrt{mA}}{b}$; tum $\alpha=-b\sqrt{mA}$ et $\delta=\sqrt{(\frac{mA}{bb}+mC)}$. Habebitur ergo haec aequatio

$$\frac{y\sqrt{mA}}{b}+\frac{x\sqrt{m(A+Cbb)}}{A}=\sqrt{m(A+Cxx)}$$

quae praebet

$$y=-x\sqrt{\frac{A+Cbb}{A}}b+b\sqrt{\frac{A+Cxx}{A}}$$

quae est integrale completum aequationis illius differentialis. Quare si x capiatur negatiue huius aequationis differentialis:

$$\frac{dx}{\sqrt{(A+Cxx)}}=\frac{dy}{\sqrt{(A+Cyy)}}$$

integrale completum est:

$$y=x\sqrt{\frac{A+Cbb}{A}}+b\sqrt{\frac{A+Cxx}{A}}.$$

Quodsi simili modo calculus in genere tractetur, aequationis differentialis:

$$\frac{dx}{\sqrt{(A+2Bx+Cxx)}}+\frac{dy}{\sqrt{(A+2By+Cyy)}}=0$$

si breuitatis gratia ponatur $\sqrt{(A+2Bb+Cbb)}=\mathfrak{B}$ erit integrale completum:

$$y\left(\sqrt{A}+\frac{Bb}{\sqrt{A}\cdot\mathfrak{B}}\right)+x\left(\mathfrak{B}+\frac{Bb}{\sqrt{A}\cdot\mathfrak{B}}\right)=\frac{Bbb}{\sqrt{A}\cdot\mathfrak{B}}+b\sqrt{(A+2Bx+Cxx)}$$

vnde casus praecedens manifesto sequitur, si ponatur $B=0$. Verum ope leuis substitutionis hae formulae, vbi adest B, ad illum casum vbi $B=0$ reduci possunt.

Problema 74.

585. Si $\Pi:z$ significet eam functionem ipsius z, quae oritur ex integratione formulae $\int\frac{dz}{\sqrt{(A+Czz)}}$, integrali hoc ita sumto, vt euanescat posito $z=0$, comparationem inter huiusmodi functiones instituere.

Solutio.

Solutio.

Confideretur haec aequatio differentialis:

$$\frac{dx}{\sqrt{(A+Cxx)}} = \frac{dy}{\sqrt{(A+Cyy)}}$$

vnde cum fit per hypothefin:

$$\int \frac{dx}{\sqrt{(A+Cxx)}} = \Pi : x \text{ et } \int \frac{dy}{\sqrt{(A+Cyy)}} = \Pi : y$$

vtroque integrali ita fumto, vt euanefcat illud pofito $x=0$, hoc vero pofito $y=0$, integrale completum erit

$$\Pi : y = \Pi : x + C.$$

Ante autem vidimus hoc integrale effe:

$$y = x\sqrt{\frac{A+Cbb}{A}} + b\sqrt{\frac{A+Cxx}{A}}$$

vbi pofito $x=0$ fit $y=b$; quare cum $\Pi : 0 = 0$, erit

$$\Pi : y = \Pi : x + \Pi : b$$

cui ergo aequationi tranfcendentali fatisfacit haec algebraica:

$$y = x\sqrt{\frac{A+Cbb}{A}} + b\sqrt{\frac{A+Cxx}{A}}.$$

Simili modo fumto b negatiuo haec aequatio

$$\Pi : y = \Pi : x - \Pi : b$$

conuenit cum hac

$$y = x\sqrt{\frac{A+Cbb}{A}} - b\sqrt{\frac{A+Cxx}{A}}$$

ficque tam fumma, quam differentia duarum huius-

 modi

modi functionum per similem functionem exprimi potest. Hic iam nullo habito discrimine inter quantitates variabiles et constantes, dum $\Pi:z$ functionem determinatam ipsius z significat, scilicet $\Pi:z=\int\frac{dz}{\sqrt{(A+Czz)}}$, quae vt assumsimus euanescat posito $z=0$, vt hoc signandi modo recepto sit

$$\Pi:r=\Pi:p+\Pi:q$$

debet esse

$$r=p\sqrt{\frac{A+Cqq}{A}}+q\sqrt{\frac{A+Cpp}{A}}$$

vt vero sit

$$\Pi:r=\Pi:p-\Pi:q$$

debet esse

$$r=p\sqrt{\frac{A+Cqq}{A}}-q\sqrt{\frac{A+Cpp}{A}}$$

vtrinque autem sublata irrationalitate prodit inter p, q, r haec aequatio:

$$p^4+q^4+r^4-2ppqq-2pprr-2qqrr=\frac{4Cppqqrr}{A}$$

cuius forma hanc suppeditat proprietatem, vt si p, q, r sint latera cuiusdam trianguli, eique circumscribatur circulus, cuius diameter vocetur $=T$, semper sit $A+CTT=0$. Illa autem aequatio ob plures quas complectitur radices, satisfacit huic relationi

$$\Pi:p\pm\Pi:q\pm\Pi:r=0.$$

Coroll. 1.

Coroll. 1.

585. Hinc ſtatim deducitur nota arcuum circularium comparatio ponendo A=1 et C=−1. Tum enim fit

$$\Pi : z = \int \frac{dz}{\sqrt{(1-zz)}} = \text{Ang. ſin. } z$$

hincque vt ſit

$$\text{Ang. ſin. } r = \text{Ang. ſin. } p + \text{Ang. ſin. } q$$

oportet eſſe

$$r = p\sqrt{(1-qq)} + q\sqrt{(1-pp)}$$

et vt ſit

$$\text{Ang. ſin. } r = \text{Ang. ſin. } p - \text{Ang. ſin. } q$$

debet eſſe

$$r = p\sqrt{(1-qq)} - q\sqrt{(1-pp)}$$

vti conſtat.

Coroll. 2.

587 Si ſit A=1 et C=1 erit

$$\Pi : z = \int \frac{dz}{\sqrt{(1+zz)}} = l(z+\sqrt{(1+zz)})$$

vnde vt ſit

$$l(r+\sqrt{(1+rr)}) = l(p+\sqrt{(1+pp)}) + l(q+\sqrt{(1+qq)})$$

erit

$$r = p\sqrt{(1+qq)} + q\sqrt{(1+pp)}$$

vt autem ſit

$$l(r+\sqrt{(1+rr)}) = l(p+\sqrt{(1+pp)}) - l(q+\sqrt{(1+qq)})$$

erit

erit

$$r = p\sqrt{(1+qq)} - q\sqrt{(1+pp)}$$

vti ex indole logarithmorum sponte liquet.

Coroll. 3.

588. Si ponamus in priori formula generali $q=p$, vt fit

$$\Pi : r = 2\,\Pi : p \text{ erit}$$

$$r = 2p\sqrt{\tfrac{A+Cpp}{A}}.$$

Hinc porro si fiat

$$q = 2p\sqrt{\tfrac{A+Cpp}{A}} \text{ erit}$$

$$\Pi : r = \Pi : p + 2\,\Pi : p = 3\,\Pi : p,$$

sumto

$$r = p\sqrt{\tfrac{A+Cqq}{A}} + q\sqrt{\tfrac{A+Cpp}{A}}.$$

Est vero

$$\sqrt{\tfrac{A+Cqq}{A}} = \sqrt{(1 + \tfrac{4Cpp}{A}(1+\tfrac{Cpp}{A}))} = 1 + \tfrac{2Cpp}{A}$$

vnde vt fit

$$\Pi : r = 3\,\Pi : p \text{ fit}$$

$$r = p(1+\tfrac{2Cpp}{A}) + 2p(1+\tfrac{Cpp}{A}) = 3p + \tfrac{4Cp^3}{A}.$$

Scholion.

589. Quo haec multiplicatio facilius continuari queat, praeter relationem aequationi

$$\Pi : r = \Pi : p + \Pi : q$$

respon-

respondentem, quae est

$$r = p\sqrt{\frac{A+Cqq}{A}} + q\sqrt{\frac{A+Cpp}{A}}$$

notetur aequatio

$$\Pi : p = \Pi : r - \Pi : q$$

cui respondet relatio

$$p = r\sqrt{\frac{A+Cqq}{A}} + q\sqrt{\frac{A+Crr}{A}}$$

vnde fit

$$\sqrt{\frac{A+Crr}{A}} = \frac{r}{q}\sqrt{\frac{A+Cqq}{A}} - \frac{p}{q} = \frac{p}{q}\left(\frac{A+Cqq}{A}\right) + \sqrt{\left(\frac{A+Cpp}{A}\right)\left(\frac{A+Cqq}{A}\right)} - \frac{p}{q}$$

seu $\sqrt{\frac{A+Crr}{A}} = \frac{Cpq}{A} + \sqrt{\left(\frac{A+Cpp}{A}\right)\left(\frac{A+Cqq}{A}\right)}$.

Quare vt sit

$$\Pi : r = \Pi : p + \Pi : q$$

habemus non solum

$$r = p\sqrt{(1 + \tfrac{C}{A}qq)} + q\sqrt{(1 + \tfrac{C}{A}pp)}$$

sed etiam

$$\sqrt{(1 + \tfrac{C}{A}rr)} = \tfrac{C}{A}pq + \sqrt{(1 + \tfrac{C}{A}pp)(1 + \tfrac{C}{A}qq)}.$$

Ponamus breuitatis gratia $\sqrt{(1 + \frac{C}{A}pp)} = P$, et sumto $q = p$ vt sit

$$\Pi : r = 2\,\Pi : p$$

erit $r = 2Pp$ et $\sqrt{(1 + \frac{C}{A}rr)} = \frac{C}{A}pp + PP$

qui valor ipsius r pro q sumtus dabit

$$\Pi : r = 3\,\Pi : p$$

existente

$$r = \frac{C}{A}p^3 + 3PPp \text{ et}$$

$$\sqrt{(1 + \frac{C}{A}rr)} = \frac{3C}{A}Ppp + P^3.$$

Hic valor ipsius r denuo pro q sumtus dabit

$$\Pi : r = 4\,\Pi : p$$

existente

$$r = \frac{4C}{A}Pp^3 + 4P^3p \text{ et}$$

$$\sqrt{(1 + \frac{C}{A}rr)} = \frac{CC}{AA}p^4 + \frac{6C}{A}PPpp + P^4.$$

Loco q substituatur hic valor ipsius r, vt prodeat

$$\Pi : r = 5\,\Pi : p$$

existente

$$r = \frac{CC}{AA}p^5 + \frac{10C}{A}PPp^3 + 5P^4p \text{ et}$$

$$\sqrt{(1 + \frac{C}{A}rr)} = \frac{5CC}{AA}Pp^4 + \frac{10C}{A}P^3pp + P^5.$$

Atque hinc generatim concludere licet, vt sit

$$\Pi : r = n\,\Pi : p$$

esse debere

$$r\sqrt{\frac{C}{A}} = \tfrac{1}{2}(P + p\sqrt{\tfrac{C}{A}})^n - \tfrac{1}{2}(P - p\sqrt{\tfrac{C}{A}})^n \text{ et}$$

$$\sqrt{(1 + \frac{C}{A}rr)} = \tfrac{1}{2}(P + p\sqrt{\tfrac{C}{A}})^n + \tfrac{1}{2}(P - p\sqrt{\tfrac{C}{A}})^n \text{ seu}$$

$$r = \frac{\sqrt{A}}{2\sqrt{C}}(P + p\sqrt{\tfrac{C}{A}})^n - \frac{\sqrt{A}}{2\sqrt{C}}(P - p\sqrt{\tfrac{C}{A}})^n.$$

Haec igitur relatio inter p et r satisfaciet huic aequationi differentiali:

$$\frac{dr}{\sqrt{(A + Crr)}} = \frac{n\,dp}{\sqrt{(A + Cpp)}}$$

dum meminerimus esse $P = \sqrt{(1 + \frac{Cpp}{A})}$.

Proble-

Problema 75.

590. Si ponatur $\int \frac{dz}{\sqrt{(A+Czz)}} = \Pi : z$ integrali ita sumto vt euanescat posito $z = f$, vnde $\Pi : z$ sit functio determinata ipsius z, comparationem inter huiusmodi functiones instituere.

Solutio.

Consideretur haec aequatio differentialis

$$\frac{dx}{\sqrt{(A+Cxx)}} + \frac{dy}{\sqrt{(A+Cyy)}} = 0$$

vnde integrando fit

$$\Pi : x + \Pi : y = \text{Const.}$$

Integrale autem fit quoque

$$\alpha + \gamma(xx+yy) + 2\delta xy = 0,$$

quod vt locum habeat necesse est, sit

$$-\alpha\gamma = Am; \text{ et } \delta\delta - \gamma\gamma = Cm$$

tum vero erit

$$\gamma x + \delta y = \sqrt{m}(A+Cyy) \text{ et } \gamma y + \delta x = \sqrt{m}(A+Cxx).$$

Ponamus constantem integratione ingressam ita definiri vt posito $x = a$ fiat $y = b$, et integrale erit

$$\Pi : x + \Pi : y = \Pi : a + \Pi : b.$$

Pro forma autem algebraica inuenienda, sit breuitatis gratia:

$$\sqrt{(A+Caa)} = \mathfrak{A} \text{ et } \sqrt{(A+Cbb)} = \mathfrak{B}$$

 eritque

eritque

$$\gamma a + \delta b = \mathfrak{B}\sqrt{m} \text{ et } \gamma b + \delta a = \mathfrak{A}\sqrt{m}$$

vnde colligitur:

$$\gamma = \frac{\mathfrak{A}b - \mathfrak{B}a}{bb - aa}\sqrt{m} \text{ et } \delta = \frac{\mathfrak{B}b - \mathfrak{A}a}{bb - aa}\sqrt{m}.$$

Quocirca aequatio integralis algebraica erit

$$(\mathfrak{A}b - \mathfrak{B}a)x + (\mathfrak{B}b - \mathfrak{A}a)y = (bb - aa)\sqrt{(A + Cyy)}$$

seu

$$(\mathfrak{A}b - \mathfrak{B}a)y + (\mathfrak{B}b - \mathfrak{A}a)x = (bb - aa)\sqrt{(A + Cxx)}.$$

Hinc y per x ita definitur, vt sit

$$y = \frac{(\mathfrak{A}a - \mathfrak{B}b)x + (bb - aa)\sqrt{(A + Cxx)}}{\mathfrak{A}b - \mathfrak{B}a}$$

quae fractio supra et infra per $\mathfrak{A}b + \mathfrak{B}a$ multiplicando ob

$$\mathfrak{A}\mathfrak{A}bb - \mathfrak{B}\mathfrak{B}aa = A(bb - aa) \text{ et } (\mathfrak{A}a - \mathfrak{B}b)(\mathfrak{A}b + \mathfrak{B}a) = (\mathfrak{A}\mathfrak{A} - \mathfrak{B}\mathfrak{B})ab - \mathfrak{A}\mathfrak{B}(bb - aa) = -(bb - aa)(Cab + \mathfrak{A}\mathfrak{B})$$

abit in

$$y = -\frac{(Cab + \mathfrak{A}\mathfrak{B})x}{A} + \frac{(\mathfrak{A}b + \mathfrak{B}a)\sqrt{(A + Cxx)}}{A}.$$

Hinc porro colligitur:

$$(bb - aa)\sqrt{(A + Cyy)} = (\mathfrak{A}b - \mathfrak{B}a)x - \frac{(\mathfrak{B}b - \mathfrak{A}a)^2 x}{\mathfrak{A}b - \mathfrak{B}a} + \frac{(\mathfrak{B}b - \mathfrak{A}a)(bb - aa)}{\mathfrak{A}b - \mathfrak{B}a}\sqrt{(A + Cxx)}$$

seu $\sqrt{(A + Cyy)} = -\frac{C(bb - aa)}{\mathfrak{A}b - \mathfrak{B}a}x + \frac{\mathfrak{B}b - \mathfrak{A}a}{\mathfrak{A}b - \mathfrak{B}a}\sqrt{(A + Cxx)}$

vbi iterum supra et infra multiplicando per $\mathfrak{A}b + \mathfrak{B}a$ fit

$$\sqrt{(A + Cyy)} = -\frac{C(\mathfrak{A}b + \mathfrak{B}a)}{A}x + \frac{(Cab + \mathfrak{A}\mathfrak{B})}{A}\sqrt{(A + Cxx)}.$$

Necesse

Necesse autem est valorem formulae $\sqrt{(A+Cyy)}$ hoc modo potius definiri quam extractione radicis, qua ambiguitas implicaretur. Quocirca haec aequatio transcendens:

$$\Pi:r+\Pi:s=\Pi:p+\Pi:q$$

praebet sequentem determinationem algebraicam, si quidem breuitatis gratia ponamus $\sqrt{(A+Cpp)}=P$, $\sqrt{(A+Cqq)}=Q$ et $\sqrt{(A+Crr)}=R$, scilicet vt sit

$$\Pi:s=\Pi:p+\Pi:q-\Pi:r$$

erit

$$s=\frac{-PQr-Cpqr+PRq+QRp}{A} \text{ et}$$

$$\sqrt{(A+Css)}=\frac{-CPqr-CQpr+CRpq+PQR}{A} \text{ seu}$$

$$\sqrt{(A+Css)}=\frac{PQR+C(Rpq+PQr-Qpr)}{A}.$$

Coroll. 1.

591. Quoniam est per hypothesin $\Pi:f=0$, si ponamus breuitatis gratia $\sqrt{(A+Cff)}=F$, et $r=f$ vt sit $R=F$, haec aequatio

$$\Pi:s=\Pi:p+\Pi:q$$

praebet:

$$s=\frac{F(Pq+Qp)-PQf-Cfpq}{A} \text{ et}$$

$$\sqrt{(A+Css)}=\frac{FPQ+CFpq-Cf(Pq+Qp)}{A}.$$

 Coroll. 2.

Coroll. 2.

592. Si ponamus $q=f$ et $Q=F$, vt fit $\Pi:q=0$, haec aequatio

$$\Pi:s=\Pi:p-\Pi:r$$

praebet

$$s=\frac{F(Rp-Pr)+fPR-Cfpr}{A} \text{ et}$$

$$\sqrt{(A+Css)}=\frac{FPR-CFpr+Cf(Rp-Pr)}{A}.$$

Coroll. 3.

593. Si fit $C=0$ et $A=1$, erit

$$\Pi:z=\int dz=z-f$$

quia integrale ita capi debet, vt euanefcat pofito $z=f$. Tum ergo erit $P=1$, $Q=1$ et $R=1$, vnde vt fit

$$\Pi:s=\Pi:p+\Pi:q-\Pi:r$$

feu $s=p+q-r$, oportet effe $s=-r+q+p$ et $\sqrt{(1+0ss)}=1$ vti per fe conftat.

Coroll. 4.

594. Si fumatur $A=1$ et $C=-1$, fiatque $\Pi:z=\text{Ang.cof.}z$ vt fit $f=1$, erit

$$\text{Arc.cof.}s=\text{Arc.cof.}p+\text{Arc.cof.}q-\text{Arc.cof.}r$$

fi fuerit

$$s=pqr-PQr+PRq+QRp \text{ et}$$
$$\sqrt{(1-ss)}=PQR+Pqr+Qpr-Rpq$$

vnde

vnde sumto $r=1$, vt sit $R=0$, et $\text{Arc.cos.}r=0$, erit $s=pq-PQ$ et $\sqrt{(1-ss)}=Pq+Qp$.

Scholion.

595. Hinc notae regulae pro cosinibus deducuntur, quas fusius non prosequor. Verum casus facillimus, quo $A=0$ et $C=1$, hincque fit $\Pi:z=\int\frac{dz}{z}=lz$ existente $f=1$, insigni difficultate premi videtur, ob expressiones pro s et $\sqrt{(A+Czz)}=z$ in infinitum abeuntes. Cui incommodo vt occurratur, primo quidem numerus A vt infinite paruus spectetur, eritque

$$P=\sqrt{(pp+A)}=p+\frac{A}{2p};\ Q=q+\frac{A}{2q};\ R=r+\frac{A}{2r}.$$

Quare vt fiat $ls=lp+lq-lr$, reperitur:

$$As=-r(p+\tfrac{A}{2p})(q+\tfrac{A}{2q})-pqr+q(p+\tfrac{A}{2p})(r+\tfrac{A}{2r})$$
$$+p(q+\tfrac{A}{2q})(r+\tfrac{A}{2r})$$

ac singulis membris euolutis

$$As=-\frac{Aqr}{2p}-\frac{Apr}{2q}+\frac{Aqr}{2p}+\frac{Apq}{2r}+\frac{Apr}{2q}+\frac{Apq}{2r}$$

seu $s=\frac{pq}{r}$, vti natura logarithmorum exigit. Caeterum ex formulis inuentis haud difficulter multiplicatio huiusmodi functionum transcendentium colligitur, veluti vt sit $\Pi:y=n\Pi:x$ relatio inter x et y algebraice assignari poterit.

Proble-

Problema 76.

596. Si ponatur $\Pi : z = \int \frac{dz(L+Mzz)}{\sqrt{(A+Czz)}}$, fumto hoc integrali ita vt euanefcat pofito $z=0$, comparationem inter huiusmodi functiones tranfcendentes inueftigare.

Solutio.

Statuatur inter binas variabiles x et y ifta relatio

$$\alpha + \gamma(xx+yy) + 2\delta xy = 0$$

vnde fit

$$y = \frac{-\delta x + \sqrt{(-\alpha\gamma + (\delta\delta - \gamma\gamma)xx)}}{\gamma}.$$

Ponatur $-\alpha\gamma = Am$ et $\delta\delta - \gamma\gamma = Cm$, vt fit

$$\gamma y + \delta x = \sqrt{m(A+Cxx)} \text{ et}$$

$$\gamma x + \delta y = \sqrt{m(A+Cyy)}.$$

At illam aequationem differentiando fit

$$dx(\gamma x + \delta y) + dy(\gamma y + \delta x) = 0$$

feu $\frac{dx}{\sqrt{(A+Cxx)}} + \frac{dy}{\sqrt{(A+Cyy)}} = 0.$

Iam ftatuatur

$$\frac{dx(L+Mxx)}{\sqrt{(A+Cxx)}} + \frac{dy(L+Myy)}{\sqrt{(A+Cyy)}} = dV\sqrt{m}$$

vt fit integrando:

$$\Pi : x + \Pi : y = \text{Conft.} + V\sqrt{m}.$$

Cum

Cum igitur fit

$$\frac{dy}{\sqrt{(A+Cyy)}} = \frac{-dx}{\sqrt{(A+Cxx)}} \text{ erit}$$

$$dV\sqrt{m} = \frac{Mdx(xx-yy)}{\sqrt{(A+Cxx)}}$$

hincque ob

$$y = \frac{\sqrt{m(A+Cxx)} - \delta x}{\gamma} \text{ erit}$$

$$xx-yy = \tfrac{1}{\gamma\gamma}(\gamma\gamma xx - mA - mCxx - \delta\delta xx + 2\delta x\sqrt{m(A+Cxx)}).$$

At $\gamma\gamma - \delta\delta = -mC$ ergo

$$dV\sqrt{m} = \frac{Mdx(2\delta x\sqrt{m(A+Cxx)} - mA - 2mCxx)}{\gamma\gamma\sqrt{(A+Cxx)}}$$

cuius integrale commode capi poteſt, dum fit

$$V\sqrt{m} = \frac{\delta Mxx\sqrt{m}}{\gamma\gamma} - \frac{Mmx}{\gamma\gamma}\sqrt{(A+Cxx)}$$

quae formula ob

$$\sqrt{m(A+Cxx)} = \gamma y + \delta x \text{ abit in}$$

$$V\sqrt{m} = \frac{\delta Mxx - \gamma Mxy - \delta Mxx}{\gamma\gamma}\sqrt{m} = -\frac{Mxy}{\gamma}\sqrt{m}.$$

Quocirca habebimus

$$\Pi : x + \Pi : y = \text{Conſt.} - \frac{Mxy}{\gamma}\sqrt{m}$$

exiſtente

$$\gamma y + \delta x = \sqrt{m(A+Cxx)} \text{ et } \gamma x + \delta y = \sqrt{m(A+Cyy)}$$

ac praeterea

$$-\alpha\gamma = Am \text{ et } \delta\delta - \gamma\gamma = Cm.$$

 Ad

Ad constantem definiendam sumamus posito $x=0$ fieri $y=b$, vt sit

$$\Pi:x+\Pi:y=\Pi:b-\frac{\mathfrak{M}xy}{\gamma}\sqrt{m}.$$

Tum vero est

$$\gamma b=\sqrt{m}\mathrm{A} \text{ et } \delta b=\sqrt{(m\mathrm{A}+m\mathrm{C}bb)}$$

ergo

$$\gamma=\frac{\sqrt{m\mathrm{A}}}{b} \text{ et } \delta=\frac{\sqrt{(m\mathrm{A}+m\mathrm{C}bb)}}{b}.$$

Hinc ergo concludimus, si fuerit

$$y\sqrt{\mathrm{A}}+x\sqrt{(\mathrm{A}+\mathrm{C}bb)}=b\sqrt{(\mathrm{A}+\mathrm{C}xx)}$$

et quod eodem redit

$$x\sqrt{\mathrm{A}}+y\sqrt{(\mathrm{A}+\mathrm{C}bb)}=b\sqrt{(\mathrm{A}+\mathrm{C}yy)}$$

fore

$$\Pi:x+\Pi:y=\Pi:b-\frac{\mathfrak{M}bxy}{\sqrt{\mathrm{A}}}$$

denotante Π eiusmodi functionem quantitatis suffixae vt sit

$$\Pi:z=\int\frac{dz(\mathrm{L}+\mathrm{M}zz)}{\sqrt{(\mathrm{A}+\mathrm{C}zz)}}$$

integrali hoc ita sumto, vt euanescat posito $z=0$. Natura harum functionum stabilita, ac sublato discrimine inter quantitates constantes ac variabiles, erit

$$\Pi:r=\Pi:p+\Pi:q+\frac{\mathfrak{M}pqr}{\sqrt{\mathrm{A}}}$$

si fuerit

$$q\sqrt{\mathrm{A}}+p\sqrt{(\mathrm{A}+\mathrm{C}rr)}=r\sqrt{(\mathrm{A}+\mathrm{C}pp)} \text{ et}$$
$$p\sqrt{\mathrm{A}}+q\sqrt{(\mathrm{A}+\mathrm{C}rr)}=r\sqrt{(\mathrm{A}+\mathrm{C}qq)}$$

vnde

vnde fit

$$r = \frac{p\sqrt{(A+Cqq)} + q\sqrt{(A+Cpp)}}{\sqrt{A}} \text{ et}$$

$$\sqrt{(A+Crr)} = \frac{Cpq + \sqrt{(A+Cpp)(A+Cqq)}}{\sqrt{A}}.$$

Coroll. 1.

597. Sumto z negatiuo est

$$\Pi : -z = -\Pi : z,$$

vnde capiendo quantitates p et q negatiue, fiet

$$\Pi : p + \Pi : q + \Pi : r = \frac{Mpqr}{\sqrt{A}}$$

si fuerit

$$p\sqrt{A} + q\sqrt{(A+Crr)} + r\sqrt{(A+Cqq)} = 0 \text{ seu}$$

$$q\sqrt{A} + p\sqrt{(A+Crr)} + r\sqrt{(A+Cpp)} = 0 \text{ seu}$$

$$r\sqrt{A} + p\sqrt{(A+Cqq)} + q\sqrt{(A+Cpp)} = 0 \text{ vel}$$

$$Cpq - \sqrt{A(A+Crr)} + \sqrt{(A+Cpp)(A+qq)} = 0$$

ex qua formatur haec relatio

$$Cpqr + p\sqrt{(A+Cqq)(A+Crr)} + q\sqrt{(A+Cpp)(A+Crr)} + r\sqrt{(A+Cpp)(A+Cqq)} = 0.$$

Coroll. 2.

598. Hac ergo methodo tres huiusmodi functiones $\Pi : z$ exhiberi possunt, quarum summam algebraice exprimere licet; quod autem de summa

 osten-

oſtendimus, valet quoque de ſumma binarum demta tertia.

Coroll. 3.

599. Si ponamus $L = A$ et $M = C$, functio propoſita $\Pi : z = \int dz \sqrt{(A + Czz)}$, exprimit arcum curuae, cuius abſciſſae z conuenit applicata $\sqrt{(A + Czz)}$; et ſumma trium huiusmodi arcuum ita algebraice dabitur:

$$\Pi : p + \Pi : q + \Pi : r = \frac{Cpqr}{\sqrt{A}}$$

ſi inter p, q, r ſuperior relatio ſtatuatur.

Scholion.

600. Haec proprietas inde eſt nata, quod differentiale dV integrationem admiſit. Cum nempe eſſet

$$dV \sqrt{m} = \frac{M dx (xx - yy)}{\sqrt{(A + Cxx)}} \text{ ob}$$

$$\sqrt{m}(A + Cxx) = \gamma y + \delta x \text{ erit}$$

$$dV = \frac{M dx (xx - yy)}{\gamma y + \delta x}$$

cuius integrale commode ex aequatione aſſumta

$$\alpha + \gamma (xx + yy) + 2\delta xy = 0$$

definiri poteſt. Ponatur enim

$$xx + yy = tt \text{ et } xy = u \text{, erit}$$

$$\alpha + \gamma tt + 2\delta u = 0$$

et

et differentialibus sumendis

$xdx+ydy=tdt$; $xdy+ydx=du$ et $\gamma tdt+\delta du=0$

ex binis prioribus colligitur:

$$(xx-yy)dx=xtdt-ydu\text{, et ob } tdt=-\frac{\delta du}{\gamma}$$

erit $(xx-yy)dx=-\frac{du}{\gamma}(\delta x+\gamma y)$,

ita vt sit

$$\frac{dx(xx-yy)}{\gamma y+\delta x}=-\frac{du}{\gamma}\text{ hincque}$$

$$dV=-\frac{Mdu}{\gamma},$$

vnde manifesto sequitur

$$V=-\frac{Mu}{\gamma}=-\frac{Mxy}{\gamma},$$

vti in solutione operosius eruimus. Verum hac operatione commode vti licebit in sequente problemate, vbi formulas magis complexas sumus contemplaturi.

Problema 77.

601. Si ponatur $\Pi:z=\int\frac{dz(L+Mz^2+Nz^4+Oz^6\ etc.)}{\surd(A+Czz)}$ integrali hoc ita sumto vt euanescat posito $z=0$, comparationem inter huiusmodi functiones transcendentes inuestigare.

Solutio.

Posita vt ante inter variabiles x et y hac relatione

$$\alpha+\gamma(xx+yy)+2\delta xy=0$$

 sit

fit

$$-\alpha\gamma = Am \text{ et } \delta\delta - \gamma\gamma = Cm$$

fietque

$$\gamma y + \delta x = \surd m(A + Cxx) \text{ et } \gamma x + \delta y = \surd m(A + Cyy)$$

famtisque differentialibus

$$\frac{dx}{\surd(A + Cxx)} + \frac{dy}{\surd(A + Cyy)} = 0.$$

Iam ftatuatur:

$$\frac{dx(L + Mx^2 + Nx^4 + Ox^6)}{\surd(A + Cxx)} + \frac{dy(L + Myy + Ny^4 + Oy^6)}{\surd(A + Cyy)} = dV\surd m$$

vt fit

$$\Pi : x + \Pi : y = \text{Conft.} + V\surd m.$$

At ob $\frac{dy}{\surd(A + Cyy)} = -\frac{dx}{\surd(A + Cxx)}$ ifta acquatio abit in

$$\frac{dx(M(xx - yy) + N(x^4 - y^4) + O(x^6 - y^6))}{\surd(A + Cxx)} = dV\surd m$$

et ob $\surd m(A + Cxx) = \gamma y + \delta x$ in hanc

$$\frac{dx(xx - yy)(M + N(xx + yy) + O(x^4 + xxyy + y^4))}{\gamma y + \delta x} = dV$$

Sit nunc $xx + yy = tt$ et $xy = u$, vt habeatur:

$$\alpha + \gamma tt + \delta u = 0 \text{ et } \gamma t dt + \delta du = 0$$

feu $t dt = -\frac{\delta du}{\gamma}$

atque ob

$$x dx + y dy = t dt \text{ et } x dy + y dx = du$$

hinc colligimus

$$(xx - yy)dx = x t dt - y du = -\frac{du}{\gamma}(\gamma y + \delta x)$$

ideoque

ideoque

$$\frac{dx(xx-yy)}{\gamma y+\delta x}=-\frac{du}{\gamma},$$

vnde habebimus:

$$dV=-\frac{du}{\gamma}(M+N(xx+yy)+O(x^4+xxyy+y^4))$$

At eft

$$xx+yy=tt=\frac{-\alpha-2\delta u}{\gamma} \text{ et}$$
$$x^4+xxyy+y^4=t^4-uu.$$

Notetur autem effe $\frac{du}{\gamma}=-\frac{tdt}{\delta}$, vnde concludimus:

$$dV=-\frac{Mdu}{\gamma}+\frac{Nt^3dt}{\delta}+\frac{Ot^5dt}{\delta}+\frac{Ouudu}{\gamma}$$

ficque prodit integrando:

$$V=-\frac{Mu}{\gamma}+\frac{Nt^4}{4\delta}+\frac{Ot^6}{6\delta}+\frac{Ou^3}{3\gamma}.$$

Quodfi iam ponamus fieri $y=b$ fi $x=0$, erit $\gamma=\frac{\sqrt{mA}}{b}$, $\delta=\frac{\sqrt{m(A+Cbb)}}{b}$; et $\alpha=-b\sqrt{mA}$, tum vero

$$y\sqrt{A}+x\sqrt{(A+Cbb)}=b\sqrt{(A+Cxx)}$$
$$x\sqrt{A}+y\sqrt{(A+Cbb)}=b\sqrt{(A+Cyy)} \text{ et}$$
$$b\sqrt{A}=x\sqrt{(A+Cyy)}+y\sqrt{(A+Cxx)}.$$

Hinc cum fit

$$V=-\frac{Mbxy}{\sqrt{mA}}+\frac{Nb(xx+yy)^2}{4\sqrt{m(A+Cbb)}}+\frac{Ob(xx+yy)^3}{6\sqrt{m(A+Cbb)}}+\frac{Obx^3y^3}{3\sqrt{mA}}$$

noftra relatio, cui fatisfaciunt praecedentes determinationes, inter functiones tranfcendentes, erit

$$\Pi:x+\Pi:y=\Pi:b-\frac{Mbxy}{\sqrt{A}}+\frac{Nb(xx+yy)^2}{4\sqrt{(A+Cbb)}}+\frac{Ob(xx+yy)^3}{6\sqrt{(A+Cbb)}}+\frac{Obx^3y^3}{3\sqrt{A}}-\frac{Nb^5}{4\sqrt{(A+Cbb)}}-\frac{Ob^7}{6\sqrt{(A+Cbb)}}$$

vbi

vbi notandum est esse in rationalibus

$$-b\sqrt{A}+\frac{(xx+yy)\sqrt{A}}{b}+\frac{2xy\sqrt{(A+Cbb)}}{b}=0$$

seu

$$xx+yy=bb-\frac{2xy\sqrt{(A+Cbb)}}{\sqrt{A}}.$$

Hinc colligitur:

$$(xx+yy)^2-b^4=-\frac{4bbxy\sqrt{(A+Cbb)}}{\sqrt{A}}+\frac{4xxyy(A+Cbb)}{A}$$

et

$$(xx+yy)^3-b^6=-\frac{6b^4xy\sqrt{(A+Cbb)}}{\sqrt{A}}+\frac{12bbxxyy(A+Cbb)}{A}-\frac{8x^3y^3(A+Cbb)^{\frac{3}{2}}}{A\sqrt{A}}$$

ita vt nostra aequatio fit

$$\Pi:x+\Pi:y=\Pi:b-\frac{Mbxy}{\sqrt{A}}-\frac{Nb^3xy}{\sqrt{A}}+\frac{Nb^2xxyy}{A}\sqrt{(A+Cbb)}$$
$$-\frac{Ob^5xy}{\sqrt{A}}+\frac{2Ob^3xxyy}{A}\sqrt{(A+Cbb)}-\frac{O^5x^3y^3}{3A\sqrt{A}}(3A+4Cbb).$$

Coroll. 1.

602. Si ponamus $b=r$, $x=-p$, $y=-q$, erit nostra aequatio

$$\Pi:p+\Pi:q+\Pi:r=\frac{pqr}{\sqrt{A}}(M+Nrr+Or^4)-\frac{ppqq\sqrt{(A+Crr)}}{A}(Nr+2Or^3)$$
$$+\frac{Op^3q^3r}{3A\sqrt{A}}(3A+4Crr)$$

existente $pp+qq=rr-\frac{2pq}{\sqrt{A}}\sqrt{(A+Crr)}$, vnde fit

$$\frac{\sqrt{(A+Crr)}}{\sqrt{A}}=\frac{rr-pp-qq}{2pq}.$$

Coroll. 2.

Coroll. 2.

603. Subſtituto hoc valore pro $\frac{\sqrt{(A+Crr)}}{\sqrt{A}}$, ſequens obtinebitur aequatio, in quam ternae quantitates p, q, r aequaliter ingrediuntur:

$$\Pi:p+\Pi:q+\Pi:r=\frac{Mpqr}{\sqrt{A}}+\frac{Npqr}{2\sqrt{A}}(pp+qq+rr)$$
$$+\frac{Opqr}{3\sqrt{A}}(p^4+q^4+r^4+ppqq+pprr+qqrr)$$

cui ſatisfaciunt formulae ſupra (602.) datae vel haec rationalis

$$\frac{4Cppqqrr}{A}=p^4+q^4+r^4-2ppqq-2pprr-2qqrr.$$

Coroll. 3.

604. Si numeratori formulae integralis adhuc adieciſſemus terminum Pz^8 vt eſſet

$$\Pi:z=\int\frac{dz(L+Mz^2+Nz^4+Oz^6+Pz^8)}{\sqrt{(A+Czz)}}$$

ad aequationem modo inuentam adhuc acceſſiſſet terminus:

$$\frac{Ppqr}{4\sqrt{A}}(p^6+q^6+r^6+ppq^4+ppr^4+p^4qq+p^4rr+q^4rr+qqr^4+3ppqqrr)$$

Scholion.

605. Iſtae relationes quoque ex ſuperioribus reductionibus deriuari poſſunt, cum enim inde ſit $\Pi:z=E\int\frac{dz}{\sqrt{(A+Czz)}}+$. *quantitate algebraica* ſi hic pro z ſucceſſiue quantitates p, q, r ſubſtituamus, ita a ſe inuicem pendentes, vt ante declarauimus

vimus, erit

$$\int\frac{dp}{\sqrt{(A+Cpp)}}+\int\frac{dq}{\sqrt{(A+Cqq)}}+\int\frac{dr}{\sqrt{(A+Crr)}}=0$$

vnde concludimus:

$$\Pi:p+\Pi:q+\Pi:r=f:p+f:q+f:r$$

denotante f functionem quandam algebraicam quantitatis suffixae: atque summa harum trium functionum rediret ad expressionem ante inuentam, si modo relationis inter p, q, r datae ratio habeatur: scilicet inde littera C eliminari deberet. Haec autem reductio ingentem laborem requireret. Hic vero imprimis methodum, qua hic sum vsus, spectari conuenit, quae cum sit prorsus singularis, ad magis arduam deducere videtur. Certe comparatio functionum transcendentium, quam in capite sequente sum traditurus, vix alia methodo inuestigari posse videtur, vnde huius methodi vtilitas in sequenti capite potissimum cernetur.

CAPVT VI.

CAPVT VI.

DE COMPARATIONE QVANT[illegible]

[illegible]

$$\int \frac{P\,dz}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}}$$

Problema 78.

606.

Proposita relatione inter x et y hac

$$\alpha+\gamma(xx+yy)+2\delta xy+\zeta xxyy=0$$

inde elicere functiones transcendentes formae praescriptae quas inter se comparare liceat.

Solutio.

Ex proposita aequatione definiatur vtraque variabilis

$$y=\frac{-\delta x+\sqrt{(-\alpha\gamma+(\delta\delta-\gamma\gamma-\alpha\zeta)xx-\gamma\zeta x^4)}}{\gamma+\zeta xx} \text{ et}$$

$$x=\frac{-\delta y+\sqrt{(-\alpha\gamma+(\delta\delta-\gamma\gamma-\alpha\zeta)yy-\gamma\zeta y^4)}}{\gamma+\zeta yy}$$

 quae

quae radicalia ad formam praescriptam reuocentur ponendo:

$$-\alpha\gamma = Am;\ \delta\delta - \gamma\gamma - \alpha\zeta = Cm \text{ et } -\gamma\zeta = Em$$

vnde fit:

$$[illegible]\ \zeta = -\frac{Em}{\gamma} \text{ et } \delta\delta = Cm + \gamma\gamma + \frac{AEmm}{\gamma\gamma}.$$

$$\gamma y + [illegible]$$

$$\gamma x + \delta y + \zeta xyy = \sqrt{m(A + Cyy + Ey^4)}$$

Ipsa autem aequatio proposita, si differentietur, dat

$$dx(\gamma x + \delta y + \zeta xyy) + dy(\gamma y + \delta x + \zeta xxy) = 0$$

vbi illi valores substituti praebent

$$\frac{dx}{\sqrt{(A + Cxx + Ex^4)}} + \frac{dy}{\sqrt{(A + Cyy + Ey^4)}} = 0.$$

Vicissim ergo proposita hac aequatione differentiali ei satisfaciet haec aequatio finita:

$$-Am + \gamma\gamma(xx + yy) + 2xy\sqrt{(\gamma^4 + Cm\gamma\gamma + AEmm)} - Emxxyy = 0$$

seu ponendo $\frac{\gamma\gamma}{m} = k$ haec:

$$-A + k(xx + yy) + 2xy\sqrt{(kk + kC + AE)} - Exxyy = 0$$

quae cum inuoluat constantem k in aequatione differentiali non contentam, simul erit integrale completum. Hinc autem fit

$$ky + x\sqrt{(kk + kC + AE)} - Exxy = \sqrt{k(A + Cxx + Ex^4)} \text{ et}$$

$$kx + y\sqrt{(kk + kC + AE)} - Exyy = \sqrt{k(A + Cyy + Ey^4)}$$

Coroll. 1.

Coroll. 1.

607. Conſtans k ita aſſumi poteſt, vt poſito $x=0$, fiat $y=b$; oritur autem:

$$bk=\sqrt{A}k \text{ et } b\sqrt{(kk+kC+AE)}=\sqrt{k(A+Cbb+Eb^4)}$$

ergo $k=\frac{A}{bb}$ et $\sqrt{(kk+kC+AE)}=\frac{1}{}\sqrt{A(A}$ [illegible]

$$Ay+x\sqrt{A(A+Cbb+Eb^4)}-Ebbxxy=b\sqrt{A(A+Cxx+Ex^4)}$$

et

$$Ax+y\sqrt{A(A+Cbb+Eb^4)}-Ebbxyy=b\sqrt{A(A+Cyy+Ey^4)}.$$

Coroll. 2.

608. Haec igitur relatio finita inter x et y erit integrale completum aequationis differentialis:

$$\frac{dx}{\sqrt{(A+Cxx+Ex^4)}}+\frac{dy}{\sqrt{(A+Cyy+Ey^4)}}=0$$

quod rationaliter inter x et y expreſſum erit:

$$A(xx+yy-bb)+2xy\sqrt{A(A+Cbb+Eb^4)}-Ebbxxyy=0.$$

Coroll. 3.

609. Hinc ergo y ita per x exprimetur, vt ſit:

$$y=\frac{b\sqrt{A(A+Cxx+Ex^4)}-x\sqrt{A(A+Cbb+Eb^4)}}{A-Ebbxx}$$

atque ex hoc valore elicitur:

$$\frac{\sqrt{(A+Cyy+Ey^4)}}{A}=\frac{(A+Ebbxx)\sqrt{(A+Cbb+Eb^4)(A+Cxx+Ex^4)}-2AEbx(bb+xx)-Cbx(A+Ebbxx)}{(A-Ebbxx)^2}$$

Coroll. 4.

610. Hinc constantem b pro lubitu determinando infinita integralia particularia exhiberi possunt, quorum praecipua sunt: 1) sumendo $b=0$, vnde 3) Si $A+Cbb+Eb^4=0$, $b=\infty$, vnde fit $y=\frac{\sqrt{A}}{x\sqrt{E}}$ vnde fit $y=\frac{b\sqrt{A(A+Cxx+Ex^4)}}{A-Ebbxx}$.

Scholion.

611. Hic iam vsus istius methodi, qua retrogrediendo ab aequatione finita ad aequationem differentialem peruenimus, luculenter perspicitur. Cum enim integratio formulae $\int\frac{dx}{\sqrt{(A+Cxx+Ex^4)}}$ nullo modo neque per logarithmos neque arcus circulares perfici posset, mirum sane est talem aequationem differentialem adeo algebraice integrari posse; quae quidem in praecedente capite ope eiusdem methodi sunt tradita, etiam methodo ordinaria erui possunt, dum singulae formulae differentiales vel per logarithmos vel arcus circulares exprimuntur, quorum deinceps comparatio ad aequationem algebraicam reducitur. Verum quia hic talis integratio plane non locum inuenit, nulla certe alia methodus patet, qua idem integrale, quod hic exhibuimus, inuestigari posset. Quare hoc argumentum diligentius euoluamus.

Problema 79.

612. Si $\Pi:z$ denotet eiusmodi functionem ipsius z, vt sit $\Pi:z=\int\frac{dz}{\sqrt{(A+Czz+Ez^4)}}$ integrali ita sumto vt euanescat posito $z=0$, comparationem inter huiusmodi functiones inuestigare.

Solutio.

Posita inter binas variabiles x et y relatione supra definita vidimus fore:

$$\frac{dx}{\sqrt{(A+Cxx+Ex^4)}}+\frac{dy}{\sqrt{(A+Cyy+Ey^4)}}=0.$$

Hinc cum posito $x=0$ fiat $y=b$, elicitur integrando

$$\Pi:x+\Pi:y=\Pi:b.$$

Cum iam nullum amplius discrimen inter variabiles x, y et constantem b intercedat, statuamus $x=p$, $y=q$, et $b=-r$, vt sit $\Pi:b=-\Pi:r$, atque haec relatio inter functiones transcendentes

$$\Pi:p+\Pi:q+\Pi:r=0$$

per sequentes formulas algebraicas exprimetur:

$$(A-Epprr)q+p\sqrt{A(A+Crr+Er^4)}+r\sqrt{A(A+Cpp+Ep^4)}=0 \text{ seu}$$

$$(A-Eppqq)r+q\sqrt{A(A+Cpp+Ep^4)}+p\sqrt{A(A+Cqq+Eq^4)}=0 \text{ seu}$$

$$(A-Eqqrr)p+r\sqrt{A(A+Cqq+Eq^4)}+q\sqrt{A(A+Crr+Er^4)}=0$$

quae oriuntur ex hac aequatione:

$$A(pp+qq-rr)-Eppqqrr+2pq\sqrt{A(A+Crr+Er^4)}=0.$$

Haec

Haec vero ad rationalitatem perducta fit

$$AA(p^4+q^4+r^4-2ppqq-2pprr-2qqrr)$$
$$-2AEpqr(pp+qq+rr)-4ACppqqrr$$
$$+EEp^4q^4r^4=0$$

quae autem ob pluralitatem radicum satisfacit omnibus signorum variationibus in superiori aequatione transcendente.

Coroll. 1.

613. Sumamus r negatiue, vt fiat:

$$\Pi:r=\Pi:p+\Pi:q$$

eritque

$$r=\frac{p\sqrt{A(A+Cqq+Eq^4)}+q\sqrt{A(A+Cpp+Ep^4)}}{A-Eppqq}$$

vnde colligitur fore:

$$\sqrt{\frac{A+Crr+Er^4}{A}}=\frac{(A+Eppqq)\sqrt{(A+Cpp+Ep^4)(A+Cqq+Eq^4)}+2AEpq(pp+qq)+Cpq(A+Eppqq)}{(A-Eppqq)^2}.$$

Coroll. 2.

614. Quodsi ergo ponamus $q=p$ vt sit

$$\Pi:r=2\,\Pi:p$$

erit

$$r=\frac{2p\sqrt{A(A+Cpp+Ep^4)}}{A-Ep^4}$$

atque

$$\sqrt{\frac{A+Crr+Er^4}{A}}=\frac{AA+2ACpp+6AEp^4+2CEp^6+EEp^8}{(A-Ep^4)^2}.$$

Hoc igitur modo functio assignari potest aequalis duplo similis functionis.

Coroll. 3.

Coroll. 3.

615. Si ponatur $q = \frac{2p\sqrt{A(A+Cpp+Ep^4)}}{A-Ep^4}$ et $\sqrt{A(A+Cqq+Eq^4)} = \frac{A(AA+2ACpp+6AEp^4+2CEp^6+EEp^8)}{(A-Ep^4)^2}$, vt sit $\Pi:q = 2\Pi:p$ fiet ex primo Coroll. $\Pi:r = 3\Pi:p$. Tum igitur erit $r = \frac{p(3AA+4ACpp+6AEp^4-EEp^8)}{AA-6AEp^4-4CEp^6-3EEp^8}$.

Scholion 1.

616. Nimis operosum est hanc functionum multiplicationem vlterius continuare, multoque minus legem in earum progressione deprehendere licet. Quodsi ponamus breuitatis gratia

$$\sqrt{A(A+Cpp+Ep^4)} = AP \text{ et } A-Ep^4 = A\mathfrak{P},$$

vt sit

$$Cpp = APP - A - Ep^4 \text{ et } Ep^4 = A(1-\mathfrak{P}),$$

hae multiplicationes vsque ad quadruplum ita se habebunt; scilicet si statuamus:

$$\Pi:r = 2\Pi:p;\ \Pi:s = 3\Pi:p \text{ et } \Pi:t = 4\Pi:p$$

reperietur:

$$r = \frac{2Pp}{\mathfrak{P}};\ s = \frac{p(4PP-\mathfrak{P}\mathfrak{P})}{\mathfrak{P}\mathfrak{P}-4PP(1-\mathfrak{P})};\ t = \frac{4pP\mathfrak{P}(2PP(2-\mathfrak{P})-\mathfrak{P}\mathfrak{P})}{\mathfrak{P}^4-16P^4(1-\mathfrak{P})}.$$

Quodsi simili modo ponamus:

$$\sqrt{A(A+Crr+Er^4)} = AR \text{ et } A-Er^4 = A\mathfrak{R}$$

erit

$$R = \frac{2PP(2-\mathfrak{P})-\mathfrak{P}\mathfrak{P}}{\mathfrak{P}\mathfrak{P}} \text{ et } \mathfrak{R} = \frac{\mathfrak{P}^4-16P^4(1-\mathfrak{P})}{\mathfrak{P}^4}$$

 vnde

vnde pro quadruplicatione fit

$$t = \frac{2Rr}{\mathfrak{R}}; \quad T = \frac{2RR(2-\mathfrak{R}) - \mathfrak{R}\mathfrak{R}}{\mathfrak{R}\mathfrak{R}}; \quad \mathfrak{T} = \frac{\mathfrak{R}^4 - 16R^4(1-\mathfrak{R})}{\mathfrak{R}^4}.$$

Quare si pro octuplicatione statuamus $\Pi : z = 8\,\Pi : p$ erit

$$z = \frac{2Tt}{\mathfrak{T}} = \frac{4rR\mathfrak{R}(2RR(2-\mathfrak{R}) - \mathfrak{R}\mathfrak{R})}{\mathfrak{R}^4 - 16R^4(1-\mathfrak{R})}.$$

Hinc intelligitur quomodo in continua duplicatione versari oporteat, neque tamen legem progressionis obseruare licet. Caeterum cognitio huius legis ad incrementum Analyseos maxime esset optanda, vt inde generatim relatio inter z et p, pro aequalitate $\Pi : z = n\Pi : p$ definiri posset, quemadmodum hoc in capite praecedente successit, hinc enim eximias proprietates circa integralia formae $\int \frac{dz}{\sqrt{(A + Czz + Ez^4)}}$ cognoscere liceret; quibus scientia analytica haud mediocriter promoueretur.

Scholion 2.

617 Modus maxime idoneus in legem progressionis inquirendi, videtur, si ternos terminos se ordine excipientes contemplemur hoc modo:

$$\Pi : x = (n-1)\Pi : p; \quad \Pi : y = n\Pi : p; \quad \Pi : z = (n+1)\Pi : p$$

vbi cum sit

$$\Pi : x = \Pi : y - \Pi : p \text{ et } \Pi : z = \Pi : y + \Pi : p \text{ erit}$$

$$x = \frac{y\sqrt{A(A + Cpp + Ep^4)} - p\sqrt{A(A + Cyy + Ey^4)}}{A - Eppyy}$$

$$z = \frac{y\sqrt{A(A + Cpp + Ep^4)} + p\sqrt{A(A + Cyy + Ey^4)}}{A - Eppyy}$$

vnde

vnde concludimus

$$(A-Eppyy)(x+z)=2y\sqrt{A(A+Cpp+Ep^4)}.$$

Ponamus vt ante:

$$\sqrt{A(A+Cpp+Ep^4)}=AP \text{ et } A-Ep^4=A\mathfrak{P}$$

et quia singulae quantitates x, y, z factorem p simpliciter inuoluunt, fit

$$x=pX;\ y=pY \text{ et } z=pZ,$$

erit

$$(1-(1-\mathfrak{P})YY)(X+Z)=2PY$$

seu

$$Z=\frac{2PY}{1-(1-\mathfrak{P})YY}-X$$

cuius formulae ope ex binis terminis contiguis X et Y sequens Z haud difficulter inuenitur. Quod quo facilius appareat ponatur $2P=Q$ et $1-\mathfrak{P}=\Omega$ vt sit $Z=\frac{QY}{1-\Omega YY}-X$. Iam progressio quaesita ita se habebit

1) 1;

2) $\frac{Q}{\mathfrak{P}}$;

3) $\frac{QQ-\mathfrak{P}\mathfrak{P}}{\mathfrak{P}\mathfrak{P}-QQ\Omega}$;

4) $\frac{Q^3\mathfrak{P}(1+\Omega)-2Q\mathfrak{P}^3}{\mathfrak{P}^4-Q^4\Omega}$;

5) $\frac{\mathfrak{P}^6-3QQ\mathfrak{P}^4+Q^4\mathfrak{P}^2(1+2\Omega)-Q^6\Omega\Omega}{\mathfrak{P}^6-3QQ\mathfrak{P}^4\Omega+Q^4\mathfrak{P}\mathfrak{P}\Omega(2+\Omega)-Q^6\Omega}$ etc.

Quaestio ergo huc redit, vt inuestigetur progressio, ex data relatione inter ternos terminos successiuos X, Y, Z, quae sit $Z=\frac{QY}{1-\Omega YY}-X$; existente termino primo $=1$ et secundo $=\frac{Q}{1-\Omega}$.

 Proble-

Problema 80.

618. Si $\Pi : z$ eiusmodi denotet functionem ipsius z, vt sit $\Pi : z = \int \frac{dz(L + Mzz + Nz^4)}{\sqrt{(A + Czz + Ez^4)}}$, integrali ita sumto vt euanescat posito $z = o$, comparationem inter huiusmodi functiones transcendentes inuestigare.

Solutio.

Stabilita inter binas variabiles x et y hac relatione vt sit

$$Ay + \mathfrak{B}x - Ebbxxy = b\sqrt{A(A + Cxx + Ex^4)} \text{ seu}$$

$$Ax + \mathfrak{B}y - Ebbxyy = b\sqrt{A(A + Cyy + Ey^4)}$$

siue sublata irrationalitate

$$A(xx + yy - bb) + 2\mathfrak{B}xy - Ebbxxyy = o$$

existente breuitatis gratia $\mathfrak{B} = \sqrt{A(A + Cbb + Eb^4)}$ erit vti ante vidimus:

$$\frac{dx}{\sqrt{(A + Cxx + Ex^4)}} + \frac{dy}{\sqrt{(A + Cyy + Ey^4)}} = o.$$

Ponamus igitur:

$$\frac{dx(L + Mxx + Nx^4)}{\sqrt{(A + Cxx + Ex^4)}} + \frac{dy(L + Myy + Ny^4)}{\sqrt{(A + Cyy + Ey^4)}} = bdV \sqrt{A}$$

vt sit nostro signandi more

$$\Pi : x + \Pi : y = \text{Const.} + bV\sqrt{A}$$

vbi constans ita definiri debet, vt posito $x = o$ fiat $y = b$. Quaestio ergo ad inuentionem functionis V reuo-

reuocatur; quem in finem loco dy valore ex priori aequatione substituto erit:

$$bdV\sqrt{A} = \frac{dx(M(xx-yy)+N(x^4-y^4))}{\sqrt{(A+Cxx+Ex^4)}}$$

verum quia

$$b\sqrt{A(A+Cxx+Ex^4)} = Ay+\mathfrak{B}x-Ebbxxy$$

habebimus

$$dV = \frac{dx(xx-yy)(M+N(xx+yy))}{Ay+\mathfrak{B}x-Ebbxxy}.$$

Sumamus iam aequationem rationalem:

$$A(xx+yy-bb)+2\mathfrak{B}xy-Ebbxxyy=0$$

et ponamus

$$xx+yy=tt \text{ et } xy=u$$

vt fit

$$A(tt-bb)+2\mathfrak{B}u-Ebbuu=0$$

ideoque

$$Atdt=-\mathfrak{B}du+Ebbudu.$$

Cum porro fit

$$xdx+ydy=tdt \text{ et } xdy+ydx=du$$

erit

$$(xx-yy)dx=xtdt-ydu$$

seu

$$A(xx-yy)dx=-du(Ay+\mathfrak{B}x-Ebbxxy)$$

ita vt fit

$$\frac{dx(xx-yy)}{Ay+\mathfrak{B}x-Ebbxxy}=-\frac{du}{A}$$

 ex

ex quo deducitur:

$$dV = -\frac{du}{A}(M+Ntt)$$

et ob

$$tt = bb - \frac{2\mathfrak{B}u}{A} + \frac{Ebbuu}{A}$$

erit

$$dV = -\frac{du}{AA}(AM + ANbb - 2\mathfrak{B}Nu + ENbbuu)$$

vnde integrando elicitur:

$$V = -\frac{Mu}{A} - \frac{Nbbu}{A} + \frac{\mathfrak{B}Nuu}{AA} - \frac{ENbbu^3}{3AA}.$$

Hoc ergo valore substituto ob $u = xy$ habebimus

$$\Pi:x + \Pi:y = \Pi:b - \frac{Mbxy}{\sqrt{A}} - \frac{Nb^3xy}{\sqrt{A}} + \frac{\mathfrak{B}Nbx^2y^2}{A\sqrt{A}} - \frac{ENb^3x^3y^3}{3A\sqrt{A}}.$$

Cum autem sit

$$\mathfrak{B}xy = \tfrac{1}{2}Abb - \tfrac{1}{2}A(xx+yy) + \tfrac{1}{2}Ebbxxyy$$

erit

$$\Pi:x + \Pi:y = \Pi:b - \frac{Mbxy}{\sqrt{A}} - \frac{Nbxy}{2A\sqrt{A}}(Abb + xx + yy) - \tfrac{1}{3}Ebbxxyy)$$

cui ergo aequationi satisfit per formulas algebraicas supra exhibitas, quibus relatio inter x, y et b exprimitur. Quodsi ergo statuatur haec aequatio:

$$\Pi:p + \Pi:q + \Pi:r = \frac{Mpqr}{\sqrt{A}} + \frac{Npqr}{2A\sqrt{A}}(A(pp+qq+rr) - \tfrac{1}{3}Eppqqrr)$$

ea efficitur sequenti relatione inter p, q, r constituta:

$$(A - Eppqq)r + p\sqrt{A(A+Cqq+Eq^4)} + q\sqrt{A(A+Cpp+Ep^4)} = 0 \text{ seu}$$
$$(A - Epprr)q + p\sqrt{A(A+Crr+Er^4)} + r\sqrt{A(A+Cpp+Ep^4)} = 0 \text{ seu}$$
$$(A - Eqqrr)p + q\sqrt{A(A+Crr+Er^4)} + r\sqrt{A(A+Cqq+Eq^4)} = 0$$

siue

fiue per fimplicem irrationalitatem

$$A(pp+qq-rr)+2pq\sqrt{A(A+Crr+Er^4)}-Eppqqrr=0 \text{ feu}$$
$$A(pp+rr-qq)+2pr\sqrt{A(A+Cqq+Eq^4)}-Eppqqrr=0 \text{ feu}$$
$$A(qq+rr-pp)+2qr\sqrt{A(A+Cpp+Ep^4)}-Eppqqrr=0$$

penitusque irrationalitate fublata:

$$EEp^4q^4r^4-2AEppqqrr(pp+qq+rr)-4ACppqqrr$$
$$+AA(p^4+q^4+r^4-2ppqq-2pprr-2qqrr)=0.$$

Coroll. 1.

619. Sit $q=r=s$, vt habeamus hanc aequationem:

$$\Pi:p+2\Pi:s=\frac{Mpss}{\sqrt{A}}+\frac{Npss}{2A\sqrt{A}}(A(pp+2ss)-\tfrac{1}{3}Epps^4)$$

cui fatisfacit haec relatio:

$$(A-Es^4)p+2s\sqrt{A(A+Css+Es^4)}=0.$$

Coroll. 2.

620. Sumamus s negatiue, et loco p fubftituamus ibi hunc valorem, vt habeamus:

$$2\Pi:s+\Pi:q+\Pi:r+\frac{Mpss}{\sqrt{A}}+\frac{Npss}{2A\sqrt{A}}(A(pp+2ss)-\tfrac{1}{3}Epps^4)$$
$$=\frac{Mpqr}{\sqrt{A}}+\frac{Npqr}{2A\sqrt{A}}(A(pp+qq+rr)-\tfrac{1}{3}Eppqqrr)$$

exiftente

$$p=\frac{2s\sqrt{A(A+Css+Es^4)}}{A-Es^4},$$

vnde

vnde fit

$$\sqrt{A(A+Cpp+Ep^4)}=\frac{A(A+Css+Es^4)^2+A(4AE-CC)s^4}{(A-Es^4)^2}$$

qui valores in superioribus formulis substitui debent.

Coroll. 3.

621. Hoc modo effici poterit, vt partes algebraicae euanescant, atque functiones transcendentes solae inter se comparentur. Veluti si esset $N=0$, statui oporteret $ss=qr$, vt fieret:

$$2\Pi:s+\Pi:q+\Pi:r=0.$$

At posito $ss=qr$ fit

$$p=\frac{2\sqrt{Aqr}(A+Cqr+Eqqrr)}{A-Eqqrr}.$$

Est vero etiam

$$p=\frac{-q\sqrt{A(A+Crr+Er^4)}-r\sqrt{A(A+Cqq+Eq^4)}}{A-Eqqrr}$$

quibus valoribus aequatis oritur haec aequatio:

$$(AA+EEq^4r^4)(qq-6qr+rr)-8Cqqrr(A+Eqqrr)$$
$$-2AEqqrr(qq+10qr+rr)=0.$$

Scholion.

622. Si $\Pi:z$ exprimat arcum cuiuspiam lineae curuae respondentem abscissae vel cordae z, hinc plures arcus eiusdem curuae inter se comparare licet,

licet, vt vel differentia binorum arcuum fiat algebraica, vel arcus exhibeantur datam rationem inter se tenentes. Hoc modo eiusmodi insignes curuarum proprietates eruuntur, quarum ratio aliunde vix perspici queat. Comparatio quidem arcuum circularium ex elementis nota per caput praecedens, vt vidimus, facile expeditur, vnde etiam comparatio arcuum parabolicorum deriuatur. Ex hoc autem capite comparatio arcuum ellipticorum et hyperbolicorum simili modo institui potest; cum enim in genere arcus sectionis conicae tali formula exprimatur $\int dx\sqrt{\frac{a+bxx}{c+exx}}$, haec transformata in istam $\int\frac{dx(a+bxx)}{\sqrt{(ac+(ae+bc)xx+bex^4)}}$, per praecepta tradita tractari potest, ponendo $A=ac$, $C=ab+bc$, $E=be$ et $L=a$, $M=b$ atque $N=0$. Haec autem inuestigatio ad formulas, quarum denominator est $\sqrt{(A+2Bz+Czz+Dz^3+Ez^4)}$ extendi potest, similisque est praecedenti, quam idcirco hic sum expositurus, vnde simul patebit, hunc esse vltimum terminum, quousque progredi liceat. Formulae enim integrales magis complicatae, vbi post signum radicale altiores potestates ipsius z occurrunt, vel ipsum signum radicale altiorem dignitatem inuoluit, hoc modo non videntur inter se comparari posse, paucissimis casibus exceptis, qui per quampiam substitutionem ad huiusmodi formam reduci queant.

Problema 81.

623. Si $\Pi:z$ eiusmodi functionem ipsius z denotet, vt sit $\Pi:z = \int \frac{dz}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4)}}$, huiusmodi functiones inter se comparare.

Solutio.

Inter binas variabiles x et y statuatur relatio hac aequatione expressa:

$$\alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy + 2\varepsilon xy(x+y) + \zeta xxyy = 0$$

vnde cum fiat

$$yy = \frac{-2y(\beta + \delta x + \varepsilon xx) - \alpha - 2\beta x - \gamma xx}{\gamma + 2\varepsilon x + \zeta xx}$$

erit radice extracta:

$$y = \frac{-\beta - \delta x - \varepsilon xx + \sqrt{((\beta + \delta x + \varepsilon xx)^2 - (\alpha + 2\beta x + \gamma xx)(\gamma + 2\varepsilon x + \zeta xx))}}{\gamma + 2\varepsilon x + \zeta xx}.$$

Reducatur signum radicale ad formam propositam, ponendo:

$$\beta\beta - \alpha\gamma = \mathrm{A}m;\quad \beta\delta - \alpha\varepsilon - \beta\gamma = \mathrm{B}m$$

$$\delta\delta - 2\beta\varepsilon - \alpha\zeta - \gamma\gamma = \mathrm{C}m;\quad \delta\varepsilon - \beta\zeta - \gamma\varepsilon = \mathrm{D}m$$

$$\varepsilon\varepsilon - \gamma\zeta = \mathrm{E}m$$

vnde ex sex coefficientibus $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ quinque definiuntur, atque ad sextum insuper accedit littera m, ita vt aequatio assumta adhuc constantem

ſtantem arbitrariam inuoluat. Inde ergo ſi breuitatis gratia ponamus:

$$\sqrt{(A + 2Bx + Cxx + 2Dx^3 + Ex^4)} = X \text{ et}$$
$$\sqrt{(A + 2By + Cyy + 2Dy^3 + Ey^4)} = Y$$

habebimus:

$$\beta + \gamma y + \delta x + \varepsilon xx + 2\varepsilon xy + \zeta xxy = X\sqrt{m} \text{ et}$$
$$\beta + \gamma x + \delta y + \varepsilon yy + 2\varepsilon xy + \zeta xyy = Y\sqrt{m}.$$

At aequatio aſſumta per differentiationem dat:

$$+dx(\beta + \gamma x + \delta y + 2\varepsilon xy + \varepsilon yy + \zeta xyy)$$
$$+dy(\beta + \gamma y + \delta x + \varepsilon xx + 2\varepsilon xy + \zeta xxy) = 0$$

quae expreſſiones quia cum ſuperioribus conueniunt, dant:

$$Y dx \sqrt{m} + X dy \sqrt{m} = 0 \text{ ſeu } \frac{dx}{X} + \frac{dy}{Y} = 0$$

vnde integrando colligimus:

$$\Pi : x + \Pi : y = \text{Conſt.}$$

quae conſtans, ſi poſito $x = 0$ fiat $y = b$, erit $= \Pi : 0 + \Pi : b$ vel in genere ſi poſito $x = a$, fiat $y = b$, ea erit $\Pi : a + \Pi : b$. Quodſi ergo litterae α, β, γ, δ, ε, ζ per conditiones ſuperiores definiantur, aequatio aſſumta algebraica inter x et y erit integrale completum huius aequationis differentialis:

$$\frac{dx}{\sqrt{(A + 2Bx + Cxx + 2Dx^3 + Ex^4)}} + \frac{dy}{\sqrt{(A + 2By + Cyy + 2Dy^3 + Ey^4)}} = 0.$$

Coroll. 4.

610. Hinc constantem b pro lubitu determinando infinita integralia particularia exhiberi possunt, quorum praecipua sunt: 1) sumendo $b = 0$, vnde
3) Si $A + Cbb + Eb^4 = 0$ [illegible], vnde fit $y = \frac{\sqrt{A}}{x\sqrt{E}}$.
vnde fit $y = \frac{b\sqrt{A(A + Cxx + Ex^4)}}{A - Ebbxx}$.

Scholion.

611. Hic iam vsus istius methodi, qua retrogrediendo ab aequatione finita ad aequationem differentialem peruenimus, luculenter perspicitur. Cum enim integratio formulae $\int \frac{dx}{\sqrt{(A + Cxx + Ex^4)}}$ nullo modo neque per logarithmos neque arcus circulares perfici posset, mirum sane est talem aequationem differentialem adeo algebraice integrari posse; quae quidem in praecedente capite ope eiusdem methodi sunt tradita, etiam methodo ordinaria erui possunt, dum singulae formulae differentiales vel per logarithmos vel arcus circulares exprimuntur, quorum deinceps comparatio ad aequationem algebraicam reducitur. Verum quia hic talis integratio plane non locum inuenit, nulla certe alia methodus patet, qua idem integrale, quod hic exhibuimus, inuestigari posset. Quare hoc argumentum diligentius euoluamus.

Proble-

Problema 79.

612. Si $\Pi:z$ denotet eiusmodi functionem ipsius z, vt sit $\Pi:z = \int \frac{dz}{\sqrt{(A+Czz+Ez^4)}}$ integrali ita sumto vt euanescat posito $z=0$, comparationem inter huiusmodi functiones inuestigare.

Solutio.

Posita inter binas variabiles x et y relatione supra definita vidimus fore:

$$\frac{dx}{\sqrt{(A+Cxx+Ex^4)}} + \frac{dy}{\sqrt{(A+Cyy+Ey^4)}} = 0.$$

Hinc cum posito $x=0$ fiat $y=b$, elicitur integrando

$$\Pi:x + \Pi:y = \Pi:b.$$

Cum iam nullum amplius discrimen inter variabiles x, y et constantem b intercedat, statuamus $x=p$, $y=q$, et $b=-r$, vt sit $\Pi:b=-\Pi:r$, atque haec relatio inter functiones transcendentes

$$\Pi:p + \Pi:q + \Pi:r = 0$$

per sequentes formulas algebraicas exprimetur:

$$(A-Epprr)q+p\sqrt{A(A+Crr+Er^4)}+r\sqrt{A(A+Cpp+Ep^4)}=0 \text{ seu}$$

$$(A-Eppqq)r+q\sqrt{A(A+Cpp+Ep^4)}+p\sqrt{A(A+Cqq+Eq^4)}=0 \text{ seu}$$

$$(A-Eqqrr)p+r\sqrt{A(A+Cqq+Eq^4)}+q\sqrt{A(A+Crr+Er^4)}=0$$

quae oriuntur ex hac aequatione:

$$A(pp+qq-rr)-Eppqqrr+2pq\sqrt{A(A+Crr+Er^4)}=0.$$

Haec

Coroll. 4.

610. Hinc constantem b pro lubitu determinando infinita integralia particularia exhiberi possunt, quorum praecipua sunt: 1) sumendo $b=0$, vnde 3) Si $A+Cbb+Eb^4=0$; $b=\infty$, vnde fit $y=\frac{\sqrt{A}}{x\sqrt{E}}$. vnde fit $y=\frac{b\sqrt{A(A+Cxx+Ex^4)}}{A-Ebbxx}$.

Scholion.

611. Hic iam vsus istius methodi, qua retrogrediendo ab aequatione finita ad aequationem differentialem peruenimus, luculenter perspicitur. Cum enim integratio formulae $\int\frac{dx}{\sqrt{(A+Cxx+Ex^4)}}$ nullo modo neque per logarithmos neque arcus circulares perfici posset, mirum sane est talem aequationem differentialem adeo algebraice integrari posse; quae quidem in praecedente capite ope eiusdem methodi sunt tradita, etiam methodo ordinaria erui possunt, dum singulae formulae differentiales vel per logarithmos vel arcus circulares exprimuntur, quorum deinceps comparatio ad aequationem algebraicam reducitur. Verum quia hic talis integratio plane non locum inuenit, nulla certe alia methodus patet, qua idem integrale, quod hic exhibuimus, inuestigari posset. Quare hoc argumentum diligentius euoluamus.

Proble-

Problema 79.

612. Si $\Pi:z$ denotet eiusmodi functionem ipsius z, vt sit $\Pi:z=\int\frac{dz}{\sqrt{(A+Czz+Ez^4)}}$ integrali ita sumto vt euanescat posito $z=0$, comparationem inter huiusmodi functiones inuestigare.

Solutio.

Posita inter binas variabiles x et y relatione supra definita vidimus fore:

$$\frac{dx}{\sqrt{(A+Cxx+Ex^4)}}+\frac{dy}{\sqrt{(A+Cyy+Ey^4)}}=0.$$

Hinc cum posito $x=0$ fiat $y=b$, elicitur integrando

$$\Pi:x+\Pi:y=\Pi:b.$$

Cum iam nullum amplius discrimen inter variabiles x, y et constantem b intercedat, statuamus $x=p$, $y=q$, et $b=-r$, vt sit $\Pi:b=-\Pi:r$, atque haec relatio inter functiones transcendentes

$$\Pi:p+\Pi:q+\Pi:r=0$$

per sequentes formulas algebraicas exprimetur:

$$(A-Epprr)q+p\sqrt{A(A+Crr+Er^4)}+r\sqrt{A(A+Cpp+Ep^4)}=0 \text{ seu}$$

$$(A-Eppqq)r+q\sqrt{A(A+Cpp+Ep^4)}+p\sqrt{A(A+Cqq+Eq^4)}=0 \text{ seu}$$

$$(A-Eqqrr)p+r\sqrt{A(A+Cqq+Eq^4)}+q\sqrt{A(A+Crr+Er^4)}=0$$

quae oriuntur ex hac aequatione:

$$A(pp+qq-rr)-Eppqqrr+2pq\sqrt{A(A+Crr+Er^4)}=0.$$

Haec

Haec vero ad rationalitatem perducta fit

$$AA(p^4+q^4+r^4-2ppqq-2pprr-2qqrr)$$
$$-2AEppqqrr(pp+qq+rr)-4ACppqqrr$$
$$+EEp^4q^4r^4=0$$

quae autem ob pluralitatem radicum satisfacit omnibus signorum variationibus in superiori aequatione transcendente.

Coroll. 1.

613. Sumamus r negatiue, vt fiat:

$$\Pi : r = \Pi : p + \Pi : q$$

eritque

$$r = \frac{p\sqrt{A(A+Cqq+Eq^4)}+q\sqrt{A(A+Cpp+Ep^4)}}{A-Eppqq}$$

vnde colligitur fore:

$$\sqrt{\frac{A+Crr+Er^4}{A}} = \frac{(A+Eppqq)\sqrt{(A+Cpp+Ep^4)(A+Cqq+Eq^4)}+2AEpq(pp+qq)+Cpq(A+Eppqq)}{(A-Eppqq)^2}.$$

Coroll. 2.

614. Quodsi ergo ponamus $q=p$ vt sit

$$\Pi : r = 2\,\Pi : p$$

erit

$$r = \frac{2p\sqrt{A(A+Cpp+Ep^4)}}{A-Ep^4}$$

atque

$$\sqrt{\frac{A+Crr+Er^4}{A}} = \frac{AA+2ACpp+6AEp^4+2CEp^6+EEp^8}{(A-Ep^4)^2}.$$

Hoc igitur modo functio assignari potest aequalis duplo similis functionis.

Coroll. 3.

Coroll. 3.

615. Si ponatur $q = \frac{2p\sqrt{A(A+Cpp+Ep^4)}}{A-Ep^4}$ et $\sqrt{A(A+Cqq+Eq^4)} = \frac{A(AA+2ACpp+6AEp^4+2CEp^6+EEp^8)}{(A-Ep^4)^2}$, vt sit $\Pi:q = 2\Pi:p$ fiet ex primo Coroll. $\Pi:r = 3\Pi:p$. Tum igitur erit $r = \frac{p(3AA+8ACpp+6AEp^4-EEp^8)}{AA-6AEp^4-4CEp^6-3EEp^8}$.

Scholion 1.

616. Nimis operosum est hanc functionum multiplicationem vlterius continuare, multoque minus legem in earum progressione deprehendere licet. Quodsi ponamus breuitatis gratia

$$\sqrt{A(A+Cpp+Ep^4)} = AP \text{ et } A-Ep^4 = A\mathfrak{P},$$

vt sit

$$Cpp = APP - A - Ep^4 \text{ et } Ep^4 = A(1-\mathfrak{P}),$$

hae multiplicationes vsque ad quadruplum ita se habebunt; scilicet si statuamus:

$$\Pi:r = 2\Pi:p;\ \Pi:s = 3\Pi:p \text{ et } \Pi:t = 4\Pi:p$$

reperietur:

$$r = \frac{2Pp}{\mathfrak{P}};\ s = \frac{p(4PP-\mathfrak{P}\mathfrak{P})}{\mathfrak{P}\mathfrak{P}-4PP(1-\mathfrak{P})};\ t = \frac{4pP\mathfrak{P}(2PP(2-\mathfrak{P})-\mathfrak{P}\mathfrak{P})}{\mathfrak{P}^4-16P^4(1-\mathfrak{P})}.$$

Quodsi simili modo ponamus:

$$\sqrt{A(A+Crr+Er^4)} = AR \text{ et } A-Er^4 = A\mathfrak{R}$$

erit

$$R = \frac{2PP(2-\mathfrak{P})-\mathfrak{P}\mathfrak{P}}{\mathfrak{P}\mathfrak{P}} \text{ et } \mathfrak{R} = \frac{\mathfrak{P}^4-16P^4(1-\mathfrak{P})}{\mathfrak{P}^4}$$

vnde pro quadruplicatione fit

$$t = \frac{2Rr}{\mathfrak{R}}\,;\quad T = \frac{2RR(2-\mathfrak{R})-\mathfrak{R}\mathfrak{R}}{\mathfrak{R}\mathfrak{R}}\,;\quad \mathfrak{T} = \frac{\mathfrak{R}^4 - 16R^4(1-\mathfrak{R})}{\mathfrak{R}^4}.$$

Quare si pro octuplicatione statuamus $\Pi:z = 8\,\Pi:p$ erit

$$z = \frac{2Tt}{\mathfrak{T}} = \frac{4rR\mathfrak{R}(2RR(2-\mathfrak{R})-\mathfrak{R}\mathfrak{R})}{\mathfrak{R}^4 - 16R^4(1-\mathfrak{R})}.$$

Hinc intelligitur quomodo in continua duplicatione versari oporteat, neque tamen legem progressionis obseruare licet. Caeterum cognitio huius legis ad incrementum Analyseos maxime esset optanda, vt inde generatim relatio inter z et p, pro aequalitate $\Pi:z = n\,\Pi:p$ definiri posset, quemadmodum hoc in capite praecedente successit, hinc enim eximias proprietates circa integralia formae $\int \frac{dz}{\sqrt{(A+Czz+Ez^4)}}$ cognoscere liceret; quibus scientia analytica haud mediocriter promoueretur.

Scholion 2.

617 Modus maxime idoneus in legem progressionis inquirendi, videtur, si ternos terminos se ordine excipientes contemplemur hoc modo:

$$\Pi:x = (n-1)\Pi:p\,;\quad \Pi:y = n\Pi:p\,;\quad \Pi:z = (n+1)\Pi:p$$

vbi cum sit

$$\Pi:x = \Pi:y - \Pi:p \text{ et } \Pi:z = \Pi:y + \Pi:p \text{ erit}$$

$$x = \frac{y\sqrt{A(A+Cpp+Ep^4)} - p\sqrt{A(A+Cyy+Ey^4)}}{A-Eppyy}$$

$$z = \frac{y\sqrt{A(A+Cpp+Ep^4)} + p\sqrt{A(A+Cyy+Ey^4)}}{A-Eppyy}$$

vnde

vnde concludimus

$$(A-Eppyy)(x+z)=2y\sqrt{A(A+Cpp+Ep^4)}.$$

Ponamus vt ante:

$$\sqrt{A(A+Cpp+Ep^4)}=AP \text{ et } A-Ep^4=A\mathfrak{P}$$

et quia singulae quantitates x, y, z factorem p simpliciter inuoluunt, sit

$$x=pX;\ y=pY \text{ et } z=pZ,$$

erit

$$(1-(1-\mathfrak{P})YY)(X+Z)=2PY$$

seu

$$Z=\frac{2PY}{1-(1-\mathfrak{P})YY}-X$$

cuius formulae ope ex binis terminis contiguis X et Y sequens Z haud difficulter inuenitur. Quod quo facilius appareat ponatur $2P=Q$ et $1-\mathfrak{P}=\Omega$ vt sit $Z=\frac{QY}{1-\Omega YY}-X$. Iam progressio quaesita ita se habebit

1) 1;

2) $\frac{Q}{\mathfrak{P}}$;

3) $\frac{QQ-\mathfrak{P}\mathfrak{P}}{\mathfrak{P}\mathfrak{P}-QQ\Omega}$;

4) $\frac{Q^3\mathfrak{P}(1+\Omega)-2Q\mathfrak{P}^3}{\mathfrak{P}^4-Q^4\Omega}$;

5) $\frac{\mathfrak{P}^6-3QQ\mathfrak{P}^4+Q^4\mathfrak{P}^2(1+2\Omega)-Q^6\Omega\Omega}{\mathfrak{P}^6-3QQ\mathfrak{P}^4\Omega+Q^4\mathfrak{P}\mathfrak{P}\Omega(2+\Omega)-Q^6\Omega}$ etc.

Quaestio ergo huc redit, vt inuestigetur progressio, ex data relatione inter ternos terminos successiuos X, Y, Z, quae sit $Z=\frac{QY}{1-\Omega YY}-X$; existente termino primo $=1$ et secundo $=\frac{Q}{1-\Omega}$.

 Proble-

Problema 80.

618. Si $\Pi:z$ eiusmodi denotet functionem ipsius z, vt sit $\Pi:z = \int \frac{dz(L+Mzz+Nz^4)}{\sqrt{(A+Czz+Ez^4)}}$, integrali ita sumto vt euanescat posito $z=0$, comparationem inter huiusmodi functiones transcendentes inuestigare.

Solutio.

Stabilita inter binas variabiles x et y hac relatione vt sit

$$Ay+\mathfrak{B}x-Ebbxxy=b\sqrt{A(A+Cxx+Ex^4)} \text{ seu}$$

$$Ax+\mathfrak{B}y-Ebbxyy=b\sqrt{A(A+Cyy+Ey^4)}$$

siue sublata irrationalitate

$$A(xx+yy-bb)+2\mathfrak{B}xy-Ebbxxyy=0$$

existente breuitatis gratia $\mathfrak{B}=\sqrt{A(A+Cbb+Eb^4)}$ erit vti ante vidimus:

$$\frac{dx}{\sqrt{(A+Cxx+Ex^4)}}+\frac{dy}{\sqrt{(A+Cyy+Ey^4)}}=0.$$

Ponamus igitur:

$$\frac{dx(L+Mxx+Nx^4)}{\sqrt{(A+Cxx+Ex^4)}}+\frac{dy(L+Myy+Ny^4)}{\sqrt{(A+Cyy+Ey^4)}}=bdV\cdot\sqrt{A}$$

vt sit nostro signandi more

$$\Pi:x+\Pi:y=\text{Const.}+bV\sqrt{A}$$

vbi constans ita definiri debet, vt posito $x=0$ fiat $y=b$. Quaestio ergo ad inuentionem functionis V reuo-

reuocatur; quem in finem loco dy valore ex priori aequatione substituto erit:

$$bdV\sqrt{A} = \frac{dx(M(xx-yy)+N(x^4-y^4))}{\sqrt{(A+Cxx+Ex^4)}}$$

verum quia

$$b\sqrt{A(A+Cxx+Ex^4)} = Ay+\mathfrak{B}x-Ebbxxy$$

habebimus

$$dV = \frac{dx(xx-yy)(M+N(xx+yy))}{Ay+\mathfrak{B}x-Ebbxxy}.$$

Sumamus iam aequationem rationalem:

$$A(xx+yy-bb)+2\mathfrak{B}xy-Ebbxxyy=0$$

et ponamus

$$xx+yy=tt \text{ et } xy=u$$

vt fit

$$A(tt-bb)+2\mathfrak{B}u-Ebbuu=0$$

ideoque

$$Atdt=-\mathfrak{B}du+Ebbudu.$$

Cum porro fit

$$xdx+ydy=tdt \text{ et } xdy+ydx=du$$

erit

$$(xx-yy)dx=xtdt-ydu$$

seu

$$A(xx-yy)dx=-du(Ay+\mathfrak{B}x-Ebbxxy)$$

ita vt fit

$$\frac{dx(xx-yy)}{Ay+\mathfrak{B}x-Ebbxxy} = -\frac{du}{A}$$

ex quo deducitur:

$$dV = -\frac{du}{A}(M + Ntt)$$

et ob

$$tt = bb - \frac{2\mathfrak{B}u}{A} + \frac{Ebbuu}{A}$$

erit

$$dV = -\frac{du}{AA}(AM + ANbb - 2\mathfrak{B}Nu + ENbbuu)$$

vnde integrando elicitur:

$$V = -\frac{Mu}{A} - \frac{Nbbu}{A} + \frac{\mathfrak{B}Nuu}{AA} - \frac{ENbbu^3}{3AA}.$$

Hoc ergo valore substituto ob $u = xy$ habebimus

$$\Pi : x + \Pi : y = \Pi : b - \frac{Mbxy}{\sqrt{A}} - \frac{Nb^3xy}{\sqrt{A}} + \frac{\mathfrak{B}Nbx^2y^2}{A\sqrt{A}} - \frac{ENb^3x^3y^3}{3A\sqrt{A}}.$$

Cum autem sit

$$\mathfrak{B}xy = \tfrac{1}{2}Abb - \tfrac{1}{2}A(xx + yy) + \tfrac{1}{2}Ebbxxyy$$

erit

$$\Pi : x + \Pi : y = \Pi : b - \frac{Mbxy}{\sqrt{A}} - \frac{Nbxy}{2A\sqrt{A}}(A'bb + xx + yy) - \tfrac{1}{3}Ebbxxyy)$$

cui ergo aequationi satisfit per formulas algebraicas supra exhibitas, quibus relatio inter x, y et b exprimitur. Quodsi ergo statuatur haec aequatio:

$$\Pi : p + \Pi : q + \Pi : r = \frac{Mpqr}{\sqrt{A}} + \frac{Npqr}{2A\sqrt{A}}(A(pp + qq + rr) - \tfrac{1}{3}Eppqqrr)$$

ea efficitur sequenti relatione inter p, q, r constita:

$$(A - Eppqq)r + p\sqrt{A(A + Cqq + Eq^4)} + q\sqrt{A(A + Cpp + Ep^4)} = 0 \quad \text{seu}$$
$$(A - Epprr)q + p\sqrt{A(A + Crr + Er^4)} + r\sqrt{A(A + Cpp + Fp^4} = 0 \quad \text{seu}$$
$$(A - Eqqrr)p + q\sqrt{A(A + Crr + Er^4)} + r\sqrt{A(A + Cqq + Eq^4)} = 0$$

siue

fiue per fimplicem irrationalitatem

$$A(pp+qq-rr)+2pq\sqrt{A(A+Crr+Er^4)}-Eppqqrr=0 \text{ feu}$$

$$A(pp+rr-qq)+2pr\sqrt{A(A+Cqq+Eq^4)}-Eppqqrr=0 \text{ feu}$$

$$A(qq+rr-pp)+2qr\sqrt{A(A+Cpp+Ep^4)}-Eppqqrr=0$$

penitusque irrationalitate fublata:

$$EEp^4q^4r^4-2AEppqqrr(pp+qq+rr)-4ACppqqrr$$
$$+AA(p^4+q^4+r^4-2ppqq-2pprr-2qqrr)=0.$$

Coroll. 1.

619. Sit $q=r=s$, vt habeamus hanc aequationem:

$$\Pi:p+2\Pi:s=\frac{Mpss}{\sqrt{A}}+\frac{Npss}{2A\sqrt{A}}(A(pp+2ss)-\tfrac{1}{3}Epps^4)$$

cui fatisfacit haec relatio:

$$(A-Es^4)p+2s\sqrt{A(A+Css+Es^4)}=0.$$

Coroll. 2.

620. Sumamus s negatiue, et loco p fubftituamus ibi hunc valorem, vt habeamus:

$$2\Pi:s+\Pi:q+\Pi:r+\frac{Mpss}{\sqrt{A}}+\frac{Npss}{2A\sqrt{A}}(A(pp+2ss)-\tfrac{1}{3}Epps^4)$$
$$=\frac{Mpqr}{\sqrt{A}}+\frac{Npqr}{2A\sqrt{A}}(A(pp+qq+rr)-\tfrac{1}{3}Eppqqrr)$$

exiftente

$$p=\frac{2s\sqrt{A(A+Css+Es^4)}}{A-Es^4},$$

vnde

vnde fit

$$\sqrt{A(A+Cpp+Ep^4)} = \frac{A(A+Css+Es^4)^2 + A(4AE-CC)s^4}{(A-Es^4)^2}$$

qui valores in superioribus formulis substitui debent.

Coroll. 3.

621. Hoc modo effici poterit, vt partes algebraicae euanescant, atque functiones transcendentes solae inter se comparentur. Veluti si esset $N=0$, statui oporteret $ss=qr$, vt fieret:

$$2\Pi:s+\Pi:q+\Pi:r=0.$$

At posito $ss=qr$ fit

$$p=\frac{2\sqrt{Aqr}(A+Cqr+Eqqrr)}{A-Eqqrr}.$$

Est vero etiam

$$p=\frac{-q\sqrt{A(A+Crr+Er^4)}-r\sqrt{A(A+Cqq+Eq^4)}}{A-Eqqrr}$$

quibus valoribus aequatis oritur haec aequatio:

$$(AA+EEq^4r^4)(qq-6qr+rr)-8Cqqrr(A+Eqqrr)$$
$$-2AEqqrr(qq+10qr+rr)=0.$$

Scholion.

622. Si $\Pi:z$ exprimat arcum cuiuspiam lineae curuae respondentem abscissae vel cordae z, hinc plures arcus eiusdem curuae inter se comparare licet,

licet, vt vel differentia binorum arcuum fiat algebraïca, vel arcus exhibeantur datam rationem inter se tenentes. Hoc modo eiusmodi insignes curuarum proprietates eruuntur, quarum ratio aliunde vix perspici queat. Comparatio quidem arcuum circularium ex elementis nota per caput praecedens, vt vidimus, facile expeditur, vnde etiam comparatio arcuum parabolicorum deriuatur. Ex hoc autem capite comparatio arcuum ellipticorum et hyperbolicorum simili modo institui potest; cum enim in genere arcus sectionis conicae tali formula exprimatur $\int dx \sqrt{\frac{a+hxx}{c+exx}}$, haec transformata in istam $\int \frac{dx(a+hxx)}{\sqrt{(ac+(ae+bc)xx+bex^4)}}$, per praecepta tradita tractari potest, ponendo $A = ac$, $C = ab + bc$, $E = be$ et $L = a$, $M = b$ atque $N = 0$. Haec autem inuestigatio ad formulas, quarum denominator est $\sqrt{(A + 2Bz + Czz + Dz^3 + Ez^4)}$ extendi potest, similisque est praecedenti, quam idcirco hic sum expositurus, vnde simul patebit, hunc esse vltimum terminum, quousque progredi liceat. Formulae enim integrales magis complicatae, vbi post signum radicale altiores potestates ipsius z occurrunt, vel ipsum signum radicale altiorem dignitatem inuoluit, hoc modo non videntur inter se comparari posse, paucissimis casibus exceptis, qui per quampiam substitutionem ad huiusmodi formam reduci queant.

ſtantem arbitrariam inuoluat. Inde ergo ſi breuitatis gratia ponamus:

$$\surd(A + 2Bx + Cxx + 2Dx^3 + Ex^4) = X \text{ et}$$

$$\surd(A + 2By + Cyy + 2Dy^3 + Ey^4) = Y$$

habebimus:

$$\beta + \gamma y + \delta x + \varepsilon xx + 2\varepsilon xy + \zeta xxy = X\surd m \text{ et}$$

$$\beta + \gamma x + \delta y + \varepsilon yy + 2\varepsilon xy + \zeta xyy = Y\surd m.$$

At aequatio aſſumta per differentiationem dat:

$$+dx(\beta + \gamma x + \delta y + 2\varepsilon xy + \varepsilon yy + \zeta xyy)$$

$$+dy(\beta + \gamma y + \delta x + \varepsilon xx + 2\varepsilon xy + \zeta xxy) = 0$$

quae expreſſiones quia cum ſuperioribus conueniunt, dant:

$$Y dx \surd m + X dy \surd m = 0 \text{ ſeu } \frac{dx}{X} + \frac{dy}{Y} = 0$$

vnde integrando colligimus:

$$\Pi : x + \Pi : y = \text{Conſt.}$$

quae conſtans, ſi poſito $x = 0$ fiat $y = b$, erit $= \Pi:0 + \Pi:b$ vel in genere ſi poſito $x = a$, fiat $y = b$, ea erit $\Pi:a + \Pi:b$. Quodſi ergo litterae α, β, γ, δ, ε, ζ per conditiones ſuperiores definiantur, aequatio aſſumta algebraica inter x et y erit integrale completum huius aequationis differentialis:

$$\frac{dx}{\surd(A + 2Bx + Cxx + 2Dx^3 + Ex^4)} + \frac{dy}{\surd(A + 2By + Cyy + 2Dy^3 + Ey^4)} = 0.$$

 Coroll. 1.

Coroll. 1.

624. Ad has litteras α, β, γ, δ, ε, ζ definiendas ſumantur primo aequationes binae ad dextram poſitae, quae ſunt:

$$(\delta-\gamma)\beta-\alpha\varepsilon=Bm \text{ et } (\delta-\gamma)\varepsilon-\zeta\beta=Dm$$

vnde quaerantur binae β et ε, reperieturque

$$\beta=\frac{(\delta-\gamma)B+\alpha D}{(\delta-\gamma)^2-\alpha\zeta}m \text{ et } \varepsilon=\frac{(\delta-\gamma)D+\zeta B}{(\delta-\gamma)^2-\alpha\zeta}m.$$

Coroll. 2.

625. Sit breuitatis gratia $\delta-\gamma=\lambda$ ſeu $\delta=\gamma+\lambda$ erit

$$\beta=\frac{D\alpha+B\lambda}{\lambda\lambda-\alpha\zeta}m \text{ et } E=\frac{B\zeta+D\lambda}{\lambda\lambda-\alpha\zeta}m.$$

Iam ex conditione prima et vltima oritur

$$\beta\beta\zeta-\alpha\varepsilon\varepsilon=(A\zeta-E\alpha)m,$$

vbi illi valores ſubſtituti praebent

$$\frac{BB\zeta-DD\alpha}{\lambda\lambda-\alpha\zeta}m=A\zeta-E\alpha,$$

vnde fit

$$m=\frac{(\lambda\lambda-\alpha\zeta)(A\zeta-E\alpha)}{BB\zeta-DD\alpha}.$$

At ex prima et vltima ſequitur

$$DD\beta\beta-BB\varepsilon\varepsilon+\gamma(BB\zeta-DD\alpha)=(ADD-BBE)m$$

vnde colligitur

$$\gamma=\frac{(A\zeta-E\alpha)(ADD-BBE)\lambda\lambda+2BD(A\zeta-E\alpha)\lambda+ABB\zeta\zeta-DDE\alpha\alpha}{(BB\zeta-DD\alpha)^2}.$$

Coroll. 3.

Coroll. 3.

626. Supereſt tertia aequatio

$$2\gamma\lambda+\lambda\lambda-2\beta\varepsilon-\alpha\zeta=Cm$$

quae cum pro m ſubſtituto valore ſit

$$\beta=\frac{(A\zeta-E\alpha)(D\alpha+B\lambda)}{BB\zeta-DD\alpha} \text{ et } \varepsilon=\frac{(A\zeta-E\alpha)(B\zeta+D\lambda)}{BB\zeta-DD\alpha}$$

ſi iſti valores ſubſtituantur, commode inde colligitur:

$$\lambda=\frac{C(A\zeta-E\alpha)(BB\zeta-DD\alpha)-BD(A\zeta-E\alpha)^2-(BB\zeta-DD\alpha)^2}{2(A\zeta-A\alpha)(EDD-BBE)}.$$

Scholion 1.

627. Quia his valoribus vti non licet, quoties fuerit $ADD-BBE=0$, aliam reſolutionem huic incommodo non obnoxiam tradam: Poſito $\delta=\gamma+\lambda$, ſit inſuper $\lambda\lambda=\alpha\zeta+\mu$ vt primae formulae fiant

$$\beta=\tfrac{m}{\mu}(D\alpha+B\lambda) \text{ et } \varepsilon=\tfrac{m}{\mu}(B\zeta+D\lambda).$$

Iam prima et vltima iunctis prodit

$$A\zeta-E\alpha=\tfrac{m}{\mu}(BB\zeta-DD\alpha)$$

qua aequatione ratio inter α et ζ definitur, quae cum ſufficiat, erit

$$\alpha=\mu A-BBm \text{ et } \zeta=\mu E-DDm$$

hincque

$$\lambda\lambda=\mu+(\mu A-BBm)(\mu E-DDm)$$

 vnde

vnde colligimus:

$$\gamma = \frac{mm}{\mu\mu}(2BD\lambda + (ADD - BBE)\mu) - \frac{2BBDDm^3}{\mu\mu} - \frac{m}{\mu}.$$

Valores α et ζ in formula Coroll. 3. substituti dant

$$\lambda = \frac{\mu\mu}{2m} + BDm - \frac{1}{2}C\mu$$

cuius quadratum illi valori $\alpha\zeta + \mu$ aequatum, perducit ad hanc aequationem

$$\mu(\mu - Cm)^2 + 4(BD - AE)mm\mu + 4(ADD - BCD + BCE)m^3 = 4mm$$

ad quam resoluendam ponatur $\mu = Mm$, fietque

$$m = \frac{4}{M(M-C)^2 + 4M(BD - AE) + 4(ADD - BCD + EBE)}$$

atque hic est M constans illa arbitraria pro integrali completo requisita. Hoc modo omnes litterae $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ eodem denominatore affecti prodibunt, quo omisso habebimus:

$$\alpha = 4(AM - BB);\ \beta = 2B(M - C) + 4AD;\ \gamma = 4AE - (M - C)^2$$

$$\zeta = 4(EM - DD);\ \varepsilon = 2D(M - C) + 4BE;\ \delta = MM - CC + 4(AE + BD)$$

quibus inuentis aequatio nostra canonica

$$0 = \alpha + 2\beta(x + y) + \gamma(xx + yy) + 2\delta xy + 2\varepsilon xy(x + y) + \zeta xxyy$$

si breuitatis gratia ponamus

$$M(M - C)^2 + 4M(BD - AE) + 4(ADD - BCD + BBE) = \Delta$$

resoluta dabit

$$\beta + \delta x + \varepsilon xx + y(\gamma + 2\varepsilon x + \zeta xx) = \pm 2\sqrt{\Delta(A + 2Bx + Cxx + 2Dx^3 + Ex^4)}$$

$$\beta + \delta y + \varepsilon yy + x(\gamma + 2\varepsilon y + \zeta yy) = \pm 2\sqrt{\Delta(A + 2By + Cyy + 2Dy^3 + Ey^4)}.$$

quae

quae ergo eſt integrale completum huius aequationis differentialis:

$$0 = \frac{dx}{\pm\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}} + \frac{dy}{\pm\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}}.$$

Scholion 2.

628. Cum hic ab idonea coefficientium determinatione totum negotium pendeat, operae pretium erit, eam luculentius exponere. Poſito igitur ſtatim $\delta=\gamma+\lambda$ et $\lambda\lambda-\alpha\zeta=Mm$, quinque conditiones adimplendae ſunt:

I. $\beta\beta-\alpha\gamma=Am$;

II. $\varepsilon\varepsilon-\gamma\zeta=Em$;

III. $\beta\lambda-\alpha\varepsilon=Bm$;

IV. $\varepsilon\lambda-\beta\zeta=Dm$;

V. $Mm+2\gamma\lambda-2\beta\varepsilon=Cm$.

Hinc ex tertia et quarta combinando deducitur:

$m(B\lambda+D\alpha)=\beta(\lambda\lambda-\alpha\zeta)=\beta Mm$ ergo $\beta=\frac{B\lambda+D\alpha}{M}$

$m(D\lambda+B\zeta)=\varepsilon(\lambda\lambda-\alpha\zeta)=\varepsilon Mm$ ergo $\varepsilon=\frac{D\lambda+B\zeta}{M}$.

Iam ex prima et ſecunda elidendo γ oritur

$$m(A\zeta-E\alpha)=\beta\beta\zeta-\varepsilon\varepsilon\alpha=\frac{BB\zeta-DD\alpha}{M}.m$$

hincque

$$\zeta(AM-BB)=\alpha(EM-DD)$$

quare

quare ftatuatur:

$$\alpha = n(AM - BB) \text{ et } \zeta = n(EM - DD).$$

Tum vero indidem eft

$$E\beta\beta - E\alpha\gamma = A\varepsilon\varepsilon - A\gamma\zeta \text{ feu}$$
$$\gamma(A\zeta - E\alpha) = A\varepsilon\varepsilon - E\beta\beta$$

pro qua tractanda cum fit pro α et ζ fubftitutis valoribus:

$$\beta = nAD + \tfrac{B}{M}(\lambda - nBD) \text{ et } \varepsilon = nBE + \tfrac{D}{M}(\lambda - nBD)$$

fit breuitatis ergo $\lambda - nBD = nMN$ vt habeamus:

$$\beta = n(AD + BN) \text{ et } \varepsilon = n(BE + DN)$$

et quia

$$A\zeta - E\alpha = n(BBE - ADD)$$

atque

$$A\varepsilon\varepsilon - E\beta\beta = nn(ABBEE + ADDNN - AADDE - BBENN)$$

feu $A\varepsilon\varepsilon - E\beta\beta = nn(BBE - ADD)(AE - NN)$

fiet $\gamma = n(AE - NN)$.

Cum autem fit

$$\lambda = n(BD + MN) \text{ et } \lambda\lambda = nn(AM - BB)(EM - DD) + Mm$$

erit

$$Mm = nn(2BDMN + MMNN - AEMM + M(ADD + BBE))$$

feu $m = nn(2BDN + MNN - AEM + ADD + BBE)$.

Denique

Denique aequatio quinta $\beta\varepsilon-\gamma\lambda=\frac{1}{2}m(M-C)$ euoluta praebet

$$\beta\varepsilon-\gamma\lambda=nn((AD+BN)(BE+DN)-(AE-NN)(BD+MN))$$
$$-nnN(2BDN+MNN-AEM+ADD+BBE)=Nm$$

vnde fit $N=\frac{1}{2}(M-C)$ ac propterea

$$m=nn(BD(M-C)+\tfrac{1}{4}M(M-C)^2-AEM+ADD+BBE).$$

Hincque fumendo $n=4$ fuperiores valores obtinentur.

Exemplum 1.

629. *Inuenire integrale completum huius aequationis differentialis*: $\frac{dp}{\pm\sqrt{(a+bp)}}+\frac{dq}{\pm\sqrt{(a+bq)}}=0$.

Hic eft $x=p$; $y=q$; $A=a$; $B=\frac{1}{2}b$; $C=0$; $D=0$; $E=0$ vnde fiunt coefficientes:

$$\alpha=4aM-bb;\ \beta=bM;\ \gamma=-MM$$
$$\zeta=0;\qquad \varepsilon=0;\quad \delta=MM$$

et $\Delta=M^3$, vnde integrale completum erit:

$$bM+MMp-MMq=\pm 2M\sqrt{M(a+bp)}$$

feu $b+M(p-q)=\pm 2\sqrt{M(a+bp)}$

vel $b+M(q-p)=\pm 2\sqrt{M(a+bq)}$

quae figna ambigua radicalium cum fignis in aequatione differentiali conuenire debent.

Exemplum 2.

630. *Inuenire integrale completum huius aequationis differentialis*: $\frac{dp}{\pm\sqrt{(a+bp^2)}} + \frac{dq}{\pm\sqrt{(a+bq^2)}} = 0$

Sumto $x = p$ et $y = q$ erit $A = a$, $B = 0$; $C = b$; $D = 0$; ergo:

$$\alpha = 4aM;\ \beta = 0;\ \gamma = -(M-b)^2$$
$$\zeta = 0;\quad \varepsilon = 0;\ \delta = MM - bb$$

atque $\Delta = M(M-b)^2$

vnde integrale completum in his aequationibus continebitur:

$$(MM-bb)p-(M-b)^2q = \pm 2(M-b)\sqrt{M(a+bpp)}$$

seu $(M+b)p-(M-b)q = \pm 2\sqrt{M(a+bpp)}$

et $(M+b)q-(M-b)p = \pm 2\sqrt{M(a+bqq)}$.

Exemplum 3.

631. *Inuenire integrale completum huius aequationis differentialis*: $\frac{dp}{\pm\sqrt{(a+bp^4)}} + \frac{dq}{\pm\sqrt{(a+bq^4)}} = 0$.

Sumto $x = p$; $y = q$; erit $A = a$; $B = 0$; $C = 0$; $D = \frac{1}{2}b$; $E = 0$; ergo

$$\alpha = 4aM;\ \beta = 2ab;\ \gamma = -MM$$
$$\zeta = -bb;\ \varepsilon = bM;\ \delta = MM$$

et $\Delta = M^3 + abb$

vnde

vndè integrale completum

$$2ab+MMp+bMpp+q(-MM+2bMp-bbpp)=\pm 2\sqrt{(M^2+abb)(a+bp^2)}$$

fiue:

$$2ab+Mp(M+bp)-q(M-bp)^2=\pm 2\sqrt{(M^2+abb)(a+bp^2)}$$ et

$$2ab+Mq(M+bq)-p(M-bq)^2=\pm 2\sqrt{(M^2+abb)(a+bq^2)}.$$

Exemplum 4.

632. *Inuenire integrale completum huius aequationis differentialis*: $\frac{dp}{\pm\sqrt{(a+bp^4)}}+\frac{dq}{\pm\sqrt{(a+bq^4)}}=0$.

Pofito $x=p$; $y=q$ erit $A=a$; $B=0$; $C=0$; $D=0$; $E=b$; ergo

$$\alpha=4aM;\ \beta=0;\ \gamma=4ab-MM$$

$$\zeta=4bM;\ \varepsilon=0;\ \delta=MM+4ab$$

et $\Delta=M^2-4abM$

vnde integrale completum:

$$(MM+4ab)p+q(4ab-MM+4bMpp)=\pm 2\sqrt{M(MM-4ab)(a+bp^4)}$$

$$(MM+4ab)q+p(4ab-MM+4bMqq)=\pm 2\sqrt{M(MM-4ab)(a+bq^4)}.$$

Exemplum 5.

633. *Inuenire integrale completum huius aequationis differentialis*: $\frac{dp}{\pm\sqrt{(a+bp^6)}}+\frac{dq}{\pm\sqrt{(a+bq^6)}}=0$.

 Pona-

Ponatur $x = pp$ et $y = qq$, atque aequatio noſtra generalis inducet poſito $A = 0$, hanc formam

$$\frac{dp}{\pm\sqrt{(2B + Cpp + 2Dp^4 + Ep^6)}} + \frac{dq}{\pm\sqrt{(2B + Cqq + 2Dq^4 + Eq^6)}} = 0.$$

Fieri ergo oportet $B = \frac{1}{2}a$; $C = 0$; $D = 0$ et $E = b$; vnde coefficientes ita determinantur:

$$\alpha = -aa;\ \beta = aM;\ \gamma = -MM$$

$$\zeta = 4bM;\ \varepsilon = 2ab;\ \delta = MM$$

et $\Delta = M^3 + aab$;

ergo integrale completum:

$$aM + MMpp + 2abp^4 + qq(-MM + 4abpp + 4bMp^4) = \pm 2p\sqrt{(M^3 + aab)(a + bp^6)}$$

ſiue

$$aM + MMqq + 2abq^4 + pp(-MM + 4abqq + 4bMq^4) = \pm 2q\sqrt{(M^3 + aab)(a + bq^6)}.$$

Corollarium.

634. Si ſumatur conſtans $M = -\sqrt[3]{aab}$, vt ſit $M^3 + aab = 0$, prodibit integrale particulare, quod ita ſe habebit:

$$pp = \frac{qq\sqrt[3]{b} + \sqrt[3]{a}}{2qq\sqrt[3]{b} - \sqrt[3]{a}} \cdot \sqrt[3]{\frac{a}{b}} \text{ ſeu } qq = \frac{pp\sqrt[3]{b} + \sqrt[3]{a}}{2pp\sqrt[3]{b} - \sqrt[3]{a}} \cdot \sqrt[3]{\frac{a}{b}}$$

quod aequationi differentiali vtique ſatisfacit.

Proble-

Problema 82.

635. Propoſita hac aequatione differentiali:

$$\frac{dp}{\pm\sqrt{(a+bpp+cp^4+ep^6)}}+\frac{dq}{\pm\sqrt{(a+bqq+cq^4+eq^6)}}=0$$

eius integrale completum algebraice aſſignare.

Solutio.

Aequatio praecedens differentialis algebraice integrata ad hanc formam reducitur ponendo $x=pp$ et $y=qq$, atque $A=0$; prodibit enim

$$\frac{dp}{\pm\sqrt{(2B+Cpp+2Dp^4+Ep^6)}}+\frac{dq}{\pm\sqrt{(2B+Cqq+2Dq^4+Eq^6)}}=0.$$

Quare tantum opus eſt vt fiat:

$$A=0;\ B=\tfrac{1}{2}a;\ C=b;\ D=\tfrac{1}{2}c;\ E=e$$

vnde coefficientes $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ ita definientur:

$$\alpha=-aa;\qquad \beta=a(M-b);\qquad \gamma=-(M-b)^2$$

$$\zeta=4eM-cc;\ \varepsilon=c(M-b)+2ae;\ \delta=MM-bb+ac$$

$$\Delta=M(M-b)^2+acM-abc+aae=(M-b)^3+b(M-b)^2+ac(M-b)+aae$$

hincque integrale completum ob conſtantem M ab arbitrio noſtro pendentem erit:

$$\beta+\delta pp+\varepsilon p^4+qq(\gamma+2\varepsilon pp+\zeta p^4)=\pm 2p\sqrt{\Delta(a+bp^2+cp^4+ep^6)}$$

$$\beta+\delta qq+\varepsilon q^4+pp(\gamma+2\varepsilon qq+\zeta q^4)=\pm 2q\sqrt{\Delta(a+bq^2+cq^4+eq^6)}$$

quae binae quidem aequationes inter ſe conueniunt, ſed ob ambiguitatem ſignorum in ipſa aequatione diffe-

differentiali ambae notari debent, ambiguitate inde sublata. Vtrinque autem haec aequatio rationalis resultat:

$$0 = \alpha + 2\beta(pp+qq) + \gamma(p^4+q^4) + 2\delta ppqq$$
$$+ 2\varepsilon ppqq(pp+qq) + \zeta p^4 q^4.$$

Coroll. 1.

636. Si constans M ita sumatur, vt fiat $\Delta = 0$, obtinetur integrale particulare huius formae $qq = \frac{E+Fpp}{G+Hpp}$, quod etiam a posteriori cognoscere licet. Vt enim satisfaciat sumi debet

$$aG^3 + bEG^2 + cEEG + eE^3 = 0,$$

vnde ratio E : G definitur, tum vero inuenitur $F = -G$ et denique

$$H = \frac{-cEG - 2eEE}{aG} = \frac{2aGG + 2bEG + cEE}{aE}.$$

Coroll. 2.

637. Constans M ita mutetur, vt sit $M - b = \frac{a}{ff}$, fietque

$\alpha = -aa$; $\beta = \frac{aa}{ff}$; $\gamma = -\frac{aa}{f^4}$
$\zeta = 4bc - cc + \frac{4ae}{ff}$; $\varepsilon = \frac{ac}{ff} + 2ae$; $\delta = \frac{aa}{f^4} + \frac{2ab}{ff} + ac$ et
$\Delta = \frac{aa}{f^6}(a + bff + cf^4 + ef^6)$

et

et aequatio integralis erit

$$aaff+a(a+2bff+cf^4)pp+aff(c+2eff)p^4$$
$$-qq(aa-2aff(c+2eff)pp+ff(ccff-4beff-4ae)p^4)$$
$$=\pm 2afp\sqrt{(a+bff+cf^4+ef^6)(a+bpp+cp^4+ep^6)}$$

vnde patet posito $p=0$ fore $qq=ff$.

Coroll. 3.

638. Haec aequatio facile in hanc formam transmutatur

$$aff(a+bpp+cp^4+ep^6)+app(a+bff+cf^4+ef^6)$$
$$-qq(a-cffpp)^2-aeffpp(ff-pp)^2+4effppqq(aff+app+bffpp)$$
$$=\pm 2fp\sqrt{a(a+bff+cf^4+ef^6)a(a+bpp+cp^4+ep^6)}$$

vnde statim patet si sit $e=0$, fore hanc aequationem radicem extrahendo

$$f\sqrt{a(a+bpp+cp^4)}\mp p\sqrt{a(a+bff+cf^4)}=q(a-cffpp)$$

quae est integralis completa huius differentialis:

$$\frac{dp}{\pm\sqrt{(a+bpp+cp^4)}}+\frac{dq}{\pm\sqrt{(a+bqq+cq^4)}}=0$$

prorsus vt supra iam inuenimus.

Coroll. 4.

639. Simili modo patet in genere, quando e non euanescit, integrale completum ita commodius exprimi posse:

$$(f\sqrt{a(a+bpp+cp^4+ep^6)}\mp p\sqrt{a(a+bff+cf^4+ef^6)})^2=$$
$$qq(a-cffpp)^2+aeffpp(ff-pp)^2-4effppqq(aff+app+bffpp)$$

quae

quae ergo cum posito $p=0$ fiat $q=f$ respondet huic functionum transcendentium relationi

$$\pm \Pi:p \pm \Pi:q = \pm \Pi:0 \pm \Pi:f.$$

Scholion 1.

640. Genera igitur functionum transcendentium, quas hoc modo perinde atque arcus circulares inter se comparare licet, in his binis formulis integralibus continentur

$$\int \frac{dz}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}} \text{ et } \int \frac{dz}{\sqrt{(a+bzz+czz^4+ez^6)}}$$

neque haec methodus ad alias formas magis complexas extendi posse videtur. Neque etiam posterior in denominatore potestates impares ipsius z admittit: nisi forte simplex substitutio reductioni ad illam formam sufficiat. Facile autem patet huiusmodi formam

$$\int \frac{dz}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4+2Fz^5+Gz^6)}}$$

hac methodo tractari certe non posse; si enim coefficientes ita essent comparati, vt radicis extractio succederet, talis formula $\int \frac{dz}{a+bz+czz+ez^3}$ prodiret, cuius integratio, cum tam logarithmos quam arcus circulares inuoluat, fieri omnio nequit, vt plures huiusmodi functiones algebraice inter se comparentur. Caeterum prior formula latius patet quam posterior, cum haec ex illa nascatur posito $A=0$, si zz loco z scribatur. De priori autem notari meretur, quod eandem

eandem formam ſeruet, etiamſi transformetur hac ſubſtitutione $z = \frac{\alpha + \beta y}{\gamma + \delta y}$ prodit enim

$$\int \frac{(\beta\gamma - \alpha\delta)dy}{\sqrt{(A(\gamma+\delta y)^4 + 2B(\alpha+\beta y)(\gamma+\delta y)^3 + C(\alpha+\beta y)^2(\gamma+\delta y)^2 + 2D(\alpha+\beta y)^3(\gamma+\delta y) + E(\alpha+\beta y)^4)}},$$

ex quo intelligitur quantitates α, β, γ, δ ita accipi poſſe vt poteſtates impares euaneſcant. Vel etiam ita definiri poterunt, vt terminus primus et vltimus euaneſcat, tum enim poſito $y = uu$, iterum forma a poteſtatibus imparibus immuuis naſcitur.

Scholion 2.

641. Sublatio autem poteſtatum imparium ita commodiſſime inſtituitur. Cum formula

$$A + 2Bz + Czz + 2Dz^3 + Ez^4$$

certe ſemper habeat duos factores reales, ita exhibeatur formula integralis

$$\int \frac{dz}{\sqrt{(a + 2bz + czz)(f + 2gz + hzz)}}$$

quae poſito $z = \frac{\alpha + \beta y}{\gamma + \delta y}$ abit in:

$$\int \frac{(\beta\gamma - \alpha\delta)dy}{\sqrt{(a(\gamma+\delta y)^2 + 2b(\alpha+\beta y)(\gamma+\delta y) + c(\alpha+\beta y)^2)(f(\gamma+\delta y)^2 + 2g(\alpha+\beta y)(\gamma+\delta y) + h(\alpha+\beta y)^2)}}$$

vbi denominatoris factores euoluti ſunt

$$(a\gamma\gamma + 2b\alpha\gamma + c\alpha\alpha) + 2(a\gamma\delta + b\alpha\delta + b\beta\gamma + c\alpha\beta)y$$
$$+ (a\delta\delta + 2b\beta\delta + c\beta\beta)yy$$

$$(f\gamma\gamma + 2g\alpha\gamma + h\alpha\alpha) + 2(f\gamma\delta + g\alpha\delta + g\beta\gamma + h\alpha\beta)y$$
$$+ (f\delta\delta + 2g\beta\delta + h\beta\beta)yy$$

 quodſi

quodſi iam vtroque terminus medius euaneſcens reddatur fit:

$$\frac{\delta}{\beta} = \frac{-b\gamma - c\alpha}{a\gamma + b\alpha} = \frac{-g\gamma - h\alpha}{f\gamma + g\alpha}$$

hincque

$$bf\gamma\gamma + (bg + cf)\alpha\gamma + cg\alpha\alpha = ag\gamma\gamma + (ab + bg)\alpha\gamma + bh\alpha\alpha$$

ſeu

$$\gamma\gamma = \frac{(ab - cf)\alpha\gamma + (bh - cg)\alpha\alpha}{bf - ag}$$

vnde fit

$$\frac{\gamma}{\alpha} = \frac{ab - af + \sqrt{((ab - cf)^2 + 4(bf - ag)(bh - cg))}}{2(bf - ag)}.$$

Hinc ſufficere poſſet eas tantum formulas, in quibus poteſtates impares deſunt, tractaſſe, id quod initio huius capitis fecimus, ſed ſi inſuper numerator accedat, haec reductio non amplius locum habet.

Problema 83.

642. Denotante n numerum integrum quemcunque inuenire integrale completum algebraice expreſſum huius aequationis differentialis:

$$\frac{dy}{\sqrt{(A + 2By + Cyy + 2Dy^3 + Ey^4)}} = \frac{n\,dx}{\sqrt{(A + 2Bx + Cx^2 + 2Dx^3 + Ex^4)}}.$$

Solutio.

Per functiones tranſcendentes integrale completum eſt

$$\Pi : y = n\,\Pi : x + \text{Conſt.}$$

At

At vt idem algebraice expressum eruamus posito $M - C = L$, fit per formulas supra (627.) inuentas:

$\alpha = 4(AC - BB + AL)$; $\beta = 4AD + 2BL$; $\gamma = 4AE - LL$

$\zeta = 4(CE - DD + EL)$; $\varepsilon = 4BE + 2DL$; $\delta = 4AE + 4BD + 2CL + LL$

et $\Delta = L^3 + CL^2 + 4(BD - AE) + 4(ADD + BBE - ACE)$.

Quibus positis si fuerit:

$$\beta + \delta p + \varepsilon pp + q(\gamma + 2\varepsilon p + \zeta pp) = 2\sqrt{\Delta}(A + 2Bp + Cp^2 + 2Dp^3 + Ep^4)$$

$$\beta + \delta q + \varepsilon qq + p(\gamma + 2\varepsilon q + \zeta qq) = -2\sqrt{\Delta}(A + 2Bq + Cq^2 + 2Dq^3 + Eq^4)$$

erit $\Pi : q = \Pi : p +$ Const.

Cum autem hae duae aequationes inter se conueniant, et in hac rationali contineantur:

$$\alpha + 2\beta(p+q) + \gamma(pp+qq) + 2\delta pq + 2\varepsilon pq(p+q) + \zeta ppqq = 0$$

si sumamus posito $p = a$, fieri $q = b$ constans illa L ita definiri debet, vt sit

$$\alpha + 2\beta(a+b) + \gamma(aa+bb) + 2\delta ab + 2\varepsilon ab(a+b) + \zeta aabb = 0$$

eritque $\Pi : q = \Pi : p + \Pi : b - \Pi : a$

vbi iam nullum inest discrimen inter constantes et variabiles. Ponamus ergo $b = p$ vt sit

$$\Pi : q = 2\,\Pi : p - \Pi : a$$

atque huic aequationi superiores aequationes algebraicae conueniunt, si modo quantitas L ita definiatur, vt sit

$$\alpha + 2\beta(a+p) + \gamma(aa+pp) + 2\delta ap + 2\varepsilon ap(a+p) + \zeta aapp = 0$$

 vnde

vnde deducitur:

$$2L(p-a)^2 = A+B(a+p)+Cap+Dap(a+p)+Eaapp$$
$$\pm\sqrt{(A+2Ba+Caa+2Da^3+Ea^4)(A+2Bp+Cpp+2Dp^3+Ep^4)}.$$

Hoc ergo valore pro L constituto, indeque litteris α, β, γ, δ, ε, ζ per superiores formulas rite definitis, si iam p et q vt variabiles, a vero vt constantem spectemus, erit haec aequatio.

$$\alpha+2\beta(p+q)+\gamma(pp+qq)+2\delta pq+2\varepsilon pq(p+q)+\zeta ppqq=0$$

integrale completum huius aequationis differentialis

$$\frac{dq}{\sqrt{(A+2Bq+Cqq+2Dq^3+Eq^4)}} = \frac{2\,dp}{\sqrt{(A+2Bp+Cpp+2Dp^3+Ep^4)}}.$$

Postquam hoc modo q per p definiuimus, determinetur r per hanc aequationem:

$$\alpha+2\beta(q+r)+\gamma(qq+rr)+2\delta qr+2\varepsilon qr(q+r)+\zeta qqrr=0$$

erit $\Pi:r-\Pi:q=\Pi:p-\Pi:a$

quoniam posito $q=a$, et $r=p$ littera L, quae in valores α, β, γ, δ, ε, ζ ingreditur, perinde definitur vt ante. Quare cum sit

$\Pi:q=2\Pi:p-\Pi:a$ erit $\Pi:r=3\Pi:p-2\Pi:a$

vnde sumto a constante illa aequatio algebraica inter q et r, dum q per praecedentem aequationem ex p definitur, erit integrale completum huius aequationis differentialis:

$$\frac{dr}{\sqrt{(A+2Br+Crr+2Dr^3+Er^4)}} = \frac{3\,dp}{\sqrt{(A+2Bp+Cpp+2Dp^3+Ep^4)}}.$$

Hoc

Hoc valore ipſius r per p inuento quaeratur s per hanc aequationem

$$\alpha+2\beta(r+s)+\gamma(rr+ss)+2\delta rs+2\varepsilon rs(r+s)+\zeta rrss=0$$

retinente L ſemper valorem primo aſſignatum, eritque

$$\Pi:s-\Pi:r=\Pi:p-\Pi:a \text{ ſeu } \Pi:s=4\Pi:p-3\Pi:a$$

vnde iſta aequatio algebraica erit integrale completum huius aequationis differentialis:

$$\frac{ds}{\sqrt{(A+2Bs+Css+2Ds^3+Es^4)}}=\frac{4dp}{\sqrt{(A+2Bp+Cpp+2Dp^3+Ep^4)}}.$$

Cum hoc modo quousque libuerit progredi liceat, perſpicuum eſt, ad integrale completum huius aequationis differentialis inueniendum

$$\frac{dz}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}}=\frac{n\,dp}{\sqrt{(A+2Bp+Cpp+2Dp^3+Ep^4)}}$$

ſequentes operationes inſtitui oportere.

1) Quaeratur quantitas L vt ſit

$$\tfrac{1}{2}L(p-a)^2=A+B(a+p)+Cap+Dap(a+p)+Eaapp$$
$$\pm\sqrt{(A+2Ba+Caa+2Da^3+Ea^4)(A+2Bp+Cpp+2Dp^3+Ep^4)}.$$

2) Hinc determinentur litterae α, β, γ, δ, ε, ζ per has formulas

$$\alpha=4(AC-BB+AL);\ \beta=4AD+2BL;\ \gamma=4AE-LL$$
$$\zeta=4(CE-DD+EL);\ \varepsilon=4BE+2DL;\ \delta=4AE+4BD+2CL+LL.$$

3) Formetur ſeries quantitatum $p, q, r, s, t \ldots z$ quarum prima ſit p, ſecunda q, tertia r etc. vltima

vero ordine n sit z, quae successiue per has aequationes determinentur:

$$\alpha+2\beta(p+q)+\gamma(pp+qq)+2\delta pq+2\varepsilon pq(p+q)+\zeta ppqq=0$$

$$\alpha+2\beta(q+r)+\gamma(qq+rr)+2\delta qr+2\varepsilon qr(q+r)+\zeta qqrr=0$$

$$\alpha+2\beta(r+s)+\gamma(rr+ss)+2\delta rs+2\varepsilon rs(r+s)+\zeta rrss=0$$

etc.

donec ad vltimam z perueniatur.

4) Relatio quae hinc concluditur inter p et z erit integrale complctum aequationis differentialis propositae, et littera a vicem gerit constantis arbitrariae per integrationem ingressae.

Corollarium.

643. Hinc etiam integrale complctum inueniri potest huius aequationis differentialis

$$\frac{m\,dy}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}}=\frac{n\,dx}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}$$

designantibus m et n numeros integros. Statuatur enim vtrumque membrum $=\frac{du}{\sqrt{(A+2Bu+Cuu+2Du^3+Eu^4)}}$ et quaeratur relatio tam inter x et u, quam inter y et u; vnde elisa u orietur aequatio algebraica inter x et y.

Scholion.

644. Ne hic extractio radicis in singulis aequationibus repetenda ambiguitatem creet, loco vniuscuius-

cuiusque vti conueniet binis per extractionem iam erutis. Scilicet vt ex prima valor q rite per p definiatur, primo quidem habemus:

$$q = \frac{-\beta - \delta p - \varepsilon pp + 2\sqrt{\Delta(A + 2Bp + Cpp + 2Dp^3 + Ep^4)}}{\gamma + 2\varepsilon p + \zeta pp}$$

tum vero capi debet:

$$2\sqrt{\Delta(A + 2Bq + Cqq + 2Dq^3 + Eq^4)} = -\beta - \delta q - \varepsilon qq - p(\gamma + 2\varepsilon q + \zeta qq)$$

similique modo in relatione inter binas sequentes quantitates inuestiganda erit procedendum. Caeterum adhuc notari conuenit numeros integros m et n positiuos esse debere, neque hanc inuestigationem ad negatiuos extendi, propterea quod formula differentialis $\frac{dz}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4)}}$ posito z negatiuo naturam suam mutat. Interim tamen cum hanc aequalitatem

$$\Pi : x + \Pi : y = \text{Const.}$$

supra algebraice expresserimus, eius ope quoque ii casus resolui possunt, vbi est m vel n numerus negatiuus: si enim fuerit

$$\Pi : z = n\,\Pi : p + C,$$

quaeratur y vt sit

$$\Pi : y + \Pi : z = \text{Const.}$$

eritque

$$\Pi : y = -n\,\Pi : p + \text{Const.}$$

Proble-

Problema 84.

645. Si $\Pi:z$ eiusmodi functionem transcendentem ipsius z denotet vt sit

$$\Pi:z=\int\frac{dz(\mathfrak{A}+\mathfrak{B}z+\mathfrak{C}zz+\mathfrak{D}z^3+\mathfrak{E}z^4)}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}}$$

comparationem inter huiusmodi functiones inuestigare.

Solutio.

Ex coefficientibus A, B, C, D, E vna cum constante arbitraria L determinentur sequentes valores:

$$\alpha=4(AC-BB+AL);\ \beta=4AD+2BL;\ \gamma=4AE-LL$$

$$\zeta=4(CE-DD+EL);\ \varepsilon=4BE+2DL;\ \delta=4AE+4BD+2CL+LL$$

et inter binas variabiles x et y haec constituatur relatio:

$$\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy+2\varepsilon xy(x+y)+\zeta xxyy=0$$

eritque

$$\frac{dx}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}+\frac{dy}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}}=0$$

pro qua sine ambiguitate habetur:

$$\beta+\delta x+\varepsilon xx+y(\gamma+2\varepsilon x+\zeta xx)=2\sqrt{\Delta(A+2Bx+Cxx+2Dx^3+Ex^4)}$$

$$\beta+\delta y+\varepsilon yy+x(\gamma+2\varepsilon y+\zeta yy)=2\sqrt{\Delta(A+2By+Cyy+2Dy^3+Ey^4)}$$

existente $\Delta=L^3+CL^2+4(BD-AE)L+4(ADD+BBE-ACE)$.

Quare

Quare si ponamus:

$$\frac{dx(\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2+\mathfrak{D}x^3+\mathfrak{E}x^4)}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}+\frac{dy(\mathfrak{A}+\mathfrak{B}y+\mathfrak{C}y^2+\mathfrak{D}y^3+\mathfrak{E}y^4)}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}}=2dV\sqrt{\Delta}$$

vt sit

$$\Pi : x + \Pi : y = \text{Const.} + 2V\sqrt{\Delta}$$

erit

$$\frac{dx(\mathfrak{B}(x-y)+\mathfrak{C}(x^2-y^2)+\mathfrak{D}(x^3-y^3)+\mathfrak{E}(x^4-y^4))}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}=2dV\sqrt{\Delta}$$

seu $dV=\frac{dx(\mathfrak{B}(x-y)+\mathfrak{C}(x^2-y^2)+\mathfrak{D}(x^3-y^3)+\mathfrak{E}(x^4-y^4))}{(\beta+\delta x+\varepsilon yx+y(\gamma+2\varepsilon x+\zeta xx))}$.

Ponatur nunc $x+y=t$ et $xy=u$, et quia $dx+dy=dt$ et $xdy+ydx=du$ erit $dx=\frac{xdt-du}{x-y}$ seu $(x-y)dx=xdt-du$, tum vero est $x=\frac{1}{2}t\pm\sqrt{(\frac{1}{4}tt-u)}$. At his positionibus aequatio assumta induit hanc formam:

$$\alpha+2\beta t+\gamma tt+2(\delta-\gamma)u+2\varepsilon tu+\zeta uu=0$$

vnde fit differentiando

$$dt(\beta+\gamma t+\varepsilon u)+du(\delta-\gamma+\varepsilon t+\zeta u)=0$$

ergo $dt=\frac{-du(\delta-\gamma+\varepsilon t+\zeta u)}{\beta+\gamma t+\varepsilon u}$ et

$$xdt-du=\frac{-du(\beta+\gamma t+\varepsilon u+(\delta-\gamma)x+\varepsilon tx+\zeta ux)}{\beta+\gamma t+\varepsilon u}\ \text{siue}$$

$$xdt-du=\frac{-du(\beta+\delta x+\varepsilon xx+y(\gamma+2\varepsilon x+\zeta xx))}{\beta+\gamma t+\varepsilon u}$$

sicque habebimus

$$\frac{dx(x-y)}{\beta+\delta x+\varepsilon xx+y(\gamma+2\varepsilon x+\zeta xx)}=\frac{-du}{\beta+\gamma t+\varepsilon u}\ \text{ergo}$$

$$dV=\frac{-du(\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-u)+\mathfrak{E}t(tt-2u))}{\beta+\gamma t+\varepsilon u}\ \text{seu}$$

$$dV=\frac{-dt(\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-u)+\mathfrak{E}t(tt-2u))}{\delta-\gamma+\varepsilon t+\zeta u}.$$

 Est

Est vero aequatione illa resoluta

$$t = \frac{-\beta - \varepsilon u + \surd(\beta\beta - \alpha\gamma + 2(\gamma\nu + \beta\varepsilon - \gamma\delta)u + (\varepsilon\varepsilon - \gamma\zeta)uu)}{\gamma}$$

seu $t = \frac{-\beta - \varepsilon u + \surd\Delta(A + Lu + Euu)}{\gamma}$.

vnde conficitur:

$$dV = \frac{-du(\mathfrak{B} + \mathfrak{C}t + \mathfrak{D}(tt - u) + \mathfrak{E}t(tt - 2u))}{2\surd\Delta(A + Lu + Euu)}.$$

ideoque:

$$\Pi : x + \Pi : y = \text{Const.} - \int \frac{du(\mathfrak{B} + \mathfrak{C}t + \mathfrak{D}(tt - u) + \mathfrak{E}t(tt - 2u))}{\surd(A + Lu + Euu)}.$$

Vel cum reperiatur:

$$u = \frac{-(\delta - \gamma) - \varepsilon t + \surd((\delta - \gamma)^2 - \alpha\zeta + 2((\delta - \gamma)\varepsilon - \beta\zeta)t + (\varepsilon\varepsilon - \gamma\zeta)tt)}{\zeta}.$$

quae expressio abit in hanc:

$$u = \frac{-(\delta - \gamma) - \varepsilon t + 2\surd\Delta(L + 2Dt + Ett)}{\zeta}.$$

vnde fit:

$$dV = \frac{dt(\mathfrak{B} + \mathfrak{C}t + \mathfrak{D}(tt - u) + \mathfrak{E}t(tt - 2u))}{2\surd\Delta(L + C + 2Dt + Ett)}$$

sicque habebimus per t:

$$\Pi : x + \Pi : y = \text{Const.} + \int \frac{dt(\mathfrak{B} + \mathfrak{C}t + \mathfrak{D}(tt - u) + \mathfrak{E}t(tt - 2u))}{\surd(L + C + 2Dt + Ett)}$$

quae expressio, nisi sit algebraica, certe vel per logarithmos, vel arcus circulares exhiberi potest. Tum vero post integrationem tantum opus est, vt loco t restituatur eius valor $x + y$.

Coroll. 1.

Coroll. 1.

646. Si velimus vt posito $x=a$ fiat $y=b$, constans L ita debet definiri vt sit

$$\tfrac{1}{4}L(-b)^2 = A + B(a+b) + Cab + Dab(a+b) + Eaabb$$
$$\pm\sqrt{(A+2Ba+Caa+2Da^3+Ea^4)(A+2Bb+Cbb+2Db^3+Eb^4)}$$

tum igitur constans nostra erit $=\Pi:a+\Pi:b$, integrali postremo ita sumto, vt euanescat posito $t=a+b$.

Coroll. 2.

647. Eodem modo etiam differentia functionum $\Pi:x-\Pi:y$ exprimi potest, mutando alterutrius formulae radicalis signum, quo pacto formularum differentialium signum alterius conuertetur.

Coroll. 3.

648. Quantitas V comparationi harum functionum inseruiens, erit algebraica, si haec formula differentialis

$$\frac{dt(\mathfrak{B}+\mathfrak{C}\zeta t+\mathfrak{D}(\delta-\gamma+\epsilon t+\zeta tt)+\mathfrak{E}(2(\delta-\gamma)+2\epsilon t+\zeta tt)t)}{\zeta\sqrt{(L+C+2Dt+Ett)}}$$

integrationem admittat; quia altera pars $\frac{-2dt\sqrt{\Delta}}{\zeta}(\mathfrak{D}+2\mathfrak{E}t)$ per se est integrabilis.

Scholion.

649. Hoc ergo argumentum plane nouum de comparatione huiusmodi functionum transcendentium tam copiose pertractauimus, quam praesens institu-

tum poſtulare videbatur. Quando autem eius applicatio ad comparationem arcuum curuarum, quorum longitudo huiusmodi functionibus exprimitur, erit facienda, vberiori euolutione erit opus, vbi contemplatio ſingularium proprietatum, quae hoc modo eruuntur, eximium vſum afferre poterit. Commode autem hoc argumentum ad doctrinam de reſolutione aequationum differentialium referri videtur, ſiquidem inde eiusmodi aequationum integralia completa et quidem algebraice exhiberi poſſunt, quae aliis methodis fruſtra indagantur. Hunc igitur huic ſectionis finem faciet methodus generalis omnium aequationum differentialium integralia proxime determinandi.

CAPVT VII.

CAPVT VII.

DE INTEGRATIONE AEQVATIONVM DIFFERENTIALIVM PER APPROXIMATIONEM.

Problema 85.

650.

Propofita aequatione differentiali quaecunque eius integrale completum vero proxime affignare.

Solutio.

Sint x et y binae variabiles, inter quas aequatio differentialis proponitur, atque haec aequatio huiusmodi habebit formam vt fit $\frac{dy}{dx} = V$, exiftente V functione quacunque ipfarum x et y. Iam cum integrale completum defideretur, hoc ita eft interpretandum, vt dum ipfi x certus quidem valor puta $x = a$ tribuitur, altera variabilis y datum quemdam valorem puta $y = b$ adipifcatur. Quaeftionem ergo primo ita tractemus, vt inueftigemus valorem ipfius y, quando ipfi x valor paulisper ab a difcrepans tribuitur, feu pofito $x = a + \omega$, vt quaeramus y. Cum autem ω fit particula minima, etiam valor ipfius y minime a b

discrepabit; vnde dum x ab a vsque ad $a+\omega$ tantum mutatur, quantitatem V interea tanquam constantem spectare licet. Quare posito $x=a$ et $y=b$ fiat $V=A$, et pro hac exigua mutatione habebimus $\frac{dy}{dx}=A$, ideoque integrando $y=b+A(x-a)$, eiusmodi scilicet constante adiecta, vt posito $x=a$ fiat $y=b$. Statuamus ergo $x=a+\omega$, fietque $y=b+A\omega$. Quemadmodum ergo hic ex valoribus initio datis $x=a$ et $y=b$, proxime sequentes $x=a+\omega$ et $y=b+A\omega$ inuenimus, ita ab his simili modo per interualla minima vlterius progredi licet, quoad tandem ad valores a primitiuis quantumuis remotos perueniatur. Quae operationes quo clarius ob oculos ponantur, sequenti modo successiue instituantur.

Ipsius	valores successiui
x	$a\,;\ a'\,;\ a''\,;\ a'''\,;\ a^{IV}\,;\ \ldots\ {}'x\,;\ x$
y	$b\,;\ b'\,;\ b''\,;\ b'''\,;\ b^{IV}\,;\ \ldots\ {}'y\,;\ y$
V	$A\,;\ A'\,;\ A''\,;\ A'''\,;\ A^{IV}\,;\ \ldots\ {}'V\,;\ V.$

Scilicet ex primis $x=a$ et $y=b$ datis habetur $V=A$ tum vero pro secundis erit $b'=b+A(a'-a)$, differentia $a'-a$ minima pro lubitu assumta. Hinc ponendo $x=a'$ et $y=b'$ colligitur $V=A'$, indeque pro tertiis obtinebitur $b''=b'+A'(a''-a')$, vbi posito $x=a''$ et $y=b''$ inuenitur $V=A''$. Iam pro quartis habebimus $b'''=b''+A''(a'''-a'')$ hincque ponendo $x=a'''$ et $y=b'''$ colligemus $V=A'''$ sicque ad valores a primitiuis quantumuis

remotos

remotos progredi licebit. Series autem prima valores ipsius x successiuos exhibens pro lubitu accipi potest, dummodo per interualla minima ascendat vel etiam descendat.

Coroll. 1.

651. Pro singulis ergo interuallis minimis calculus eodem modo instituitur, sicque valores, a quibus sequentia pendent, obtinetur. Hoc ergo modo singulis pro x assumtis valoribus valores respondentes ipsius y assignari possunt.

Coroll. 2.

652. Quo minora accipiuntur interualla, per quae valores ipsius x progredi assumuntur, eo accuratius valores pro singulis eliciuntur. Interim tamen errores in singulis commissi, etiamsi sint multo minores, ob multitudinem coaceruatur.

Coroll. 3.

653. Errores autem in hoc calculo inde oriuntur, quod in singulis interuallis ambas quantitates x et y vt constantes spectemus, sitque functio V pro constante habeatur. Quo magis ergo valor ipsius V a quouis interuallo ad sequens immutatur, eo maiores errores sunt pertimescendi.

Scho-

Scholion 1.

654. Hoc incommodum imprimis occurrit, vbi valor ipsius V vel euanescit vel in infinitum excrescit, etiamsi mutationes ipsis x et y accidentes sint satis paruae. His autem casibus errores saltim enormes sequenti modo euitabuntur: sit pro initio huiusmodi interualli $x=a$ et $y=b$, tum vero in ipsa aequatione proposita ponatur $x=a+\omega$ et $y=b+\psi$, vt sit $\frac{d\psi}{d\omega}=V$, in V autem ita fiat substitutio $x=a+\omega$ et $y=b+\psi$, vt quantitates ω et ψ tanquam minimae spectentur, reiiciendo scilicet altiores potestates prae inferioribus, hoc enim modo plerumque integratio pro his interuallis actu institui poterit. Hac autem emendatione vix vnquam erit opus, nisi termini ex ipsis valoribus a et b nati se destruant. Veluti si habeatur haec aequatio $\frac{dy}{dx}=\frac{aa}{xx-yy}$ ac pro initio debeat esse $x=a$ et $y=a$; iam pro interuallo hinc incipiente ponatur $x=a+\omega$ et $y=a+\psi$ habebiturque $\frac{d\psi}{d\omega}=\frac{aa}{2a\omega-2a\psi}$, seu $2\omega d\psi-2\psi d\psi=ad\omega$, seu $d\omega-\frac{2\omega d\psi}{a}=\frac{-2\psi d\psi}{a}$, quae per $e^{\frac{-2\psi}{a}}=1-\frac{2\psi}{a}$ multiplicata et integrata praebet

$$\left(1-\frac{2\psi}{a}\right)\omega=\frac{-2}{a}\int\left(1-\frac{2\psi}{a}\right)\psi d\psi=-\frac{\psi\psi}{a}$$

quia posito $\omega=0$ fieri debet $\psi=0$. Hinc ergo habetur $\omega=\frac{-\psi\psi}{a-2\psi}=\frac{-\psi\psi}{a}$, seu $a(a'-a)=-(b'-b)^2$ existente $b=a$, vnde colligitur pro sequente intervallo

vallo $b' = b + \sqrt{-a(a'-a)}$, quo casu patet valorem x non vltra a augeri posse, quia y fieret imaginarium.

Scholion 2.

655. Passim traduntur regulae aequationum differentialium integralia per series infinitas exprimendi, quae autem plerumque hoc vitio laborant, vt integralia tantum particularia exhibeant, praeterquam quod series illae certo tantum casu conuergant, neque ergo aliis casibus vllum vsum praestent. Veluti si proposita sit aequatio $dy + y\,dx = ax^n dx$, iubemur huiusmodi seriem in genere fingere:

$$y = Ax^{\alpha} + Bx^{\alpha+1} + Cx^{\alpha+2} + Dx^{\alpha+3} + Ex^{\alpha+4} + \text{etc.}$$

qua substituta fit

$$\begin{array}{l} \alpha A x^{\alpha-1} + (\alpha+1)Bx^{\alpha} + (\alpha+2)Cx^{\alpha+1} + (\alpha+3)Dx^{\alpha+2} + \text{etc.} = 0 \\ \qquad\quad + \;\; A \;\; + \;\; B \;\; + \;\; C \\ -ax^n \end{array}$$

Statuatur ergo $\alpha - 1 = n$, seu $\alpha = n + 1$, eritque $A = \frac{a}{n+1}$ tum vero reliquis terminis ad nihilum reductis:

$$B = \frac{-A}{n+2};\; C = \frac{-B}{n+3};\; D = \frac{-C}{n+4};\; \text{etc.}$$

sicque habebitur haec series:

$$y = \frac{ax^{n+1}}{n+1} - \frac{ax^{n+2}}{(n+1)(n+2)} + \frac{ax^{n+3}}{(n+1)(n+2)(n+3)} - \frac{ax^{n+4}}{(n+1)(n+2)(n+3)(n+4)}\text{ etc.}$$

Verum hoc integrale tantum eſt particulare, quoniam euaneſcente x, ſimul y euaneſcit, niſi $n+1$ ſit numerus negatiuus; tum vero haec ſeries non conuergit, niſi x capiatur valde paruum. Quamobrem hinc minime cognoſcere licet valores ipſius y, qui reſpondeant valoribus quibuscunque ipſius x. Hoc autem vitio non laborat methodus, quam hic adumbrauimus, cum primo integrale completum praebeat, dum ſcilicet pro dato ipſius x valore datum ipſi y valorem tribuit, tum vero per interualla minima procedens, ſemper proxime ad veritatem accedat, et quousque libuerit progredi liceat. Sequenti autem modo haec methodus magis perfici poterit.

Problema 86.

656. Methodum praecedentem, aequationes differentiales proxime integrandi, magis perficere, vt minus a veritate aberret.

Solutio.

Propoſita aequatione integranda $\frac{dy}{dx}=V$, error methodi ſupra expoſitae inde oritur, quod per ſingula interualla functio V vt conſtans ſpectetur, cum tamen reuera mutationem ſubeat, praecipue niſi interualla ſtatuantur minima. Variabilitas autem ipſius V per quoduis interuallum ſimili modo in computum duci poteſt, quo in ſectione praecedente §. 321. vſi ſumus. Scilicet ſi iam ipſi x conueniat y,

niat y, ex natura differentialium ipsi $x-ndx$ vidimus conuenire

$$y-ndy+\frac{n(n+1)}{1.2}ddy-\frac{n(n+1)(n+2)}{1.2.3}d^3y+\text{etc.}$$

qui valor sumto n infinito erit

$$y-ndy+\frac{nnddy}{1.2}-\frac{n^3d^3y}{1.2.3}+\frac{n^4d^4y}{1.2.3.4}-\text{etc.}$$

Statuatur iam $x-ndx=a$ et

$$y-ndy+\frac{nnddy}{1.2}-\frac{n^3d^3y}{1.2.3}+\frac{n^4d^4y}{1.2.3.4}-\text{etc.}=b$$

hique valores in quouis interuallo vt primi spectentur, dum extremi per x et y indicantur. Cum igitur sit $n=\frac{x-a}{dx}$, fiet

$$y=b+\frac{(x-a)dy}{dx}-\frac{(x-a)^2ddy}{1.2dx^2}+\frac{(x-a)^3d^3y}{1.2.3dx^3}-\frac{(x-a)^4d^4y}{1.2.3.4dx^4}+\text{etc.}$$

quae expressio, si x non multum superat a, valde conuergit, ideoque admodum est idonea ad valorem y proxime inueniendum. Verum ad singulos terminos huius seriei euoluendos, notari oportet esse $\frac{dy}{dx}=V$, hincque $\frac{ddy}{dx^2}=\frac{dV}{dx}$. Cum autem V sit functio ipsarum x et y, si ponamus $dV=Mdx+Ndy$ ob $\frac{dy}{dx}=V$ erit $\frac{ddy}{dx^2}=M+NV$ seu exprimendi modo iam supra exposito $\frac{ddy}{dx^2}=(\frac{dV}{dx})+V(\frac{dV}{dy})$, quae expressio vti nata est ex praecedente $\frac{dy}{dx}=V$, ita ex ea nascetur sequens:

$$\frac{d^3y}{dx^3}=(\frac{ddV}{dx^2})+(\frac{dV}{dx})(\frac{dV}{dy})+2V(\frac{ddV}{dxdy})+V(\frac{dV}{dy})^2+VV(\frac{ddV}{dy^2}).$$

Quoniam vero ipse valor ipsius y nondum est cognitus,

gnitus, hoc modo ſaltem obtinetur aequatio algebraica, qua relatio inter x et y exprimitur; niſi forte ſufficiat in terminis minimis poſuiſſe $y=b$.

Altera autem operatio §. 322. expoſita valorem ipſius y, qui ipſi x in fine cuiusque interualli reſpondet, explicite determinabit, cum in initio eiusdem interualli fuerit $x=a$ et $y=b$. Cum enim hinc poſito $x=a+nda$, ſi quidem a et b vt variabiles ſpectemus, fiat

$$y=b+ndb+\frac{n(n-1)}{1.2}ddb+\frac{n(n-1)(n-2)}{1.2.3}d^3b+\text{etc.}$$

quia eſt $n=\frac{x-a}{da}$, ideoque numerus infinitus, erit

$$y=b+\frac{(x-a)db}{da}+\frac{(x-a)^2ddb}{1.2da^2}+\frac{(x-a)^3d^3b}{1.2.3da^3}+\text{etc.}$$

Eſt vero $\frac{db}{da}=V$ ſiquidem in functione V ſcribatur $x=a$ et $y=b$; tum vero iisdem pro x et y valoribus ſubſtitutis erit

$$\frac{ddb}{da^2}=\left(\frac{dV}{dx}\right)+V\left(\frac{dV}{dy}\right) \text{ et}$$

$$\frac{d^3b}{da^3}=\left(\frac{ddV}{dx^2}\right)+2V\left(\frac{ddV}{dxdy}\right)+VV\left(\frac{ddV}{dy^2}\right)+\left(\frac{dV}{dy}\right)\left(\left(\frac{dV}{dx}\right)+V\left(\frac{dV}{dy}\right)\right)$$

vnde ſequentes ſimili modo formari oportet. Sit igitur poſtquam, ſcripſerimus $x=a$ et $y=b$.

$$\frac{dy}{dx}=A;\ \frac{ddy}{dx^2}=B;\ \frac{d^3y}{dx^3}=C;\ \frac{d^4y}{dx^4}=D;\ \text{etc.}$$

ac valori $x=a+\omega$ conueniet iſte valor

$$y=b+A\omega+\tfrac{1}{2}B\omega^2+\tfrac{1}{6}C\omega^3+\tfrac{1}{24}D\omega^4+\text{etc.}$$

qui

qui duo valores iam pro ſequente interuallo erunt initiales, ex quibus ſimili modo finales erui oportet.

Coroll. 1.

657. Quoniam hic variabilitatis functionis V rationem habuimus, interualla iam maiora ſtatuere licet, ac ſi illas formulas A, B, C, D, etc. in infinitum continuare vellemus, interualla quantumvis magna aſſumi poſſent, tum autem pro y orire-tur ſeries infinita.

Coroll. 2.

658. Si ſeriei inuentae tantum binos terminos primos ſumamus, vt ſit $y = b + A\omega$, habebitur determinatio praecendens, vnde ſimul patet errorem ibi commiſſum ſequentibus terminis iunctim ſumtis aequari.

Coroll. 3.

659. Etiamſi autem ſeriei inuentae plures terminos capiamus, conſultum tamen non erit interualla nimis magna conſtitui, vt ω valorem modicum obtineat, praecipue ſi quantitates B, C, D etc. euadant valde magnae.

Scholion.

660. Maximo incommodo hae operationes turbantur, ſi quando horum coefficientium A, B, C, D etc. quidam in infinitum excreſcant. Euenit autem hoc tantum in certis interuallis, vbi ipſa

 quan-

quantitas V vel in nihilum abit vel in infinitum, cui incommodo, quemadmodum sit occurrendum, iam innuimus et mox accuratius ostendemus. Caeterum calculus pro singulis interuallis pari modo instituitur, ita vt cum eius ratio pro interuallo primo fuerit inuenta, quod incipit a valoribus pro lubitu assumtis $x=a$ et $y=b$, eadem pro sequentibus interuallis sit valitura. Cum enim pro fine interualli primi fiat

$$x=a+\omega=a' \text{ et}$$

$$y=b+A\omega+\tfrac{1}{2}B\omega^2+\tfrac{1}{6}C\omega^3+\tfrac{1}{24}D\omega^4+\text{etc.}=b'$$

hi erunt valores initiales pro interuallo secundo, ex quibus simili modo finales elici oportet; hic scilicet calculus innitetur perinde litteris a' et b', ac prior litteris a et b, id quod clarius ex exemplis subiunctis patebit.

Exemplum 1.

661. *Aequationis differentialis* $dy=dx(x^n+cy)$ *integrale completum proxime inuestigare.*

Cum hic sit $V=\frac{dy}{dx}=x^n+cy$, erit differentiando

$$\frac{ddy}{dx^2}=nx^{n-1}+cx^n+ccy\text{; sicque porro}$$

$$\frac{d^3y}{dx^3}=n(n-1)x^{n-2}+ncx^{n-1}+ccx^n+c^3y$$

$$\frac{d^4y}{dx^4}=n(n-1)(n-2)x^{n-3}+n(n-1)cx^{n-2}+nccx^{n-1}+c^3x^n+c^4y$$

etc.

Quodsi

Quodsi ergo ponamus valori $x=a$, conuenire $y=b$, alii cuicunque valori $x=a+\omega$ conueniet:

$$y=b+\omega(a^n+cb)+\tfrac{1}{2}\omega^2(ccb+ca^n+na^{n-1})$$
$$+\tfrac{1}{6}\omega^3(c^3b+cca^n+nca^{n-1}+n(n-1)a^{n-2})$$
$$+\tfrac{1}{24}\omega^4(c^4b+c^3a^n+ncca^{n-1}+n(n-1)ca^{n-2}+n(n-1)(n-2)a^{n-3})$$

etc.

quae series sumta quantitate ω satis parua, quantumuis promte conuergit, sicque posito $a+\omega=a'$ et respondente valore ipsius $y=b'$, hinc simili modo ad sequentes perueniemus, quam operationem, quousque lubuerit, continuare licet.

Exemplum 2.

662. *Aequationis differentialis* $dy=dx(xx+yy)$ *integrale completum proxime inuestigare.*

Cum hic sit $\frac{dy}{dx}=V=xx+yy$ erit continuo differentiando

$$\frac{ddy}{dx^2}=2x+2xxy+2y^3 \text{ et}$$
$$\frac{d^3y}{dx^3}=2+4xy+2x^4+8xxyy+6y^4$$
$$\frac{d^4y}{dx^4}=4y+12x^3+20xyy+16x^4y+40xxy^3+24y^5$$
$$\frac{d^5y}{dx^5}=40x^2+24y^2+104x^3y+120xy^3+16x^6+136x^4y^2$$
$$+240x^2y^4+120y^6$$

etc.

Quare

Quare si initio sit $x = a$ et $y = b$, erit

$A = aa + bb$

$B = 2a + 2aab + 2b^3$

$C = 2 + 4ab + 2a^4 + 8aabb + 6b^4$

$D = 4b + 12a^3 + 20abb + 16a^4b + 40aab^3 + 24b^5$

$$E = 40a^2 + 24b^2 + 104a^3b + 120ab^3 + 16a^6 + 136a^4b^2 + 240a^2b^4 + 120b^6$$

vnde valori cuicunque alii $x = a + \omega$ conueniet:

$$y = b + A\omega + \tfrac{1}{2}B\omega^2 + \tfrac{1}{6}C\omega^3 + \tfrac{1}{24}D\omega^4 + \tfrac{1}{120}E\omega^5 \text{ etc.}$$

atque ex talibus binis valoribus qui sint $x = a'$ et $y = b'$ denuo sequentes elici possunt.

Scholion.

663. Quoniam totum negotium ad inuentionem horum coefficientium A, B, C, D etc. redit, obseruo eosdem sine differentiatione inueniri posse, id quod in hoc postremo exemplo $\frac{dy}{dx} = xx + yy$ ita praestabitur. Cum statuamus posito $x = a$ fieri $y = b$, ponamus in genere $x = a + \omega$ et $y = b + \psi$, et nostra aequatio induet hanc formam:

$$\frac{d\psi}{d\omega} = aa + bb + 2a\omega + \omega\omega + 2b\psi + \psi\psi$$

et quia euanescente ω simul euanescit ψ, sumimus:

$$\psi = \alpha\omega + \beta\omega^2 + \gamma\omega^3 + \delta\omega^4 + \varepsilon\omega^5 + \text{etc.}$$

hocque

hocque valore substituto prodibit:

$$\alpha + 2\beta\omega + 3\gamma\omega^2 + 4\delta\omega^3 + 5\varepsilon\omega^4 + \text{etc.} =$$
$$\begin{aligned} aa + bb &+ 2a\omega + \omega\omega \\ &+ 2\alpha b\omega + 2\beta b\omega^2 + 2\gamma b\omega^3 + 2\delta b\omega^4 \text{ etc.} \\ &\quad + \alpha^2\omega^2 + 2\alpha\beta\omega^3 + 2\alpha\gamma\omega^4 \text{ etc.} \\ &\qquad + \beta\beta\omega^4 \text{ etc.} \end{aligned}$$

singulis ergo terminis ad nihilum reductis fiet

$$\alpha = aa + bb;\ 2\beta = 2ab + 2a;\ 3\gamma = 2\beta b + \alpha\alpha + 1$$
$$4\delta = 2\gamma b + 2\alpha\beta;\ 5\varepsilon = 2\delta b + 2\alpha\gamma + \beta\beta;$$
$$6\zeta = 2\varepsilon b + 2\alpha\delta + 2\beta\gamma \text{ etc.}$$

vnde iidem valores qui supra per differentiationem eliciuntur. Vti haec methodus simplicior est praecedente, ita etiam hoc illi praestat, quod semper in vsum vocari possit, cum illa interdum frustra applicetur, veluti in exemplis allatis euenit, si valores initiales a et b euanescant, vbi plerique coefficientes in nihilum abirent. Quod idem incommodum iam supra animaduertimus, cum adeo euenire possit, vt omnes coefficientes vel euanescant, vel in infinitum abeant. Verum hoc nonnisi in certis interuallis vsu venit, pro quibus ergo calculum peculari modo institui conueniet; reliquis autem interuallis methodus hic exposita per differentiationem procedens commodius adhiberi videtur, quippe quae saepe facilius instituitur quam substitutio, certisque regulis continetur, semper locum habentibus etiam in aequationibus transcendentibus. Quare pro singularibus illis interuallis praecepta tradere oportebit.

Problema 87.

664. Si in integratione aequationis $\frac{dy}{dx} = V$ pro quopiam interuallo eueniat, vt quantitas V vel euanescat, vel fiat infinita, integrationem pro isto interuallo instituere.

Solutio.

Sit pro initio interualli, quod contemplamur $x = a$ et $y = b$, quo casu cum V vel euanescat vel in infinitum abeat ponamus $\frac{dy}{dx} = \frac{P}{Q}$, ita vt posito $x = a$ et $y = b$ vel P vel Q vel vtrumque euanescat. Statuamus ergo vt ab his terminis vlterius progrediamur $x = a + \omega$ et $y = b + \psi$, fietque $\frac{dy}{dx} = \frac{d\psi}{d\omega}$; atque tam P quam Q erit functio ipsarum ω et ψ, quarum altera saltem euanescat facto $\omega = 0$ et $\psi = 0$. Iam ad rationem inter ω et ψ proxime saltem inuestigandam, ponatur $\psi = m\omega^n$, erit $\frac{d\psi}{d\omega} = mn\omega^{n-1}$, hincque $mnQ\omega^{n-1} = P$; vbi P et Q ob $\psi = m\omega^n$ meras potestates ipsius ω continebunt, quarum tantum minimas in calculo retinuisse sufficit, cum altiores prae his vt euanescentes spectari queant. Infimae ergo potestates ipsius ω inter se aequales reddantur, simulque ad nihilum redigantur; vnde tam exponens n quam coefficiens m determinabitur. Si deinde relationem inter ω et ψ exactius cognoscere velimus, inuentis m et n ad altiores potestates ascendamus ponendo $\psi = m\omega^n + M\omega^{n+\mu} + N\omega^{n+\nu}$ etc. hinc-

que

que fimili modo fequentes partes definientur, quousque ob magnitudinem interualli feu particulae ω neceffarium vifum fuerit.

Coroll. 1.

665. Si pofito $x = a$ et $y = b$ neque P neque Q euanefcat, fubftitutione adhibita reperietur $\frac{d\psi}{d\omega} = \frac{A + etc.}{\alpha + etc.}$, hincque proxime $\alpha d\psi = A d\omega$ et $\psi = \frac{A}{\alpha}\omega$, qui eft primus terminus praecedentis approximationis, quo inuento reliqui vt ante fe habebunt.

Coroll. 2.

666. Si α tantum euanefcat, habebitur $\frac{d\psi}{d\omega}(M\omega^{\mu} + N\psi^{\nu}\text{etc.}) = A$ proxime: vnde pofito $\psi = m\omega^{n}$ fit $A = mn\omega^{n-1}(M\omega^{\mu} + Nm^{\nu}\omega^{n\nu})$; vbi fi $n\nu > \mu$ debet effe $n = 1 - \mu$, et $mnM = A$; quod autem non valet, nifi fit $\nu(1 - \mu) > \mu$ feu $\nu > \frac{\mu}{1-\mu}$. Sin autem fit $\nu < \frac{\mu}{1-\mu}$, ftatui debet $n - 1 + n\nu = 0$ feu $n = \frac{1}{1+\nu}$, altero termino vt infima poteftate fpectata. At fi fuerit $\nu = \frac{\mu}{1-\mu}$ ambo termini pro paribus poteftatibus erunt habendi fietque $n = 1 - \mu$ et $A = mn(M + Nm^{\nu})$ vnde m definiri debet.

Scholion.

667. In genere hic vix quicquam praecipere licet, fed quouis cafu oblato haud difficile eft omnia, quae ad folutionem perducunt perfpicere. Si quidem

omnes exponentes essent integri, regula illa *Neutoniana*, qua ope parallelogrammi resolutio aequationum instruitur, hic in vsum vocari posset; tum vero exponentium fractorum ad integros reductio satis est nota. Verum huiusmodi casus tam raro occurrunt, vt inutile foret in praeceptis prolixum esse, quae quouis casu ab exercitato facile conduntur. Veluti si perueniatur ad hanc aequationem $\frac{d\psi}{d\omega}(\alpha V\omega+\beta\psi)=\gamma$, ex superioribus patet primam operationem dare $\psi=mV\omega$, vnde fit $\frac{1}{2}m(\alpha+\beta m)=\gamma$, vnde m innotescit idque duplici modo. Quin etiam haec aequatio posito $V\omega=p$ ad homogeneitatem reducitur ideoque reuera integrari potest, verum haec vix vnquam vsum habitura fusius non prosequor, sed, quod adhuc in hac parte pertractandum restat exponam, quomodo eiusmodi aequationes differentiales resolui oporteat, in quibus differentialium ratio puta $\frac{dy}{dx}=p$ vel plures obtinet dimensiones, vel adeo transcendenter ingreditur, quo absoluto partem secundam, in qua differentialia altiorum graduum occurrunt, aggrediar.

CALCVLI

CALCVLI INTEGRALIS
LIBER PRIOR.

PARS PRIMA

SEV

METHODVS INVESTIGANDI FVNCTIONES VNIVS VARIABILIS EX DATA RELATIONE QVACVNQVE DIFFERENTIALIVM PRIMI GRADVS.

SECTIO TERTIA

DE

RESOLVTIONE AEQVATIONVM DIFFERENTIALIVM MAGIS COMPLICATARVM.

DE

RESOLVTIONE AEQVATIONVM DIFFERENTIALIVM IN QVIBVS DIFFERENTIALIA AD PLVRES DIMENSIONES ASSVRGVNT, VEL ADEO TRANSCENDENTER IMPLICANTVR.

Problema 88.

668.

Posita differentialium relatione $\frac{dy}{dx}=p$, si proponatur aequatio quaecunque inter binas quantitates x et p, relationem inter ipsas veriabiles x et y inuestigare.

Solutio.

Cum detur aequatio inter p et x, concessa aequationum resolutione, ex ea quaeratur p per x, ac reperietur functio ipsius x, quae ipsi p erit aequalis. Peruenietur ergo ad huiusmodi aequationem $p=X$ existente X functione quapiam ipsius x tantum. Quare cum sit $p=\frac{dy}{dx}$, habebimus $dy=Xdx$, sicque quaestio ad sectionem primam est reducta, vnde formulae Xdx integrale inuestigari oportet; quo facto integrale quaesitum erit $y=\int Xdx$.

Si

Si aequatio inter x et p data, ita fuerit comparata, vt inde facilius x per p definiri possit, quaeratur x prodeatque $x=P$ existente P functione quadam ipsius p. Hac igitur aequatione differentiata erit $dx=dP$, hincque $dy=pdx=pdP$, vnde integrando elicitur $y=\int pdP$ seu $y=pP-\int Pdp$. Hinc ergo ambae variabiles x et y per tertiam p ita determinantur, vt sit:

$$x=P \text{ et } y=pP-\int Pdp$$

vnde relatio inter x et y est manifesta.

Si neque p commode per x, neque x per p definiri queat, saepe effici potest, vt vtraque commode per nouam quantitatem u definiatur; ponamus ergo inueniri $x=U$ et $p=V$, vt U et V sint functiones eiusdem variabilis u. Hinc ergo erit $dy=pdx=VdU$, et $y=\int VdU$, sicque x et y per eandem nouam variabilem u exprimuntur.

Coroll. 1.

669. Simili modo resoluetur casus, quo aequatio quaecunque inter p et alteram variabilem y proponitur, quoniam binas variabiles x et y inter se permutare licet. Tum autem siue p per y, siue y per p, siue vtraque per nouam variabilem u definiatur, notari oportet, esse $dx=\frac{dy}{p}$.

Coroll. 2.

Coroll. 2.

670. Cum $\sqrt{(dx^2+dy^2)}$ exprimat elementum arcus curuae, cuius coordinatae rectangulae sunt x et y, si ratio $\frac{\sqrt{(dx^2+dy^2)}}{dx}=\sqrt{(1+pp)}$ seu $\frac{\sqrt{(dx^2+dy^2)}}{dy}=\frac{\sqrt{(1+pp)}}{p}$, aequetur functioni vel ipsius x vel ipsius y, hinc relatio inter x et y inueniri poterit.

Coroll. 3.

671. Quoniam hoc modo relatio inter x et y per integrationem inuenitur, simul noua quantitas constans introducitur, quo circa illa relatio pro integrali completo erit habenda.

Scholion 1.

672. Hactenus eiusmodi tantum aequationes differentiales examini subiecimus, quibus posito $\frac{dy}{dx}=p$, eiusmodi relatio inter ternas quantitatis x, y et p proponitur, vnde valor ipsius p commode per x et y exprimi potest, ita vt $p=\frac{dy}{dx}$ aequetur functioni cuipiam ipsarum x et y. Nunc igitur eiusmodi relationes inter x, y et p considerandae veniunt, ex quibus valorem ipsius p vel minus commode, vel plane non, per x et y definire liceat; atque hic simplicissimus casus sine dubio est, quando in relatione proposita altera variabilis x seu y plane deest, ita vt tantum relatio inter p et x vel p et y pro-

ponatur; quem casum in hoc problemate expediuimus. Solutionis autem vis in eo versatur, vt proposita aequatione inter x et p, non littera p per x, nisi forte hoc facile praestari queat, sed potius x per p, vel etiam vtraque per nouam variabilem u definiatur. Veluti si proponatur haec aequatio

$$xdx+ady=b\sqrt{(dx^2+dy^2)}$$

quae posito $\frac{dy}{dx}=p$ abit in hanc

$$x+ap=b\sqrt{(1+pp)}$$

hinc minus commode definiretur p per x. Cum autem sit

$$x=b\sqrt{(1+pp)}-ap, \text{ ob } y=\int pdx=px-\int xdp$$

erit

$$y=bp\sqrt{(1+pp)}-app-b\int dp\sqrt{(1+pp)}+\tfrac{1}{2}app$$

sicque relatio inter x et y constat. Sin autem peruentum fuerit ad talem aequationem

$$x^3dx^3+dy^3=axdx^2dy \text{ seu } x^3+p^3=apx$$

hinc neque x per p neque p per x commode definire licet; ex quo pono $p=ux$, vnde fit $x+u^3x=au$, hincque $x=\frac{au}{1+u^3}$ et $p=\frac{auu}{1+u^3}$. Iam ob $dx=\frac{adu(1-2u^3)}{(1+u^3)^2}$, colligitur $y=aa\int\frac{uudu(1-2u^3)}{(1+u^3)^3}$, ac reducendo hanc formam ad simpliciorem

$$y=\tfrac{1}{6}aa.\frac{2u^3-1}{(1+u^3)^2}-aa\int\frac{uudu}{(1+u^3)^2} \text{ seu}$$

$$y=\tfrac{1}{6}aa.\frac{2u^3-1}{(1+u^3)^2}+\tfrac{1}{3}aa.\frac{1}{1+u^3}+\text{Const.}$$

Scholion 2.

Scholion 2.

673. Cum igitur hunc casum, quo aequatio vel inter x et p vel inter y et p proponitur, generatim expedire licuerit, videndum est quibus casibus euolutio succedat, quando omnes tres quantitates x, y et p in aequatione proposita insunt. Ac primo quidem obseruo, dummodo binae variabiles x et y vbique eundem dimensionum numerum adimpleant, quomodocunque praeterea quantitas p ingrediatur, resolutionem semper ad casus ante tractatos reuocari posse; tales scilicet aequationes perinde tractare licet, atque aequationes homogeneas, ad quod genus etiam merito referuntur, cum dimensiones a differentialibus natae vbique debeant esse pares, et iudicium ex solis quantitatibus finitis x et y peti oporteat. Quae ergo dummodo vbique eundem dimensionum numerum constituant, aequatio pro homogenea erit habenda, veluti est $xxdy - yy\sqrt{(dx^2 + dy^2)} = 0$ seu $pxx - yy\sqrt{(1 + pp)} = 0$. Deinde etiam eiusmodi aequationes euolutionem admittunt, in quibus altera variabilis x vel y plus vna dimensione nusquam habet, vtcunque praeterea differentialium ratio $p = \frac{dy}{dx}$ ingrediatur. Hos ergo casus hic accuratius explicemus.

Problema 89.

674. Posito $p = \frac{dy}{dx}$ si in aequatione inter x, y et p proposita binae variabiles x et y vbique eundem dimensionum numerum compleant, inuenire

relationem inter x et y, quae illius aequationis sit integrale completum.

Solutio.

Cum in aequatione inter x, y et p proposita binae variabiles x et y vbique eundem dimensionum numerum constituant, si ponamus $y = ux$, quantitas x inde per diuisionem tolletur, habebiturque aequatio inter duas tantum quantitates u et p, qua earum relatio ita definietur, vt vel u per p, vel p per u determinari possit. Iam ex positione $y = ux$ sequitur $dy = udx + xdu$ cum igitur sit $dy = pdx$, erit $pdx - udx = xdu$, ideoque $\frac{dx}{x} = \frac{du}{p-u}$. Quia itaque p per u datur, formula differentialis $\frac{du}{p-u}$ vnicam variabilem complectens per regulas primae sectionis integretur, eritque $lx = \int \frac{du}{p-u}$, sicque x per u determinatur; et cum sit $y = ux$ ambae variabiles x et y per eandem tertiam variabilem u determinantur, et quia illa integratio constantem arbitrariam inducit, haec relatio inter x et y erit integrale completum.

Coroll. 1.

675. Cum sit $\frac{dx}{x} = \frac{du}{p-u}$, erit etiam $lx = -l(p-u) + \int \frac{dp}{p-u}$, quae formula commodior est, si forte ex aequatione inter p et u proposita quantitas u facilius per p definitur.

Coroll. 2.

Coroll. 2.

676. Quodsi integrale $\int\frac{du}{p-u}$ vel $\int\frac{du}{p-u}$ per logarithmos exprimi possit, vt sit $\int\frac{du}{p-u}=lU$, erit $lx=lC+lU$ hincque $x=CU$ et $y=CUu$; vnde relatio inter x et y algebraice dabitur: et cum sit $u=\frac{y}{x}$, haec tertia variabilis u facile eliditur.

Scholion.

677. Eandem hanc resolutionem supra in aequationibus homogeneis ordinariis docuimus, quae ergo ob dimensiones differentialium non turbatur; quin etiam succedit, etiamsi ratio differentialium $\frac{dy}{dx}=p$ transcendenter ingrediatur. Hoc modo scilicet resolutio ad integrationem aequationis differentialis separatae $\frac{dx}{x}=\frac{du}{p-u}$ perducitur, quemadmodum etiam supra per priorem methodum negotium fuit expeditum. Altera vero methodus, qua supra vsi sumus, quaerendo factorem qui aequationem differentialem reddat per se integrabilem, hic plane locum non habet, cum per differentiationem aequationis finitae nunquam differentialia ad plures dimensionis exsurgere queant. Non ergo hoc modo inuenitur aequatio finita inter x et y, quae differentiata ipsam aequationem propositam reproducat, sed quae saltem cum ea conueniat, et quidem non obstante arbitraria illa constante, quae per integrationem ingressa integrale completum reddit.

Exemplum 1.

678. *Si in aequationem propositam neutra variabilium* x *et* y *ipsa ingrediatur, sed tantum differentialium ratio* $\frac{dy}{dx} = p$, *integrale completum assignare.*

Posito ergo $\frac{dy}{dx} = p$ aequatio proposita solam variabilem p cum constantibus complectetur, vnde ex eius resolutione, prout plures inuoluat radices, orietur $p = \alpha$, $p = \beta$, $p = \gamma$ etc. Iam ob $p = \frac{dy}{dx}$ ex singulis radicibus integralia completa elicientur, quae erunt:

$$y = \alpha x + a;\ y = \beta x + b;\ y = \gamma x + c \text{ etc.}$$

quae singula aequationi propositae aeque satisfaciunt. Quae si velimus omnia vna aequatione finita complecti, erit integrale completum

$$(y - \alpha x - a)(y - \beta x - b)(y - \gamma x - c) \text{ etc.} = 0$$

quae vti apparet non vnam nouam constantem, sed plures a, b, c etc. comprehendit, tot scilicet, quot aequatio differentialis plurium dimensionum habuerit radices.

Coroll. 1.

679. Ita aequationis differentialis $dy^2 - dx^2 = 0$ seu $pp - 1 = 0$, ob $p = +1$ et $p = -1$, duo habemus integralia $y = x + a$ et $y = -x + b$, quae in vnum collecta dant $(y - x - a)(y + x - b) = 0$ seu $yy - xx - (a + b)y - (a - b)x + ab = 0$.

Coroll. 2.

Coroll. 2.

680. Proposita aequatione $dy^3+dx^3=0$ seu $p^3+1=0$, ob radices $p=-1$; $p=\frac{1+\sqrt{-3}}{2}$ et $p=\frac{1-\sqrt{-3}}{2}$ erit vel $y=-x+a$; vel $y=\frac{1+\sqrt{-3}}{2}x+b$; vel $y=\frac{1-\sqrt{-3}}{2}x+c$, quae collecta praebent:

$$y^3+x^3-(a+b+c)yy+\left(a-\frac{1-\sqrt{-3}}{2}b-\frac{1+\sqrt{-3}}{2}c\right)xy+\left(-a+\frac{1-\sqrt{-3}}{2}b+\frac{1+\sqrt{-3}}{2}c\right)xx$$
$$+(ab+ac+bc)y+\left(bc-\frac{1-\sqrt{-3}}{2}ac-\frac{1+\sqrt{-3}}{2}ab\right)x-abc=0$$

quae aequatio etiam ita exhiberi potest:

$$y^3+x^3-fyy-gxy-hxx+Ay+Bx+C=0$$

vbi constantes A, B, C ita debent esse comparatae, vt aequatio haec resolutionem in tres simplices admittat.

Exemplum 2.

681. *Proposita aequatione differentiali*

$$ydx-x\sqrt{(dx^2+dy^2)}=0$$

eius integrale completum inuenire.

Posito $\frac{dy}{dx}=p$ fit $y-x\sqrt{(pp+1)}=0$ fit ergo $y=ux$ erit $u=\sqrt{(pp+1)}$ et $\frac{dx}{x}=\frac{du}{p-u}$, vnde per alteram formulam

$$lx=-l(p-u)+\int\frac{dp}{p-\sqrt{(pp+1)}}=-l(p-u)-\int dp(p+\sqrt{(pp+1)})$$

at $\int dp\sqrt{(pp+1)}=\frac{1}{2}p\sqrt{(1+pp)}+\frac{1}{2}l(p+\sqrt{(1+pp)})$

vnde

inde colligitur

$$lx = C - \tfrac{1}{2}l(\surd(1+pp)-p) - \tfrac{1}{2}p\surd(1+pp) - \tfrac{1}{2}pp = C + \tfrac{1}{2}l(\surd(1+pp)+p) - \tfrac{1}{2}p\surd(1+pp) - \tfrac{1}{2}pp$$

et $y = ux = x\surd(pp+1)$.

Exemplum 3.

682. *Huius aequationis* $ydx - xdy = nx\surd(dx^2+dy^2)$ *integrale completum inuenire.*

Ob $\frac{dy}{dx} = p$ nostra aequatio est $y - px = nx\surd(1+pp)$, quae posito $y = ux$ abit in $u - p = n\surd(1+pp)$. Cum ergo sit

$lx = -l(p-u) + \int\frac{dp}{p-u}$, erit $lx = -ln\surd(1+pp) - \int\frac{dp}{n\surd(1+pp)}$

hincque

$$lx = C - ln\surd(1+pp) - \tfrac{1}{n}l(p+\surd(1+pp)).$$

Quare habetur

$$x = \frac{a}{\surd(1+pp)}(\surd(1+pp)-p)^{\frac{1}{n}} \text{ et } y = \frac{a(p+n\surd(1+pp))}{\surd(1+pp)}(\surd(1+pp)-p)^{\frac{1}{n}}.$$

Cum nunc sit $uu - 2up + pp = nn + nnpp$ erit

$$p = \frac{u - n\surd(uu+1-nn)}{1-nn} \text{ et } \surd(1+pp) = \frac{-nu+\surd(uu+1-nn)}{1-nn}$$

atque $\surd(1+pp) - p = \frac{-u+\surd(uu+1-nn)}{1-n}$

vnde fit

$$\frac{x(-nu+\surd(uu+1-nn))}{a(1-nn)} = \left(\frac{-u+\surd(uu+1-nn)}{1-n}\right)^{\frac{1}{n}} \text{ vbi } u = \frac{y}{x}.$$

At si $n = 1$, erit $p = \frac{uu-1}{2u}$; $\surd(1+pp) = \frac{uu+1}{2u}$, atque $x = \frac{2au}{uu+1}\cdot\frac{1}{u} = \frac{2axx}{xx+yy}$ seu $yy + xx = 2ax$.

Si

Si $n=-1$ eſt quidem vt ante

$$p=\frac{uu-1}{2u} \text{ et } \sqrt{(1+pp)}=\frac{-uu-1}{2u}$$

vnde

$$x=\frac{a}{\sqrt{(1+pp)}}(\sqrt{(1+pp)}+p)=\frac{-2a}{1+uu}=\frac{-2axx}{xx+yy}.$$

Ergo et $x=0$ et $xx+yy+2ax=0$.

Scholion.

683. Haec aequatio ſumendis vtrinque quadratis et radice $p=\frac{dy}{dx}$ extrahenda ad aequationem homogeneam ordinariam reducitur. Fit enim primo $yy-2pxy+ppxx=nnxx+nnppxx$, tum vero $px=\frac{xdy}{dx}=\frac{y\pm n\sqrt{(yy+xx-nnxx)}}{1-nn}$, quae poſito $y=ux$ ſeparabilis redditur. Vbi imprimis caſus quo $nn=1$ notari meretur, quo fit $yy-2pxy=xx$, ſeu $p=\frac{dy}{dx}=\frac{yy-xx}{2xy}$, ideoque $2xydy+xxdx-yydx=0$: quae etiam per partes integrari poteſt, cum $2xydy-yydx$ integrabile fiat per factorem $\frac{1}{xyy}f:\frac{yy}{x}$, quo vt etiam pars $xxdx$ integrabilis reddatur, illa forma abit in $\frac{1}{xx}$, ſicque habebitur $\frac{2xydy-yydx}{xx}+dx=0$, cuius integrale eſt $\frac{yy}{x}+x=2a$, vt ante, niſi quod altera ſolutio $x=0$ hinc non eliciatur. Verum cum aequatio illa quadrata poſito $n=1$ ſubito abeat in ſimplicem, altera radix perit, quae reperitur ponendo $n=1-\alpha$, quo fit

$$yy-2pxy=xx-2\alpha xx-2\alpha ppxx$$

ideoque px infinitum, reiectis ergo terminis prae reliquis

reliquis euanescentibus est $-2pxy = xx - 2appxx$, quae diuisibilis per x alteram praebet solutionem $x=0$. Talis quidem resolutio succedit, quando valorem p per radicis extractionem elicere licet; sed si aequatio ad plures dimensiones ascendat, vel adeo transcendens fiat, methodo hic exposita carere non possumus.

Exemplum 4.

684. *Proposita aequatione*

$$x\,dy^2 + y\,dx^2 = dy\,dx\sqrt{xy(dx^2+dy^2)}$$

eius integrale completum inuestigare.

Posito $\frac{dy}{dx}=p$, et $y=ux$, nostra aequatio induet hanc formam $p^2+u=p\sqrt{u(1+pp)}$, vnde conficitur $\frac{dx}{x}=\frac{du}{p-u}$, seu $lx=\int\frac{du}{p-u}=-l(p-u)+\int\frac{dp}{p-u}$. Inde autem est

$$\sqrt{u}=\tfrac{1}{2}p\sqrt{(1+pp)}+\tfrac{1}{2}p\sqrt{(1-4p+pp)},$$

et quadrando

$$u=\tfrac{1}{2}pp-p^3+\tfrac{1}{2}p^4+\tfrac{1}{2}pp\sqrt{(1+pp)(1-4p+pp)}$$

hincque

$$p-u=\tfrac{1}{2}p(1+pp)(2-p)-\tfrac{1}{2}pp\sqrt{(1+pp)(1-4p+pp)}$$

vnde colligimus

$$\frac{dp}{p-u}=\frac{dp(2-p)}{2p(1-p+pp)}+\frac{dp\sqrt{(1-4p+pp)}}{2(1-p+pp)\sqrt{(1+pp)}}.$$

In quorum membrorum posteriore si ponatur $\sqrt{\frac{1-4p+pp}{1+pp}}=q$; ob

ob $p = \frac{2 + \sqrt{(4 - (1 - qq)^2)}}{1 - qq}$; $dp = \frac{4qdq(2 + \sqrt{(4 - (1 - qq)^2)})}{(1 - qq)^2 \sqrt{(4 - (1 - qq)^2)}}$ et $1 - p + pp = \frac{(3 + qq)(2 + \sqrt{(4 - (1 - qq)^2)})}{(1 - qq)^2}$ obtinebitur

$$\int \frac{dp}{p - u} = \frac{1}{2} \int \frac{dp(2 - p)}{p(1 - p + pp)} + 2 \int \frac{qqdq}{(3 + qq)\sqrt{(4 - (1 - qq)^2)}}$$

vbi membrum posterius neque per logarithmos, neque arcus circulares integrari potest.

Exemplum 5.

685. *Inuenire relationem inter* x *et* y, *vt posito* $s = \int \sqrt{(dx^2 + dy^2)}$ *fiat* $ss = 2xy$.

Cum sit $s = \sqrt{2xy}$ erit $ds = \sqrt{(dx^2 + dy^2)} = \frac{xdy + ydx}{\sqrt{2xy}}$ hincque posito $\frac{dy}{dx} = p$ et $y = ux$ fiet $\sqrt{(1 + pp)} = \frac{p + u}{\sqrt{2u}}$, seu $u = \sqrt{2u(1 + pp)} - p$, et radice extracta $\sqrt{u} = \sqrt{\frac{1 + pp}{2}} + \frac{1 - p}{\sqrt{2}} = \frac{1 - p + \sqrt{(1 + pp)}}{\sqrt{2}}$, quare

$u = 1 - p + pp + (1 - p)\sqrt{(1 + pp)}$ et $p - u = -(1 - p)(1 - p + \sqrt{(1 + pp)})$. Ergo

$$\int \frac{dp}{p - u} = \int \frac{dp}{2p(1 - p)}(1 - p - \sqrt{(1 + pp)}) = \frac{1}{2} lp - \frac{1}{2} \int \frac{dp\sqrt{(1 + pp)}}{p(1 - p)}.$$

At posito $p = \frac{1 - qq}{2q}$ fit

$$\int \frac{dp\sqrt{(1 + pp)}}{p(1 - p)} = \int \frac{-dq(1 + qq)^2}{q(1 - qq)(qq + 2q - 1)} = + \int \frac{dq}{q} - 2 \int \frac{dq}{1 - qq} - 4 \int \frac{dq}{(q + 1)^2 - 2} =$$
$$= + lq - l\frac{1 + q}{1 - q} + \sqrt{2}\, l\frac{\sqrt{2} + 1 + q}{\sqrt{2} - 1 - q}$$

hincque

$$\int \frac{dp}{p - u} = \frac{1}{2} lp - \frac{1}{2} lq + \frac{1}{2} l\frac{1 + q}{1 - q} - \frac{1}{\sqrt{2}} l\frac{\sqrt{2} + 1 + q}{\sqrt{2} - 1 - q} = l\left(\frac{1 + q}{2q}\right) - \frac{1}{\sqrt{2}} l\frac{\sqrt{2} + 1 + q}{\sqrt{2} - 1 - q}.$$

Iam $p - u = \frac{(1 + q)(1 - 2q - qq)}{2q} = + \frac{(1 + q)(2 - (1 + q)^2)}{2q}$

reliquis euanescentibus est $-2pxy = xx - 2appxx$, quae diuisibilis per x alteram praebet solutionem $x=0$. Talis quidem resolutio succedit, quando valorem p per radicis extractionem elicere licet; sed si aequatio ad plures dimensiones ascendat, vel adeo transcendens fiat, methodo hic exposita carere non possumus.

Exemplum 4.

684. *Proposita aequatione*

$$x\,dy^2 + y\,dx^2 = dy\,dx\,\sqrt{xy(dx^2+dy^2)}$$

eius integrale completum inuestigare.

Posito $\frac{dy}{dx}=p$, et $y=ux$, nostra aequatio induet hanc formam $p^2+u=p\sqrt{u(1+pp)}$, vnde conficitur $\frac{dx}{x}=\frac{du}{p-u}$, seu $lx=\int\frac{du}{p-u}=-l(p-u)+\int\frac{dp}{p-u}$. Inde autem est

$$\sqrt{u}=\tfrac{1}{2}p\sqrt{(1+pp)}+\tfrac{1}{2}p\sqrt{(1-4p+pp)},$$

et quadrando

$$u=\tfrac{1}{2}pp-p^3+\tfrac{1}{2}p^4+\tfrac{1}{2}pp\sqrt{(1+pp)(1-4p+pp)}$$

hincque

$$p-u=\tfrac{1}{2}p(1+pp)(2-p)-\tfrac{1}{2}pp\sqrt{(1+pp)(1-4p+pp)}$$

vnde colligimus

$$\frac{dp}{p-u}=\frac{dp(2-p)}{2p(1-p+pp)}+\frac{dp\sqrt{(1-4p+pp)}}{2(1-p+pp)\sqrt{(1+pp)}}.$$

In quorum membrorum posteriore si ponatur $\sqrt{\frac{1-4p+pp}{1+pp}}=q$; ob

ob $p=\frac{2+\sqrt{(4-(1-qq)^2)}}{1-qq}$; $dp=\frac{4qdq(2+\sqrt{(4-(1-qq)^2)})}{(1-qq)^2\sqrt{(4-(1-qq)^2)}}$ et $1-p+pp=\frac{(3+qq)(2+\sqrt{(4-(1-qq)^2)})}{(1-qq)^2}$ obtinebitur

$$\int\frac{dp}{p-u}=\frac{1}{2}\int\frac{dp(2-p)}{p(1-p+pp)}+2\int\frac{qqdq}{(3+qq)\sqrt{(4-(1-qq)^2)}}$$

vbi membrum posterius neque per logarithmos, neque arcus circulares integrari potest.

Exemplum 5.

685. *Inuenire relationem inter* x *et* y, *vt posito* $s=\int\sqrt{(dx^2+dy^2)}$ *fiat* $ss=2xy$.

Cum sit $s=\sqrt{2xy}$ erit $ds=\sqrt{(dx^2+dy^2)}=\frac{xdy+ydx}{\sqrt{2xy}}$ hincque posito $\frac{dy}{dx}=p$ et $y=ux$ fiet $\sqrt{(1+pp)}=\frac{p+u}{\sqrt{2u}}$, seu $u=\sqrt{2u(1+pp)}-p$, et radice extracta $\sqrt{u}=\sqrt{\frac{1+pp}{2}}+\frac{1-p}{\sqrt{2}}=\frac{1-p+\sqrt{(1+pp)}}{\sqrt{2}}$, quare

$u=1-p+pp+(1-p)\sqrt{(1+pp)}$ et $p-u=-(1-p)(1-p+\sqrt{(1+pp)})$.

Ergo

$$\int\frac{dp}{p-u}=\int\frac{dp}{2p(1-p)}(1-p-\sqrt{(1+pp)})=\frac{1}{2}lp-\frac{1}{2}\int\frac{dp\sqrt{(1+pp)}}{p(1-p)}.$$

At posito $p=\frac{1-qq}{2q}$ fit

$$\int\frac{dp\sqrt{(1+pp)}}{p(1-p)}=\int\frac{-dq(1+qq)^2}{q(1-qq)(qq+2q-1)}=+\int\frac{dq}{q}-2\int\frac{dq}{1-qq}-4\int\frac{dq}{(q+1)^2-2}$$
$$=+lq-l\frac{1+q}{1-q}+\sqrt{2}\,l\frac{\sqrt{2}+1+q}{\sqrt{2}-1-q}$$

hincque

$$\int\frac{dp}{p-u}=\frac{1}{2}lp-\frac{1}{2}lq+\frac{1}{2}l\frac{1+q}{1-q}-\frac{1}{\sqrt{2}}l\frac{\sqrt{2}+1+q}{\sqrt{2}-1-q}=l\left(\frac{1+q}{2q}\right)-\frac{1}{\sqrt{2}}l\frac{\sqrt{2}+1+q}{\sqrt{2}-1-q}.$$

Iam $p-u=\frac{(1+q)(1-2q-qq)}{2q}=+\frac{(1+q)(2-(1+q)^2)}{2q}$

 sicque

ſicque habetur

$$lx = C - l(1+q) + lq - l(2-(1+q)^2) + l\frac{1+q}{q} - \frac{1}{\sqrt{2}}\, l\frac{\sqrt{2}+1+q}{\sqrt{2}-1-q}$$

$$= la - l(2-(1+q)^2) - \frac{1}{\sqrt{2}}\, l\frac{\sqrt{2}+1+q}{\sqrt{2}-1-q}$$

vbi eſt $u = \frac{y}{x} = \frac{1}{2}(1+q)^2$, et $1+q = \sqrt{\frac{2y}{x}}$ vnde $x = \frac{ax}{x-y}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)^{\frac{1}{\sqrt{2}}}$ ſeu $x-y = a\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)^{\frac{1}{\sqrt{2}}}$ vel $(\sqrt{x}+\sqrt{y})^{1+\frac{1}{\sqrt{2}}} = a(\sqrt{x}-\sqrt{y})^{\frac{1}{\sqrt{2}}-1}$. Eſt ergo aequatio inter x et y interſcendens vti vocari ſolet.

Scholion.

686. Facilius haec reſolutio abſoluitur quaerendo ſtatim ex aequatione

$u + p = \sqrt{2u(1+pp)}$ ſeu $uu + 2up + pp = 2u + 2upp$

valorem ipſius p qui fit

$$p = \frac{u + \sqrt{(uu - 4uu + 2u + 2u^3 - uu)}}{2u-1} \text{ ſeu } p = \frac{u + (1-u)\sqrt{2u}}{2u-1}$$

et $p - u = \frac{(1-u)(2u+\sqrt{2u})}{2u-1} = \frac{(1-u)\sqrt{2u}}{\sqrt{2u}-1}$.

Quare

$$lx = \int\frac{du}{p-u} = \int\frac{du(\sqrt{2u}-1)}{(1-u)\sqrt{2u}} = C - l(1-u) - \int\frac{du}{(1-u)\sqrt{2u}}$$

ſit $u = vv$ eritque

$$\int\frac{du}{(1-u)\sqrt{2u}} = \frac{1}{\sqrt{2}}\int\frac{2dv}{1-vv} = \frac{1}{\sqrt{2}}\, l\frac{1+v}{1-v}$$

hincque

$$lx = la - l(1-u) - \frac{1}{\sqrt{2}}\, l\frac{1+\sqrt{u}}{1-\sqrt{u}}.$$

Vnde

Vnde ob $u = \frac{y}{x}$ reperitur $x = \frac{ax}{x-y}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)^{\frac{1}{\sqrt{2}}}$; vt ante. Quare si curua desideretur coordinatis rectangulis x et y determinanda, vt eius arcus s sit $= \sqrt{2xy}$, erit aequatio eius naturam definiens:

$$(\sqrt{x}+\sqrt{y})^{\frac{1}{\sqrt{2}}+1} = a(\sqrt{x}-\sqrt{y})^{\frac{1}{\sqrt{2}}-1}.$$

Caeterum euidens est simili modo quaestionem resolui posse, si arcus s functioni cuicunque homogeneae vnius dimensionis ipsarum x et y aequetur, seu si proponatur aequatio quaecunque homogenea inter x, y et s, id quod sequenti problemate ostendisse operae erit pretium.

Problema 90.

687. Si fuerit $s = \int \sqrt{(dx^2 + dy^2)}$, atque aequatio proponatur homogenea quaecunque inter x, y et s, in qua scilicet hae tres variabiles x, y et s vbique eundem dimensionum numerum constituant, inuenire aequationem finitam inter x et y.

Solutio.

Ponatur $y = ux$ et $s = vx$, vt hac substitutione ex aequatione homogenea proposita variabilis x elidatur, et aequatio obtineatur inter binas u et v, vnde v per u definiri possit. Tum vero sit $dy = pdx$; eritque $ds = dx\sqrt{(1+pp)}$ vnde fit

$$pdx = udx + xdu \text{ et } dx\sqrt{(1+pp)} = vdx + xdv$$

ergo $\frac{dx}{x} = \frac{du}{p-u} = \frac{dv}{\sqrt{(1+pp)}-v}$.

 Quia

Quia nunc v datur per u, fit $dv = q\,du$, vt habeatur $\sqrt{(1+pp)} = v + pq - qu$, et sumtis quadratis

$$1 + pp = (v - qu)^2 + 2pq(v - qu) + ppqq,$$

vnde elicitur

$$p = \frac{q(v - qu) + \sqrt{((v - qu)^2 - 1 + qq)}}{1 - qq} \text{ et}$$

$$p - u = \frac{qv - u + \sqrt{((v - qu)^2 - 1 + qq)}}{1 - qq}.$$

Quare hinc deducimus

$$\frac{dx}{x} = \frac{du(1 - qq)}{qv - u + \sqrt{((v - qu)^2 - 1 + qq)}} = \frac{du(qv - u - \sqrt{((v - qu)^2 - 1 + qq)})}{1 + uu - vv}$$

vnde cum v et q detur per u inueniri potest x per eandem u: at ob $q\,du = dv$ fiet:

$$lx = la - l\sqrt{(1 + uu - vv)} - \int \frac{du \sqrt{((v - qu)^2 - 1 + qq)}}{1 + uu - vv}$$

tum vero est $y = ux$, seu posito $\frac{y}{x}$ loco u habebitur aequatio quaesita inter x et y.

Coroll. 1.

688. Cum s exprimat arcum curuae coordinatis rectangulis x et y respondentem, sic definitur curua, cuius arcus aequatur functioni cuicunque vnius dimensionis ipsarum x et y; quae ergo erit algebraica, si integrale $\int \frac{du \sqrt{((v - qu)^2 - 1 + qq)}}{1 + uu - vv}$ per logarithmos exhiberi potest.

Coroll. 2.

689. Simili modo resolui poterit problema, si s eiusmodi formulam integralem exprimat, vt

fit

fit $ds = Q dx$ existente Q functione quacunque quantitatum p, u et v. Tum autem ex aequalitate $\frac{dx}{x} = \frac{du}{p-u} = \frac{du}{Q-v}$ valorem ipsius p elici oportet, et quia v per u datur, erit $lx = \int \frac{du}{p-u}$.

Exemplum 1.

690. Si debeat esse $s = \alpha x + \beta y$, erit $v = \alpha + \beta u$, et $q = \frac{dv}{du} = \beta$, hinc $v - qu = \alpha$, ergo:

$$lx = la - l\sqrt{(1 + uu - (\alpha + \beta u)^2)} - \int \frac{du\sqrt{(\alpha\alpha + \beta\beta - 1)}}{1 + uu - (\alpha + \beta u)^2}$$

quae postrema pars est

$$-\int \frac{du\sqrt{(\alpha\alpha + \beta\beta - 1)}}{1 - \alpha\alpha - 2\alpha\beta u + (1 - \beta\beta)uu} = (\alpha\alpha + \beta\beta - 1)^{\frac{1}{2}} \int \frac{du}{\alpha\alpha - 1 + 2\alpha\beta u + (\beta\beta - 1)uu}$$

quae transformatur in

$$\int \frac{(\beta\beta - 1)du\sqrt{(\alpha\alpha + \beta\beta - 1)}}{(u(\beta\beta - 1) + \alpha\beta - \sqrt{(\alpha\alpha + \beta\beta - 1)})(u(\beta\beta - 1) + \alpha\beta + \sqrt{(\alpha\alpha + \beta\beta - 1)})}$$
$$= \frac{1}{2} l \frac{(\beta\beta - 1)u + \alpha\beta - \sqrt{(\alpha\alpha + \beta\beta - 1)}}{(\beta\beta - 1)u + \alpha\beta + \sqrt{(\alpha\alpha + \beta\beta - 1)}}.$$

Quare posito $u = \frac{y}{x}$ aequatio integralis quaesita est, sumtis quadratis:

$$\frac{xx + yy - (\alpha x + \beta y)^2}{aa} = \frac{(\beta\beta - 1)y + \alpha\beta x - x\sqrt{(\alpha\alpha + \beta\beta - 1)}}{(\beta\beta - 1)y + \alpha\beta x + x\sqrt{(\alpha\alpha + \beta\beta - 1)}}.$$

At posito

$$(\beta\beta - 1)y + \alpha\beta x - x\sqrt{(\alpha\alpha + \beta\beta - 1)} = P$$
$$(\beta\beta - 1)y + \alpha\beta x + x\sqrt{(\alpha\alpha + \beta\beta - 1)} = Q$$

est

$$PQ = (\beta\beta - 1)^2 yy + 2\alpha\beta(\beta\beta - 1)xy + (\alpha\alpha - 1)(\beta\beta - 1)xx$$
$$= (\beta\beta - 1)((\alpha x + \beta y)^2 - xx - yy)$$

unde

vnde mutata constante fit $\frac{PQ}{bb} = \frac{P}{Q}$, ergo vel $P=0$ vel $Q=b$; solutio ergo in genere est

$$(\beta\beta-1)y+\alpha\beta x \pm x\sqrt{(\alpha\alpha+\beta\beta-1)}=c$$

quae est aequatio pro linea recta.

Exemplum 2.

691. Si debeat esse $s=\frac{nyy}{x}$, erit $v=nuu$ et $q=2nu$; vnde $1+uu-vv=1+uu-nnu^4$ et $v-qu=-nuu$ ergo

$$lx=la-l\sqrt{(1+uu-nnu^4)}-\int\frac{du\sqrt{(nnu^4-1+nnuu)}}{1+uu-nnu^4}$$

quae formula autem per logarithmos integrari nequit.

Exemplum 3.

692. Si debeat esse $ss=xx+yy$ erit $v=\sqrt{(1+uu)}$ et $q=\frac{u}{\sqrt{(1+uu)}}$; vnde fit $1+uu-vv=0$, solutionem ergo ex primis formulis repeti conuenit, vnde fit $v-qu=\frac{1}{\sqrt{(1+uu)}}$; $qq-1=\frac{-1}{1-uu}$ et $qv-u=0$; ergo $p-u=0$ seu $\frac{dy}{dx}-\frac{y}{x}=0$, ita vt prodeat $y=nx$.

Exemplum 4.

693. Si debeat esse $ss=yy+nxx$ seu $v=\sqrt{(uu+n)}$ et $q=\frac{u}{\sqrt{(uu+n)}}$ erit $1+uu-vv=1-n$; $v-qu=\frac{n}{\sqrt{(uu+n)}}$ et $qq-1=\frac{-n}{uu+n}$. Quare habebitur

$$lx=la-l\sqrt{(1-n)}-\frac{1}{1-n}\int\frac{du\sqrt{(nn-n)}}{\sqrt{(uu+n)}}=lb+\frac{\sqrt{n}}{\sqrt{(n-1)}}l(u+\sqrt{(uu+n)})$$

hinc-

hincque

$$\frac{x}{b} = \left(\frac{y + \sqrt{(yy + nxx)}}{x}\right)^{\sqrt{\frac{n-1}{n}}}.$$

Quoties ergo $\frac{n}{n-1}$ est numerus quadratus aequatio inter x et y prodit algebraica. Sit $\sqrt{\frac{n}{n-1}} = m$, erit $n = \frac{mm}{mm-1}$, et $ss = yy + \frac{mmxx}{mm-1}$ cui conditioni satisfit hac aequatione algebraica:

$$x^{m+1} = b\left(y + \sqrt{\left(yy + \frac{mmxx}{mm-1}\right)}\right)^m$$

quae transformatur in

$$x^{\frac{2}{m}} - 2b^{\frac{1}{m}} x^{\frac{1-m}{m}} y = \frac{mm}{mm-1} b^{\frac{2}{m}} \text{ seu}$$

$$y = \frac{(mm-1)x^{\frac{2}{m}} - mmb^{\frac{2}{m}}}{2(mm-1)b^{\frac{1}{m}} x^{\frac{1-m}{m}}}.$$

Corollarium.

694. Ponamus $m = \frac{1}{n}$, ac si fuerit

$$y = \frac{b^{2n} + (nn-1)x^{2n}}{2(nn-1)b^n x^{n-1}} \text{ erit}$$

$$ss = yy - \frac{xx}{nn-1}, \text{ seu } s = \sqrt{\left(yy - \frac{xx}{nn-1}\right)}.$$

Quare si

$$y = \frac{b^4 + 3x^4}{6bbx} \text{ est } s = \sqrt{\left(yy - \frac{xx}{3}\right)}.$$

Problema 91.

695. Si posito $\frac{dy}{dx}=p$ eiusmodi detur aequatio inter x, y et p, in qua altera variabilis y vnicam tantum habeat dimensionem, inuenire relationem inter binas variabiles x et y.

Solutio.

Hinc ergo y aequabitur functioni cuipiam ipsarum x et p vnde differentiando fiet $dy=Pdx+Qdp$. Cum igitur sit $dy=pdx$, habebitur haec aequatio differentialis: $(P-p)dx+Qdp=0$, quam integrari oportet. Quoniam tantum duas continet variabiles x et p, et differentialia simpliciter inuoluit, eius resolutio per methodos supra expositas est tentanda.

Primo ergo resolutio succedet, si fuerit $P=p$, ideoque $dy=pdx+Qdp$. Quod euenit, si y per x et p ita determinetur, vt sit $y=px+\Pi$, denotante Π functionem quamcunque ipsius p. Tum ergo erit $Q=x+\frac{d\Pi}{dp}$ et cum solutio ab ista aequatione $Qdp=0$ pendeat, erit vel $dp=0$, hincque $p=\alpha$ seu $y=\alpha x+\beta$ vbi altera constantium α et β per ipsam aequationem propositam determinatur, dum posito $p=\alpha$ fit $\beta=\Pi$; vel erit $Q=0$, ideoque $x=-\frac{d\Pi}{dp}$, et $y=-\frac{pd\Pi}{dp}+\Pi$, vbi ergo vtraque solutio est algebraica, si modo Π fuerit functio algebrica ipsius p.

Secundo, aequatio $(P-p)dx+Qdp=0$, resolutionem admittet, si altera variabilis x cum suo diffe-

differentiali dx vnam dimensionem non superet. Euenit hoc si fuerit $y = Px + \Pi$ dum P et Π sunt functiones ipsius p tantum, tum enim erit $P = P$ et $Q = \frac{xdP}{dp} + \frac{d\Pi}{dp}$, hincque haec habeatur aequatio integranda $(P-p)dx + xdP + d\Pi = 0$ seu $dx + \frac{xdP}{P-p} = -\frac{d\Pi}{P-p}$, quae per $e^{\int \frac{dP}{P-p}}$ multiplicata dat $e^{\int \frac{dP}{P-p}} x = -\int e^{\int \frac{dP}{P-p}} \frac{d\Pi}{P-p}$. Siue ponatur $\frac{dP}{P-p} = \frac{dR}{R}$, erit aequatio integralis $Rx = C - \int \frac{Rd\Pi}{P-p} = C - \int \frac{d\Pi\, dR}{dP}$; vnde fit $x = \frac{C}{R} - \frac{1}{R}\int \frac{d\Pi\, dR}{dP}$ et $y = \frac{CP}{R} + \Pi - \frac{P}{R}\int \frac{d\Pi\, dR}{dP}$.

Tertio resolutio nullam habebit difficultatem, si denotantibus X et V functiones quascunque ipsius x fuerit $y = X + Vp$. Tum enim erit $dy = pdx = dX + Vdp + pdV$, ideoque $dp + p\left(\frac{dV - dx}{V}\right) = -\frac{dX}{V}$, sit $\frac{dx}{V} = \frac{dR}{R}$, vt R sit etiam functio ipsius x, erit $\frac{V}{R}p = C - \int \frac{dX}{R}$ seu $p = \frac{CR}{V} - \frac{R}{V}\int \frac{dX}{R}$, et $y = X + CR - R\int \frac{dX}{R}$, quae aequatio relationem inter x et y exprimit.

Quarto aequatio $(P-p)dx + Qdp = 0$ resolutionem admittit si fuerit homogenea. Cum ergo terminus pdx duas contineat dimensiones, hoc euenit, si totidem dimensiones et in reliquis terminis insint. Vnde perspicuum est, P et Q esse debere functiones homogeneas vnius dimensionis ipsarum x et p. Quare si y ita per x et p definiatur, vt y aequetur functioni homogeneae duarum dimensionum ipsarum x et p, resolutio succedet. Quodsi enim fuerit $dy = Pdx + Qdp$, aequatio solutionem continens $(P-p)dx + Qdp = 0$,

 erit

erit homogenea, fietque per se integrabilis, si diuidatur per $(P-p)x+Qp$.

Coroll. 1.

696. Pro casu quarto si ponatur $y=zz$, aequatio proposita debet esse homogenea inter tres variabiles x, z et p. Vnde si proponatur aequatio homogenea quaecunque inter x, z et p, in qua hae ternae litterae x, z et p vbique eundem dimensionum numerum constituant, problema semper resolutionem admittit.

Coroll. 2.

697. Simili modo conuersis variabilibus, si ponatur $x=vv$ et $\frac{dx}{dy}=q$, vt sit $p=\frac{1}{q}$; ac proponatur aequatio homogenea quaecunque inter y, v et q, problema itidem resolui potest.

Scholion.

698. Pro casu quarto, vt aequatio $(P-p)dx+Qdp=0$ fiat homogenea, conditiones magis amplificari possunt. Ponatur enim $x=v^{\mu}$ et $p=q^{\nu}$, fitque facta substitutione haec aequatio $\mu(P-q^{\nu})v^{\mu-1}dv+\nu Qq^{\nu-1}dq=0$ homogenea inter v et q; eritque P functio homogenea ν dimensionum, et Q functio homogenea μ dimensionum. Cum iam sit

$$dy=Pdx+Qdp=\mu Pv^{\mu-1}dv+\nu Qq^{\nu-1}dq$$

erit y functio homogenea $\mu+\nu$ dimensionum. Quare posito $y=z^{\mu+\nu}$ problema resolutionem admittit, si inter

inter x, y et p eiusmodi relatio proponatur, vt positio $y = z^{\mu+\nu}$; $x = v^{\mu}$ et $p = q^{\nu}$ habeatur aequatio homogenea inter ternas quantitates z, v et q, ita vt dimensionum ab iis formatarum numerus vbique sit idem. Ac si proposita fuerit huiusmodi aequatio homogenea inter z, v et q, solutio problematis ita expedietur. Cum sit $dy = p\,dx$, erit

$$(\mu+\nu)z^{\mu+\nu-1}dz = \mu v^{\mu-1}q^{\nu}dv;$$

ponatur iam $z = rq$ et $v = sq$; et aequatio proposita tantum duas litteras r et s continebit, ex qua alteram per alteram definire licet, tum autem per has substitutiones prodibit haec aequatio:

$$(\mu+\nu)r^{\mu+\nu-1}q^{\mu+\nu-1}(rdq+qdr) = \mu s^{\mu-1}q^{\mu+\nu-1}(sdq+qds)$$

ex qua oritur

$$\frac{dq}{q} = \frac{\mu s^{\mu-1}ds - (\mu+\nu)r^{\mu+\nu-1}dr}{(\mu+\nu)r^{\mu+\nu} - \mu s^{\mu}}$$

quae est aequatio differentialis separata, quoniam s per r datur. Quin etiam bini casus allati manifesto continentur in formulis $y = z^{\mu+\nu}$, $x = v^{\mu}$ et $p = q^{\nu}$; prior scilicet si $\mu = 1$ et $\nu = 1$, posterior vero si $\mu = 2$ et $\nu = -1$. Hos igitur casus perinde ac praecedentes exemplis illustrari conueniet, quorum primus praecipue est memorabilis, cum per differentiationem aequationis propositae $y = px + \Pi$ statim praebeat aequationem integralem quaesitam, neque integratione omnino sit opus, siquidem alteram solutionem ex $dp = 0$ natam excludamus.

Exemplum 1.

699. *Proposita aequatione differentiali*

$$ydx - xdy = a\sqrt{(dx^2 + dy^2)}$$

eius integrale inuenire.

Posito $\frac{dy}{dx} = p$ fit $y - px = a\sqrt{(1 + pp)}$, quae aequatio differentiata ob $dy = pdx$ dat $-xdp = \frac{apdp}{\sqrt{(1+pp)}}$, quae cum sit diuisibilis per dp praebet primo $p = \alpha$, hincque $y = \alpha x + a\sqrt{(1 + \alpha\alpha)}$. Alter vero factor suppeditat $x = \frac{-ap}{\sqrt{(1+pp)}}$, hincque $y = \frac{-app}{\sqrt{(1+pp)}} + a\sqrt{(1+pp)} = \frac{a}{\sqrt{(1+pp)}}$, vnde fit $xx + yy = aa$, quae est etiam aequatio integralis, sed quia nouam constantem non inuoluit, non pro completo integrali haberi potest. Integrale autem completum duas aequationes complectitur. Scilicet

$$y = \alpha x + a\sqrt{(1 + \alpha\alpha)} \text{ et } xx + yy = aa$$

quae in hac vna comprehendi possunt:

$$((y - \alpha x)^2 - aa(1 + \alpha\alpha))(xx + yy - aa) = 0.$$

Scholion.

700. Nisi hoc modo operatio instituatur, solutio huius quaestionis fit satis difficilis. Si enim aequationem differentialem $ydx - xdy = a\sqrt{(dx^2 + dy^2)}$ quadrando ab irrationalitate liberemus, indeque rationem $\frac{dy}{dx}$ per radicis extractionem definiamus, fit

$$(xx - aa)dy - xydx = \pm adx\sqrt{(xx + yy - aa)}$$

quae

quae aequatio per methodos cognitas difficulter tractatur. Multiplicator quidem inueniri poteſt vtrumque membrum per ſe integrabile reddens; prius enim membrum $(xx-aa)dy-xydx$ diuiſum per $y(xx-aa)$ fit integrabile integrali exiſtente $l\frac{y}{\surd(xx-aa)}$: vnde in genere multiplicator id integrabile reddens eſt

$$\frac{1}{y(xx-aa)}\Phi:\frac{y}{\surd(xx-aa)}$$

quae functio ita determinari debet, vt eodem multiplicatore quoque alterum membrum $adx\surd(xx+yy-aa)$ fiat integrabile. Talis autem multiplicator eſt:

$$\frac{1}{y(xx-aa)}\cdot\frac{y}{\surd(xx+yy-aa)}=\frac{1}{(xx-aa)\surd(xx+yy-aa)}$$

quo fit

$$\frac{(xx-aa)dy-xydx}{(xx-aa)\surd(xx+yy-aa)}=\frac{+adx}{xx-aa}.$$

Iam ad integrale prioris membri inueſtigandum, ſpectetur x vt conſtans, eritque inregrale

$$=l(y+\surd(xx+yy-aa))+X$$

denotante X functionem quampiam ipſius x, ita comparatam, vt ſumta iam y conſtante fiat:

$$\frac{xdx}{(y+\surd(xx+yy-aa))\surd(xx+yy-aa)}+dX=\frac{-xydx}{(xx-aa)\surd(xx+yy-aa)}$$ ſeu

$$\frac{-xdx(y-\surd(xx+yy-aa))}{(xx-aa)\surd(xx+yy-aa)}+dX=\frac{-xydx}{(xx-aa)\surd(xx+yy-aa)}$$

vnde fit

$$dX=\frac{-xdx}{xx-aa}\text{ et }X=l\frac{C}{\surd(xx-aa)}$$

Quare

Quare integrale quaesitum est

$$l(y+\sqrt{(xx+yy-aa)})+l\frac{C}{\sqrt{(xx-aa)}}=\pm\tfrac{1}{2}l\frac{a+x}{a-x} \text{ seu}$$

$$\frac{y+\sqrt{(xx+yy-aa)}}{\sqrt{(xx-aa)}}=\alpha\sqrt{\frac{x+a}{x-a}} \text{ vel } =\alpha\sqrt{\frac{x-a}{x+a}}$$

vnde fit

$$y+\sqrt{(xx+yy-aa)}=\alpha(x\pm a)\text{, hincque}$$
$$xx-aa=\alpha\alpha(x\pm a)^2-2\alpha(x\pm a)y \text{ vel}$$
$$x\mp a=\alpha\alpha(x\pm a)-2\alpha y$$

quae autem tantum est altera binarum aequationum integralium, altera autem aequatio integralis $xx+yy=aa$ iam quasi per diuisionem de calculo sublata est censenda. Caeterum eadem solutio aequationis

$$(aa-xx)dy+xydx=\pm adx\sqrt{(xx+yy-aa)}$$

facilius instituitur ponendo $y=u\sqrt{(aa-xx)}$, vnde fit $(aa-xx)^{\frac{3}{2}}du=\pm adx\sqrt{(aa-xx)(uu-1)}$ seu $\frac{du}{\sqrt{(uu-1)}}=\frac{\pm adx}{aa-xx}$

cui quidem satisfit sumendo $u=1$, neque tamen hic casus in aequatione integrali continetur, vti supra iam ostendimus. Ex quo suspicari liceret alteram solutionem $xx+yy=aa$ adeo esse excludendam, quod tamen secus se habere deprehenditur; si ipsam aequationem primariam $\frac{ydx-xdy}{\sqrt{(dx^2+dy^2)}}=\alpha$ perpendamus. Si enim x et y sint coordinatae rectangulae lineae curuae, formula $\frac{ydx-xdy}{\sqrt{(dx^2+dy^2)}}$ exprimit perpendiculum ex origine coordinatarum in tangen-

tem

tem dimiſſum, quod ergo conſtans eſſe debet. Hoc autem euenire in circulo origine in centro conſtituta, dum aequatio fit $xx+yy=aa$, per ſe eſt manifeſtum. Atque hinc realitas harum ſolutionum, quae minus congruae videri poterant, confirmatur, etiamſi earum ratio haud ſatis clare perſpiciatur.

Exemplum 2.

701. *Propoſita aequatione differentiali*

$$ydx-xdy=\frac{a(dx^2+dy^2)}{dx}$$

eius integrale inuenire.

Poſito $dy=pdx$ fit $y-px=a(1+pp)$ et differentiando $-xdp=2apdp$, vnde concluditur vel $dp=0$, et $p=\alpha$, hincque $y=\alpha x+a(1+\alpha\alpha)$ vel $x=-2ap$ et $y=a(1-pp)$, ſicque ob $p=\frac{-x}{2a}$ habebitur $4ay=4aa-xx$, quae aequatio ad geometriam translata illam conditionem omnino adimplet.

Ex aequatione autem propoſita radicem extrahendo reperitur

$$2ady+xdx=dx\sqrt{(xx+4ay-4aa)}$$

quae poſito $y=u(4aa-xx)$ abit in

$$2adu(4aa-xx)-xdx(4au-1)=dx\sqrt{(4aa-xx)(4au-1)}$$

haecque poſito $4au-1=tt$ in

$$tdt(4aa-xx)-ttxdx=tdx\sqrt{(4aa-xx)}$$

quae cum ſit diuiſibilis per t, concludere licet $t=0$, ideoque $u=\frac{1}{4a}$, atque hinc $4ay=4aa-xx$.

Exemplum 3.

702. *Proposita aequatione differentiali*

$$ydx - xdy = a\sqrt[3]{(dx^3 + dy^3)}.$$

eius integrale assignare.

Haec aequatio more consueto, si rationem $\frac{dy}{dx}$ inde extrahere vellemus, vix tractari posset. Posito autem $dy = pdx$ fit $y - px = a\sqrt[3]{(1+p^3)}$ et differentiando $xdp = \frac{-appdp}{\sqrt[3]{(1+p^3)^2}}$ vnde duplex conclusio deducitur: vel $dp = 0$ et $p = \alpha$, sicque $y = \alpha x + a\sqrt[3]{(1+\alpha^3)}$, vel $x = \frac{-app}{\sqrt[3]{(1+p^3)^2}}$ et $y = \frac{a}{\sqrt[3]{(1+p^3)^2}}$ vnde fit $pp = -\frac{x}{y}$, et ob $y^3(1+p^3)^2 = a^3$, erit $p^3 = \frac{a\sqrt{a}}{y\sqrt{y}} - 1$, hincque $\frac{(a\sqrt{a} - y\sqrt{y})^2}{y^3} = -\frac{x^3}{y^3}$ seu $x^3 + (a\sqrt{a} - y\sqrt{y})^2 = 0$.

Exemplum 4.

703. *Proposita aequatione*

$$ydx - nxdy = a\sqrt{(dx^2 + dy^2)}$$

eius integrale inuenire.

Posito $dy = pdx$ habetur $y - npx = a\sqrt{(1+pp)}$, vnde differentiando elicitur $(1-n)pdx - nxdp = \frac{apdp}{\sqrt{(1+pp)}}$, siue $dx - \frac{nxdp}{(1-n)p} = \frac{adp}{(1-n)\sqrt{(1+pp)}}$ quae per $p^{\frac{n}{n-1}}$ multiplicata et integrata praebet

$$p^{\frac{n}{n-1}}x = \frac{a}{1-n}\int \frac{p^{\frac{n}{n-1}}dp}{\sqrt{(1+pp)}}.$$

Hinc

Hinc deducimus casus sequentes, integrationem admittentes:

si $n=\frac{3}{2}$; $p^3 x=C-\frac{2}{3}a(pp-\frac{2}{1})\surd(1+pp)$

si $n=\frac{5}{4}$; $p^5 x=C-\frac{4}{5}a(p^4-\frac{4}{3}p^2+\frac{4\cdot 2}{3\cdot 1})\surd(1+pp)$

si $n=\frac{7}{6}$; $p^7 x=C-\frac{6}{7}a(p^6-\frac{6}{5}p^4+\frac{6\cdot 4}{5\cdot 3}p^2-\frac{6\cdot 4\cdot 2}{5\cdot 3\cdot 1})\surd(1+pp)$

ac si $n=\frac{2\lambda+1}{2\lambda}$ erit $y=px+a\surd(1+pp)+\frac{px}{2\lambda}$ et

$$x=\frac{C}{p^{2\lambda+1}}-\frac{2\lambda a}{(2\lambda+1)p}(1-\frac{2\lambda}{(2\lambda-1)pp}+\frac{2\lambda(2\lambda-2)}{(2\lambda-1)(2\lambda-3)p^4}-\text{etc})\surd(1+pp).$$

Quodsi ergo sumatur $\lambda=\infty$ vt sit $n=1$, erit

$$y=px+a\surd(1+pp) \text{ et } x=\frac{C}{p^{2\lambda+1}}-\frac{ap}{\surd(1+pp)}$$

vnde si constans C sit $=0$, statim sequitur solutio superior $xx+yy=aa$. At si constans C non euanescat, minimum discrimen in quantitate p infinitam varietatem ipsi x inducit. Quantumuis ergo x varietur, quantitas p vt constans spectari potest, vnde posito $p=\alpha$ altera solutio $y=\alpha x+a\surd(1+\alpha\alpha)$ obtinetur. Hinc ergo dubium supra circa exemplum I. natum non mediocriter illustratur.

Exemplum 5.

704. *Proposita aequatione differentiali*
$A\,dy^n=(Bx^\alpha+Cy^\beta)dx^n$ *existente* $n=\frac{\alpha\beta}{\alpha-\beta}$
eius intergrale inuestigare.

 Posito

Posito $\frac{dy}{dx}=p$ erit $Ap^n=Bx^\alpha+Cy^\beta$. Ponamus iam $p=q^{\alpha\beta}$, $x=v^{\beta n}$ et $y=z^{\alpha n}$, vt habeamus hanc aequationem homogeneam $Aq^{\alpha\beta n}=Bv^{\alpha\beta n}+Cz^{\alpha\beta n}$, quae, positis $z=rq$ et $v=sq$, abit in $A=Bs^{\alpha\beta n}+Cr^{\alpha\beta n}$. Cum vero sit

$$dy=\alpha n z^{\alpha n-1}dz=\alpha n r^{\alpha n-1}q^{\alpha n-1}(r\,dq+q\,dr) \text{ et}$$

$$p\,dx=\beta n v^{\beta n-1}q^{\alpha\beta}dv=\beta n s^{\beta n-1}q^{\alpha\beta+\beta n-1}(s\,dq+q\,ds)$$

erit $\alpha r^{\alpha n-1}(r\,dq+q\,dr)=\beta s^{\beta n-1}q^{\alpha\beta+\beta n-\alpha n}(s\,dq+q\,ds)$.

Est vero per hypothesin $\alpha\beta+\beta n-\alpha n=0$, vnde oritur

$$\alpha r^{\alpha n}dq+\alpha r^{\alpha n-1}q\,dr=\beta s^{\beta n}dq+\beta s^{\beta n-1}q\,ds$$

hincque

$$\frac{dq}{q}=\frac{\alpha r^{\alpha n-1}dr-\beta s^{\beta n-1}ds}{\beta s^{\beta n}-\alpha r^{\alpha n}}.$$

At est

$$s^{\beta n}=\left(\frac{A-Cr^{\alpha\beta n}}{B}\right)^{\frac{1}{\alpha}} \text{ hincque}$$

$$\beta s^{\beta n-1}ds=-\frac{\beta C}{B}r^{\alpha\beta n-1}dr\left(\frac{A-Cr^{\alpha\beta n}}{B}\right)^{\frac{1-\alpha}{\alpha}}$$

vnde fit

$$\frac{dq}{q}=\frac{\alpha r^{\alpha n-1}dr+\frac{\beta C}{B}r^{\alpha\beta n-1}dr\left(\frac{A-Cr^{\alpha\beta n}}{B}\right)^{\frac{1-\alpha}{\alpha}}}{\beta\left(\frac{A-Cr^{\alpha\beta n}}{B}\right)^{\frac{1}{\alpha}}-\alpha r^{\alpha n}}$$

Facilins autem calculus hoc modo instituetur, sumto $A = 1$ erit

$$p = \frac{dy}{dx} = (Bx^{\alpha} + Cy^{\beta})^{\frac{\nu}{n}},$$

sit $y = x^{\frac{\alpha}{\beta}} u$, fiet

$$x^{\frac{\alpha}{\beta}} du + \frac{\alpha}{\beta} x^{\frac{\alpha-\beta}{\beta}} u dx = x^{\frac{\alpha}{n}} dx (B + Cu^{\beta})^{\frac{\nu}{n}}$$

quae aequatio, cum sit $\frac{\alpha}{n} = \frac{\alpha - \beta}{\beta}$, abit in hanc

$$\beta x du + \alpha u dx = \beta dx (B + Cu^{\beta})^{\frac{\nu}{n}}$$

vnde fit

$$\frac{dx}{x} = \frac{\beta du}{\beta (B + Cu^{\beta})^{\frac{1}{n}} - \alpha u}$$

sicque x per u determinatur, et quia $u = x^{-\frac{\alpha}{\beta}} y$ habebitur aequatio inter x et y.

Scholion.

705. Hoc igitur modo operationem institui conueniet, quando inter binas variabiles x et y vna cum differentialium ratione $\frac{dy}{dx} = p$, eiusmodi relatio proponitur, ex qua valor ipsius p commode elici non potest. Tum ergo calculum ita tractari oportet, vt per differentiationem ponendo $dy = pdx$ vel $dx = \frac{dy}{p}$ tandem perueniatur ad aequationem differentialem simplicem inter duas tantum variabiles, quem in finem etiam saepe idoneis substitutionibus vti necesse est. Atque hucusque fere Geometris in resolu-

tione aequationum differentialium primi gradus etiamnum pertingere licuit, vix enim vlla via integralia inueſtigandi adhuc quidem adhibita hic praetermiſſa videtur. Num autem multo maiorem calculi integralis promotionem ſperare liceat? vix equidem affirmauerim, cum plurima extent inuenta, quae ante vires ingenii humani ſuperare videbantur.

Cum igitur calculum integralem in duos libros ſim partitus, quorum prior circa relationem, binarum tantum variabilium, poſterior vero ternarum pluriumue verſatur, atque iam libri primi partem priorem in differentialibus primi ordinis conſtitutam hic pro viribus expoſuerim, ad eius alteram partem progredior, in qua binarum variabilium relatio ex data differentialium ſecundi altiorisue ordinis conditione requiritur.